STAT 535 Homework 2
Out October 11, 2011
Due October 18, 2011
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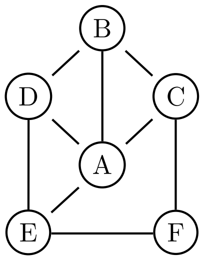
Problem 1 – Separation and factorization in MRF

a. Consider the graph below. Which of the following separation relationships hold in this graph? If a separation is not true, give an open path.



$BC \perp E \,|\, A$
$A \perp B$
$B \perp F \,|\, CE$

b. Write the general factored expression of a joint distribution P_V over $V = \{A, B, \ldots, F\}$ that has $\mathcal{G}$ as an I-map.

c. Choose one of the true statements in **a**. Prove that the corresponding independence statement holds in P_V by using the factored expression of P_V from **b.**. [Use Lecture 3, section 2.6 as an example of such a proof.]

d. Is the graph $\mathcal{G}$ an I-map for the following log-linear model?

$$\ln P_V \;=\; w_{ABC} f_{ABC} + w_{AD} f_{AD} + w_{EF} f_{EF} \tag{1}$$

where w_X are real parameters and f_X are functions of the variables in $X \subseteq V$. Explain why yes or no.

e. Is the graph $\mathcal{G}$ an I-map for the following log-linear model?

$$\ln P_V \;=\; w_{ABC} f_{ABC} + w_{ABD} f_{ABD} + w_{AED} f_{AED} + w_{ACD} f_{ACD} + w_{CE} f_{CE} + w_{EF} f_{EF} \tag{2}$$

where w_X are real parameters and f_X are functions of the variables in $X \subseteq V$. Explain why yes or no.

Problem 2 – conditional distributions in the Ising model

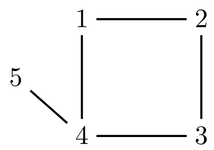
Here we will show that in an Ising model the conditional distribution of a single variable given the others has a simple expression.

Assume an Ising model over $V = \{1, 2, \dots n\}$ is defined by

$$\ln P_V \;=\; \sum_{i \in V} a_i x_i + \sum_{ij \in \mathcal{E}} b_{ij} x_i x_j \tag{3}$$

In the above, $\mathcal{E}$ is a given set of edges (i.e pairs of variables), $x_i \in \{\pm 1\}$, $i \in V$ and $a_i, i \in V, b_{ij},\ ij \in \mathcal{E}$ are real parameters.

a. We will start with an example having only 5 variables.



For the graph above, consider the corresponding Ising model and calculate the expression of

$$\frac{P(x_1 = +1, x_2, x_3, x_4, x_5)}{P(x_1 = -1, x_2, x_3, x_4, x_5)} \tag{4}$$

as a function of $a_i, i \in V, b_{ij},\ ij \in \mathcal{E}$ and the variables x_2, x_3, x_4, x_5.

b. From the expression you found in **a**, derive an expression for the conditional probability

$$P(x_1 = +1 | x_2, x_3, x_4, x_5)$$

Simplify the expression as much as possible.

c. Now for an arbitrary undirected graph with the Ising model (3), calculate the expression

$$\frac{P(x_1, x_2, \dots, x_{i-1}, +1, x_{i+1}, \dots x_n)}{P(x_1, x_2, \dots, x_{i-1}, -1, x_{i+1}, \dots x_n)} \tag{5}$$

as a function of $a_i, i \in V, b_{ij},\ ij \in \mathcal{E}$ and the variables $x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots x_n$

d. From the expression you found in **c**, derive an expression for the conditional probability

$$P(x_i = +1 | x_1, x_2, \dots, x_{i-1}, x_{i+1}, \dots x_n)$$

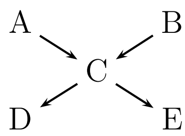
Simplify the expression as much as possible.
[Optional: where have you encountered this distribution before in your statistics studies?]

e. Use the result in **d.** to conclude that the Markov blanket of any variable X in an Ising model is represented by $n(X)$.

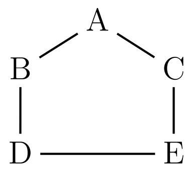
Problem 3 - I-maps [Warmup Problem – NOT GRADED]

This exercise exemplifies how directed and undirected graphical representations are not equivalent.

Let $\mathcal{D}_1$ and $\mathcal{U}_2$ be the graphs below



$\mathcal{D}_1$



$\mathcal{U}_2$

a. Draw an undirected graph $\mathcal{U}_1$ which is an I-map of $\mathcal{D}_1$. In other words, $\mathcal{U}_1$ must capture as many as possible of the independencies in $\mathcal{D}_1$, and have no extra independence relation besides those in $\mathcal{D}_1$.

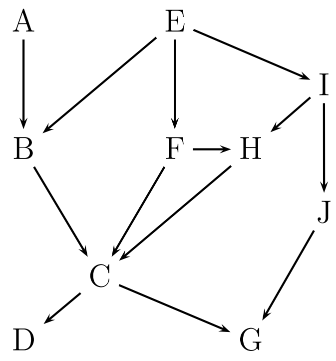
List all the independencies in $\mathcal{D}_1$ which are not in $\mathcal{U}_1$.

b. Draw a directed graph $\mathcal{D}_2$ which is an I-map of $\mathcal{U}_2$. List all the independencies in $\mathcal{U}_2$ which are not in $\mathcal{D}_2$. List all the V-structures in $\mathcal{D}_2$.

c. Draw two graphs $\mathcal{U}_3, \mathcal{D}_3$ on the five nodes A, B, C, D, E, so that the first graph is undirected, the second graph is directed, and the two graphs have the same list of independence relations. (That is, the two graphs are perfect maps of each other).

Try to find the most interesting, non-trivial example you can. In particular, the graphs cannot be: all disconnected, fully connected, or chains.

Problem 4 - D-separation



a.

Are there any converging arrows on the path $ABCG$? Enumerate them.
Are there any converging arrows on the path $HCFEI$? Enumerate them.
Enumerate all the V-structures at node C.

b. Which of the following independence statements is true? For each false statement, give an open path.

B $\perp$ H | E
B $\perp$ I | A,D,E
A $\perp$ E | G
C $\perp$ J
A $\perp$ I
E $\perp$ G | C

c. Is there any edge that can be inverted? In other words, is there any other DAG structure that is equivalent to this DAG?