STAT 535 Homework 4
Out October 26, 2011
Due November 1, 2011

mmp@stat.washington.edu

Problem 1 – Moralization, triangulation, and factorizations - Warmup, NOT GRADED

For all the DAGs below

1.1. Write the factored form of the joint distribution P_V

1.2. Moralize the DAG

1.3. Triangulate the graphs obtained in **1.2** by the elimination algorithm. List the nodes eliminated at each step and the cliques associated with them. Enclose the non-maximal cliques in parantheses as we did in class. Try to find an elimination that keeps the maximum clique size as small as possible.

Please make clear drawings.

A hint: For the triangulation of the graph in **d.** it may be easier to redraw the moral graph before starting the triangulation, omitting nodes $s1 - s12$. In this case, you need to explain clearly what must happen to $s1 - s12$ and to list all the cliques that appear, even those which you will not draw.

1.4. Draw the intersection graph for this triangulation.

1.5. Draw the junction tree $\mathcal{J}$.

1.6 Verify the running intersection property of this junction tree for: variable H_2 in graphs **a, b, c** and variables B, C, D in graph **d**. To verify the running intersection property for a variable X, draw the subtrees of $\mathcal{J}$ corresponding to X and verify that it is connected.

1.7 Write P_V in the junction tree factorization of clique marginals divided by separator marginals.

1.8 Assume that all variables in all graphs are binary. List all the P_{clique} and $P_{separator}$ tables and their sizes. (The size of a table containing a marginal distribution can be either one of: (1) number of possible configurations or, (2) number of possible configurations - 1.) What is the total storage needed for the representation of all the cliques in the junction tree? (This is lower than the total storage needed for the junction tree, since we would have to also store the separators. However, it is a good estimate, since the size of any separator is a fraction of that of the smallest neighboring clique, and there are no more separators than cliques).

a. Hidden Markov Model (HMM)

Note: In a HMM the observed, i.e visible variables are denoted by V_i, while the "hidden" variables are denoted by H_i. This is for your information only, because in this homework it will be irrellevant which variables are hidden and which are observed.

$$\begin{tikzcd}
H_1 \arrow[r] \arrow[d] & H_2 \arrow[r] \arrow[d] & H_3 \arrow[r] \arrow[d] & H_4 \arrow[d] \\
V_1 & V_2 & V_3 & V_4
\end{tikzcd}$$

b. Multiple output HMM

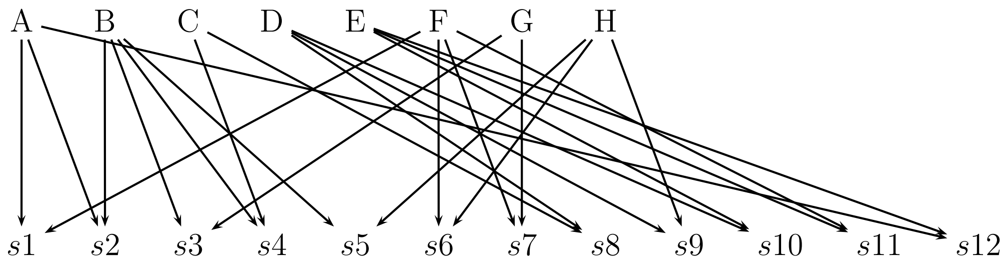
$$\begin{tikzcd}
H_1 \arrow[r] \arrow[d] \arrow[dd, bend left] & H_2 \arrow[r] \arrow[d] \arrow[dd, bend left] & H_3 \arrow[r] \arrow[d] \arrow[dd, bend left] & H_4 \arrow[d] \arrow[dd, bend left] \\
V_1 & V_2 & V_3 & V_4 \\
V_1' & V_2' & V_3' & V_4'
\end{tikzcd}$$

c. Coupled HMM's

$$\begin{tikzcd}
H_1 \arrow[r] \arrow[d] & H_2 \arrow[r] \arrow[d] & H_3 \arrow[r] \arrow[d] & H_4 \arrow[d] \\
V_1 & V_2 & V_3 & V_4 \\
H_1' \arrow[r] \arrow[u] & H_2' \arrow[r] \arrow[u] & H_3' \arrow[r] \arrow[u] & H_4' \arrow[u]
\end{tikzcd}$$

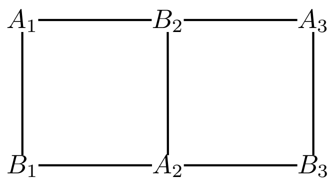
Compare the graphs that appear after triangulations in the three stuctures

d. Disease-symptoms DAG



Problem 2 – Triangulating for small state space

Consider the undirected graph below, where nodes marked A have $r_A = 1000$ values and nodes marked B take $r_B = 2$ values.



a. Triangulate this graph starting the elimination with nodes A_1, A_3. List the cliques that form during the triangulation and construct the junction tree. Calculate the total space needed to store the marginal tables of this junction tree representation of P_V.

