STAT 535 Lecture 5.1
Junction Tree - additional proofs

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Theorem 1 *Every chordal graph has a junction tree.*

Proof The proof is by induction on the number of variables $n = |V|$.

The statement is trivially true for $n = 1$. We assume that the statement is true for all graphs with n nodes. Let X be the $n+1$-th node of a given graph $\mathcal{G}$. Then $\mathcal{G} \setminus \{X\}$ has a junction tree $\mathcal{J}$. Denote by $C = \{X\} \cup n(X)$ the clique formed by X and its neighbors.

If $n(X)$ is a node in $\mathcal{J}$, then we replace $n(X)$ with C in $\mathcal{J}$. It is easy to verify that the newly obtained j.t. has the Running Intersection Property. Thus, we have constructed a j.t on $n+1$ nodes for $\mathcal{G}$.

If $n(X)$ is not a node in $\mathcal{J}$, then it must be contained in a node (maximal clique) C' of $\mathcal{J}$. We create the j.t. for $\mathcal{G}$ by adding to $\mathcal{J}$ the clique C, which we connect to C' by separator $n(X)$. Again, it is easy to see that the newly obtained j.t. has the Running Intersection Property. QED

A tree formed from the maximal cliques of a graph, with edges labeled by the intersections of the adjacent node cliques is called a **clique tree**.

Theorem 2 *A clique tree has the Running Intersection Property* $\Leftrightarrow$ *The clique tree is a Maximum Spanning Tree of the intersection graph, with weights equal to the sizes of the separators.*

Proof(after Jordan) Let $\mathcal{J}$ be the clique tree, let m be the number of nodes in $\mathcal{J}$, n be the number of variables, and let $w(\mathcal{J})$ be its weight. Then,

$$w(\mathcal{J}) = \sum_{j=1}^{m-1} |S_j| \quad \text{with } \mathrm{S_j} \text{ the separators} \tag{1}$$

$$= \sum_{j=1}^{m-1} \sum_{X \in V} \mathbf{1}_{X \in S_j} \tag{2}$$

$$= \sum_{X \in V} \sum_{j=1}^{m-1} \mathbf{1}_{X \in S_j} \tag{3}$$

$$\leq \sum_{X \in V} \left(\sum_{i=1}^{m} \mathbf{1}_{X \in C_i} - 1 \right) \tag{4}$$

$$= \sum_{i=1}^{m} \sum_{X \in V} \mathbf{1}_{X \in C_i} - n \tag{5}$$

$$= \sum_{i=1}^{m} |C_i| - n \tag{6}$$

The last quantity does not depend on the tree structure, and is an upper bound for the weight of any clique tree. Assume that $\mathcal{J}$ attains the upper bound. Then, obviously it is a Maximum Weight spanning tree. Now examine the inequality: the upper bound is attained if and only if for every X in V, the subgraph of $\mathcal{J}$ that contains it is a spanning tree (i.e the number of separators that contain X equals the number of maximal cliques that contain X minus one). QED