The 2^k Factorial Design

- Montgomery, chap 6; BHH (2nd ed), chap 5
- **Special case** of the general factorial design; *k* factors, all at two levels
- Require relatively few runs per factor studied
- Very widely used in industrial experimentation
- Interpretation of data can proceed largely by common sense, elementary arithmetic, and graphics
- For quantitative factors, can’t explore a wide region of factor space, but determine promising directions
- Designs can be suitably augmented---sequential assembly
- Basis for 2-level fractional factorial designs, especially useful for screening.

The Simplest Case: The 2^2

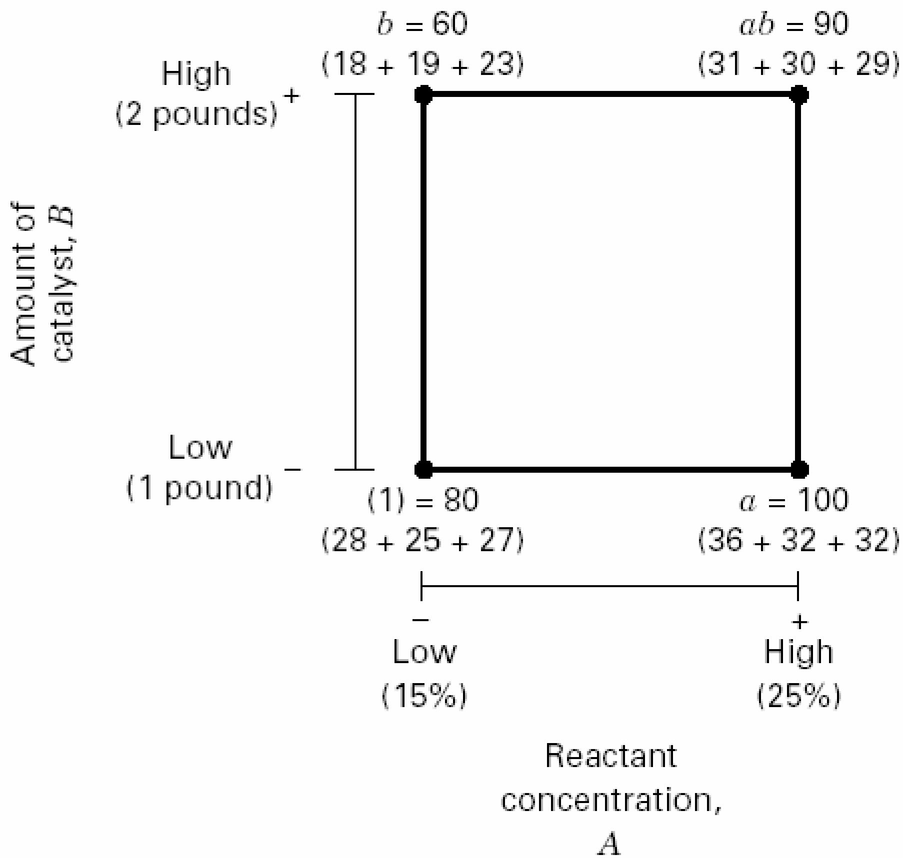


Figure 6-1 Treatment combinations in the 2^2 design.

"-" and "+" denote the low and high levels of a factor, respectively.

Note names of treatment combinations: (1), a, b, ab

Low and high are arbitrary terms

Geometrically, the four runs form the corners of a square

Factors: quantitative or qualitative; interpretation in the final model will be different

Chemical Process Example

Factor		Treatment	Replicate			
A	B	Combination	I	II	III	Total
$-$	$-$	A low, B low	28	25	27	80
$+$	$-$	A high, B low	36	32	32	100
$-$	$+$	A low, B high	18	19	23	60
$+$	$+$	A high, B high	31	30	29	90

A = reactant concentration, B = catalyst amount, y = recovery

Analysis Procedure for a Factorial Design

- Estimate factor **effects**
- **Formulate** model
  - With replication, use full model
  - With an unreplicated design, use normal probability plots
- Statistical **testing** (ANOVA)
- **Refine** the model
- Analyze **residuals** (graphical)
- **Interpret** results

