STAT 403

4/18/25

Lecture 8

Rejection sampling

HW2 out

Lecture Notes III – Importance sampling and rejection sampling

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Importance sampling

Rejection sampling

Rejection sampling

- Sampling from $F(x)$ when we only know $p(x) = F'(x)$
- (or, when X is multidimensional)

Given $p(x)$ a density

Want An algorithm to get samples from $p(x)$

- Will use $q(x)$ another distribution
- Let $\boxed{M = \sup_x \frac{p(x)}{q(x)}}$

  Note that $p(x) > 0$ implies $q(x) > 0$
- Rejection Sampling Algorithm
  1. sample $Y \sim q$
  2. sample $U \sim \mathrm{unif}[0,1]$
  3. if $\boxed{U < \frac{1}{M}\frac{p(Y)}{q(Y)}}$ then output $X = Y$

     else go to step 1.

Y, U independent

$\Pr\left[\text{"if } U < a \leq 1 \text{ then} \ldots\text{"}\right] = a$

Why does this work?

we will prove this for any $x \in \mathbb{R}$

- Denote F_{RS} the CDF of the samples from the Rejection Sampling algorithm.

any value $\in \mathbb{R}$

$$F_{RS}(x) = Pr[X \le x \mid Y \text{ accepted }] = \frac{Pr\left[X \le x, U < \frac{1}{M}\frac{p(Y)}{q(Y)}\right]}{Pr\left[U < \frac{1}{M}\frac{p(Y)}{q(Y)}\right]} \tag{3}$$

$\leftarrow$ Pr Y accepted

Pr is the probability under the joint distribution of U and Y

- the denominator

$$Pr\left[U < \frac{1}{M}\frac{p(Y)}{q(Y)}\right] = \int \left(Pr_{U\sim \mathrm{unif}[0,1]}\left[U < \frac{1}{M}\frac{p(y)}{q(y)}\right]\right) q(y)dy \tag{4}$$

$$= \int \frac{1}{M}\frac{p(y)}{q(y)} q(y)dy = \frac{1}{M}\int p(y)dy = \frac{1}{M} \tag{5}$$

- now the numerator

$$Pr\left[X \le x, U < \frac{1}{M}\frac{p(Y)}{q(Y)}\right] = \int \left(Pr_{U\sim \mathrm{unif}[0,1]}\left[U < \frac{1}{M}\frac{p(y)}{q(y)}\right]\right) I(y \le x)q(y)dy \tag{6}$$

$$= \int \frac{1}{M}\frac{p(y)}{q(y)} I(y \le x)q(y)dy \tag{7}$$

$$= \frac{1}{M}\int p(y)I(y \le x)dy = \frac{1}{M}\int_{-\infty}^{x} p(y)dy = \frac{1}{M}F(x)$$

we sample from

- Therefore $F_{RS}(x) = F(x)$ for all x QED

$\leftarrow$ want to sample from

$$\boxed{\Pr[Y \text{ accepted}]} = \int_{\mathbb{R}} \Big[\underbrace{\int_0^1 \mathbb{1}\Big(u < \frac{p(y)}{Mg(y)}\Big) 1\, du}_{\Pr[U < \cdots] = \frac{p(y)}{Mg(y)}} \Big] g(y)\, dy$$

$Y \sim g$

$U \sim \text{unif}[0,1]$ independent

$$= \int_{\mathbb{R}} \Big(\frac{p(y)}{M\cancel{g(y)}}\Big) \cancel{g(y)}\, dy = \frac{1}{M} \cdot \underbrace{\int_{\mathbb{R}} p(y)\, dy}_{1} = \boxed{\frac{1}{M}}$$

fixed

$$\boxed{\Pr[X \le x, Y \text{ accepted}]} = \quad (X = Y)$$

$$= \int_{\mathbb{R}} \Big[\underbrace{\int_0^1 \mathbb{1}\Big(u \le \frac{p(y)}{Mg(y)}\Big) 1\, du}_{\frac{p(y)}{Mg(y)}} \Big] \mathbb{1}(y \le x)\, g(y)\, dy$$

$$= \int_{\mathbb{R}} \Big(\frac{p(y)}{M\cancel{g(y)}}\Big) \mathbb{1}(y \le x)\, \cancel{g(y)}\, dy = \frac{1}{M} \int_{\mathbb{R}} \mathbb{1}(y \le x)\, p(y)\, d(y) = \frac{1}{M} \int_{-\infty}^{x} p(y)\, dy = \boxed{\frac{1}{M} F(x)}$$

$$\boxed{F_{RS}(x) = \frac{1/M\, F(x)}{1/M} = \boxed{F(x)}} \checkmark$$

Example

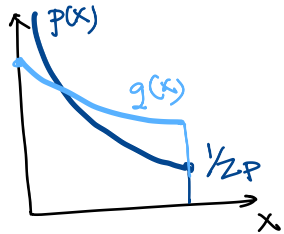
$$p(x) = \frac{1}{Z_p} x^{-1/2} e^{-x} \qquad x \in (0,1]$$

$x \in (0,\infty) \Rightarrow p(x)$ is Γ distribution

$$Z_p = \int_0^1 x^{-1/2} e^{-x} dx$$

$$g(x) = \frac{1}{Z_g} e^{-x} \qquad x \in (0,1]$$

$$Z_g = \int_0^1 e^{-x} dx = 1 - e^{-1}$$



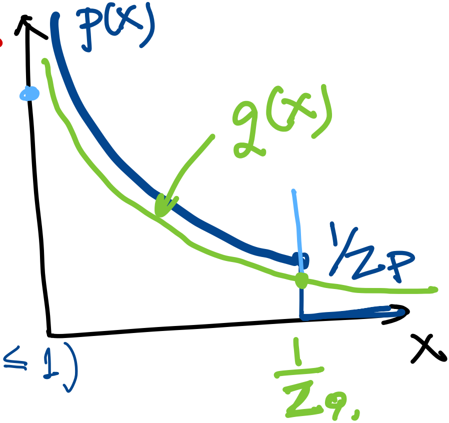
Rejection Sampling

$$M = \sup_{x \in (0,1]} \frac{p(x)}{g(x)} = \frac{Z_g}{Z_p} \frac{x^{-1/2} e^{-x}}{e^{-x}} = \frac{Z_g}{Z_p} x^{-1/2} = \infty \;!!$$

sup x

sup $x \in (0,1]$

can't use g for R.S.



Another rej sampling

$$g(x) = \left(\mathrm{Gamma}\left(\tfrac{1}{2}\right) \right) = \frac{1}{Z_g} x^{-1/2} e^{-x} \qquad x \in (0,\infty)$$

$$M = \sup_{x \in (0,\infty)} \frac{p(x)}{g(x)} = \sup_x \begin{cases} 0 & x > 1 \\ \frac{Z_g}{Z_p} & x \in (0,1] \end{cases} = \frac{Z_g}{Z_p} \Rightarrow P[\text{accept}] = \frac{Z_p}{Z_g} = P_g(X \le 1)$$

Practical Rejection Sampling. What is a good q?

- Intuitively, q should be as close to p as possible
- Acceptance probability $Pr\left[U < \frac{1}{M}\frac{p(Y)}{q(Y)}\right] = \boxed{\frac{1}{M}}$
- Therefore, a good q will have $M = \sup_x \frac{p(x)}{q(x)}$ small (Note that $M \geq 1$)

← efficient

!! $\lim_{x \to x_0} \frac{p}{q} \to \infty$ BAD