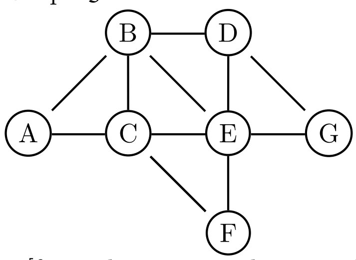
b. Now triangulate the same graph again, starting the elimination with nodes B_1, B_3. List the cliques that form during the triangulation and construct the junction tree. Calculate the total space needed to store the marginal tables of this junction tree representation of P_V.

c. Compare the numerical values you obtained in **1,2** for the total space needed. What lesson can you derive from this comparison?

Problem 3 - A graph of treewidth two

This is the continuation of Problem 2 from the previous homework.

Graph $\mathcal{G}$



Factored forms for P_V

$$P_V = \phi_{ABC}\phi_{BCE}\phi_{CEF}\phi_{BDE}\phi_{DEG}$$

$$P'_V = P_A P_{B|A} P_{C|AB} P_{E|BC} P_{F|CE} P_{D|BE} P_{G|DE}$$

a. [from the previous homework] The two factorizations above must be equal. Find a way to group the factors in P' to obtain the factorization in P. The grouping may not be unique.

Using the grouping you found, show that the potentials of P have a probabilistic interpretation. Since there will be many potentials, it is sufficient to find a probabilistic interpretation for one potential containing variable A and for one containing the variable E.

b. [from the previous homework] Construct a junction tree for the graph $\mathcal{G}$. Is this tree unique? List its separators (with multiplicities).

c. Write the junction tree marginal factorization of P_V for this graph (i.e the factorization involving marginals of cliques, separators).

d. Use the factorization of P_V obtained in **c** to prove that $A \perp F \,|\, CE$. (Note that you are asked to derive this by marginalization in the factored P_V , *as if you didn't already know it was true.*)

e. Chose as *root* for the junction tree the clique that contains A. Give an orientation to the junction tree, by directing each edge outward from the root.

f. Write the junction tree conditional factorization of P_V for this graph (this is the factorization involving conditional distribution of a clique given its parent clique).

[**g. OPTIONAL, extra credit**] Use the factorization in **f.** to prove that $A \perp E \,|\, BC$.

h. Calculate the total storage needed for the junction tree representation in **f.** Assume the variables are all binary. Compare the value you obtain with the size of the state space Ω_V.

Problem 4 – Separation relations in a junction tree

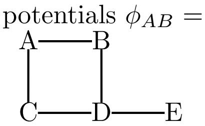
a. In a junction tree, any separator S partitions the tree into exactly two *subtrees*; let us call them the *left subtree* and the *right subtree*, and let us denote by V_r (respectively V_l) the set of nodes in the right (left) subtree, not including S. We have thus partitioned the variables in V into three disjoint sets, S, V_r, V_l, with $V = S \cup V_r \cup V_l$.

Prove that $V_r \perp V_l \,|\, S$. *(Note: This should be a rigurous proof, not merely an explanation)*

b. Let $C - S - C'$ be an edge in a junction tree, with end cliques C, C' and separator S. Prove that $C \setminus S \perp C' \setminus S \,|\, S$. *(Note: This should be a rigurous proof, not merely an explanation)*

Problem 5 – can we sometimes do inference without triangulation?

Consider the MRF below, where all variable take values in $\{1, 2\}$, with clique potentials $\phi_{AB} = \phi_{BD} = \phi_{AC} = \phi_{CD} = \phi_{DE} = \phi$.



$$\phi = \begin{bmatrix} 6 & 3 \\ 2 & 6 \end{bmatrix}$$

(Read the ϕ table as, for instance, $\phi_{AB}(1, 2) = 3,\ \phi_{AB}(2, 2) = 6$)

a. Can you obtain at "at a glance" $P[B = 1|A = 2]$? Can you obtain at "at a glance" $P[E = 2|D = 1]$?

Your answer here should be short: "No" or the value

b. If you did obtain any of the probabilities above at a glance, prove now that your answer is correct.