Estimation of Factor Effects

$$A = \bar{y}_{A^+} - \bar{y}_{A^-}$$
$$= \frac{ab+a}{2n} - \frac{b+(1)}{2n}$$
$$= \tfrac{1}{2n}[ab + a - b - (1)]$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-}$$
$$= \frac{ab+b}{2n} - \frac{a+(1)}{2n}$$
$$= \tfrac{1}{2n}[ab + b - a - (1)]$$

$$AB = \frac{ab+(1)}{2n} - \frac{a+b}{2n}$$
$$= \tfrac{1}{2n}[ab + (1) - a - b]$$

Computation: difference between averages of "+" and "-" sign observations

The effect estimates are:

A = 8.33, B = -5.00, AB = 1.67

Practical interpretation?

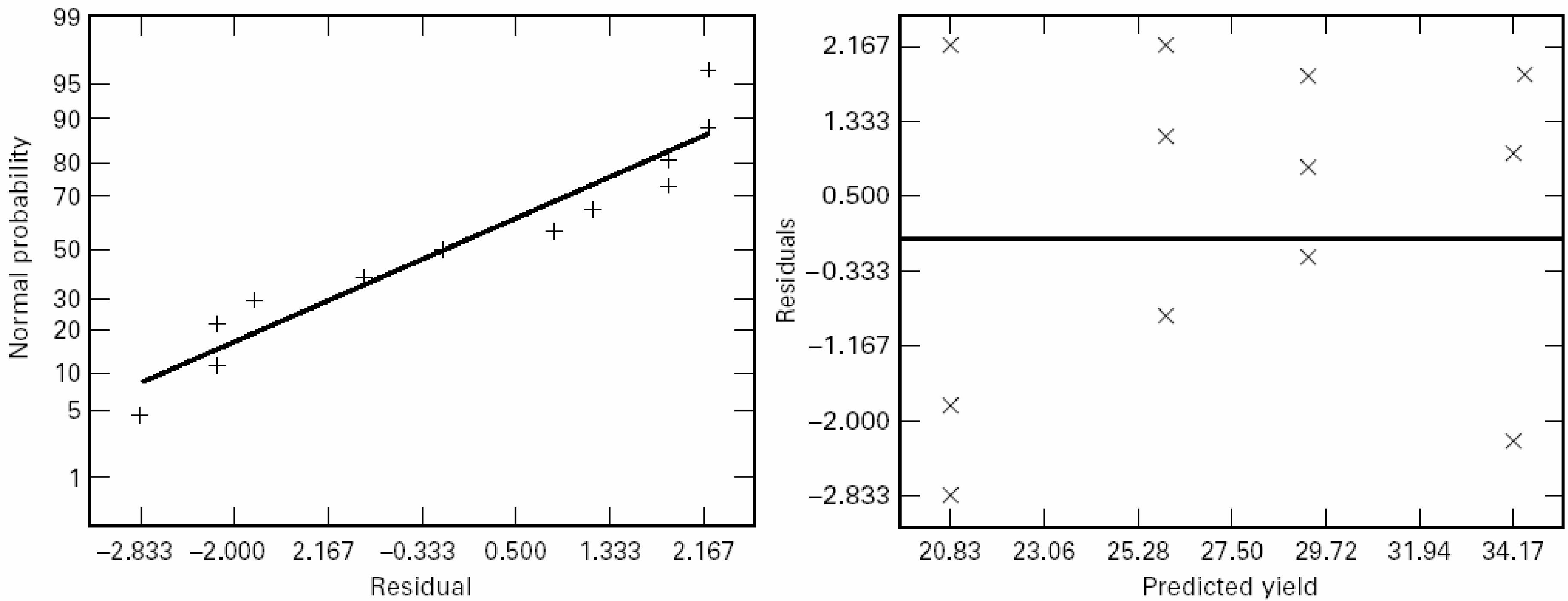
- Increasing reactant concentration increases yield
- Catalyst effect is negative
- Interaction effect is relatively smaller

Statistical Testing - ANOVA

Table 6-1 Analysis of Variance for the Experiment in Figure 6-1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
A	208.33	1	208.33	53.15	0.0001
B	75.00	1	75.00	19.13	0.0024
AB	8.33	1	8.33	2.13	0.1826
Error	31.34	8	3.92		
Total	323.00	11			

Residuals and Diagnostic Checking

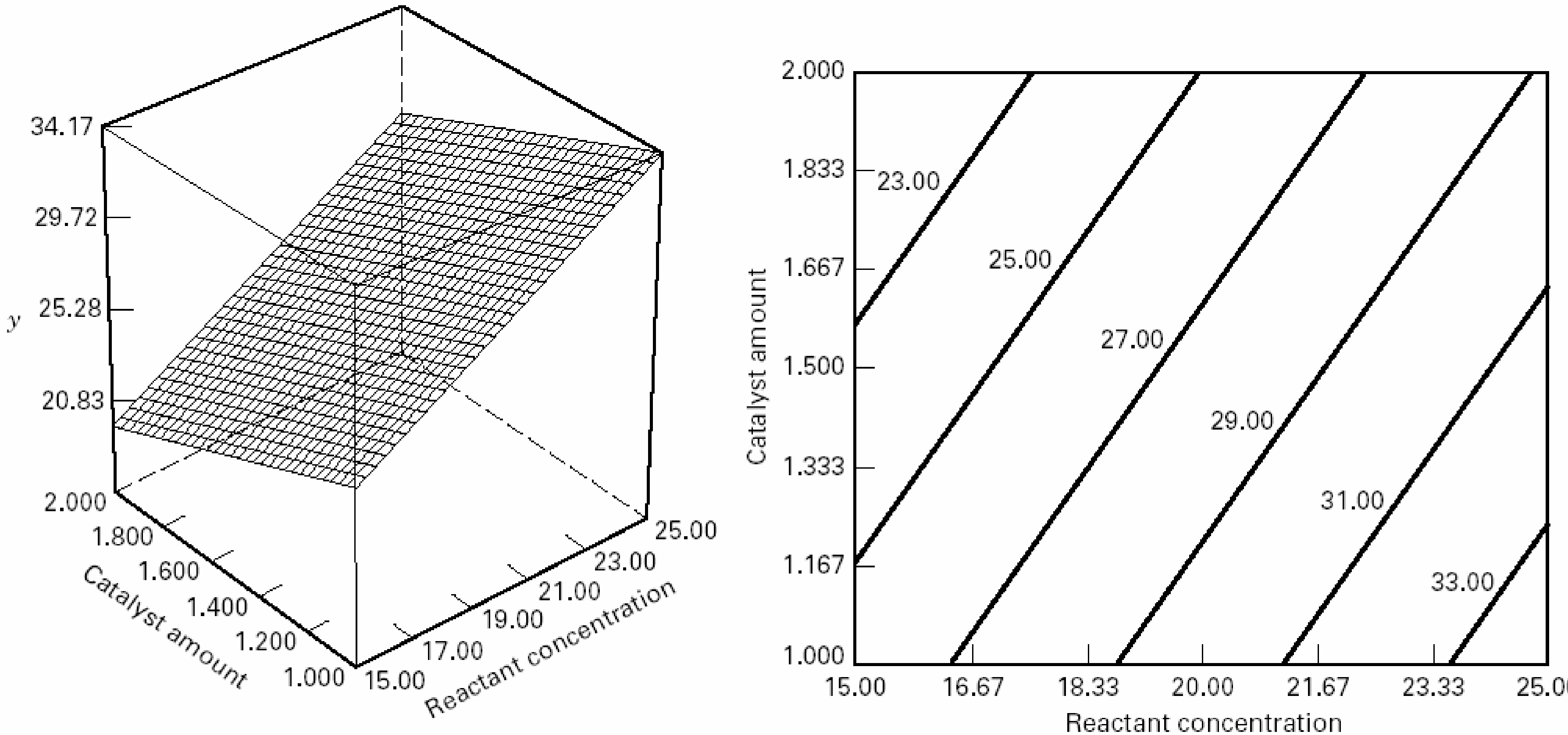


(*a*) Normal probability plot

(*b*) Residuals versus predicted yield

Figure 6-2 Residual plots for the chemical process experiment.

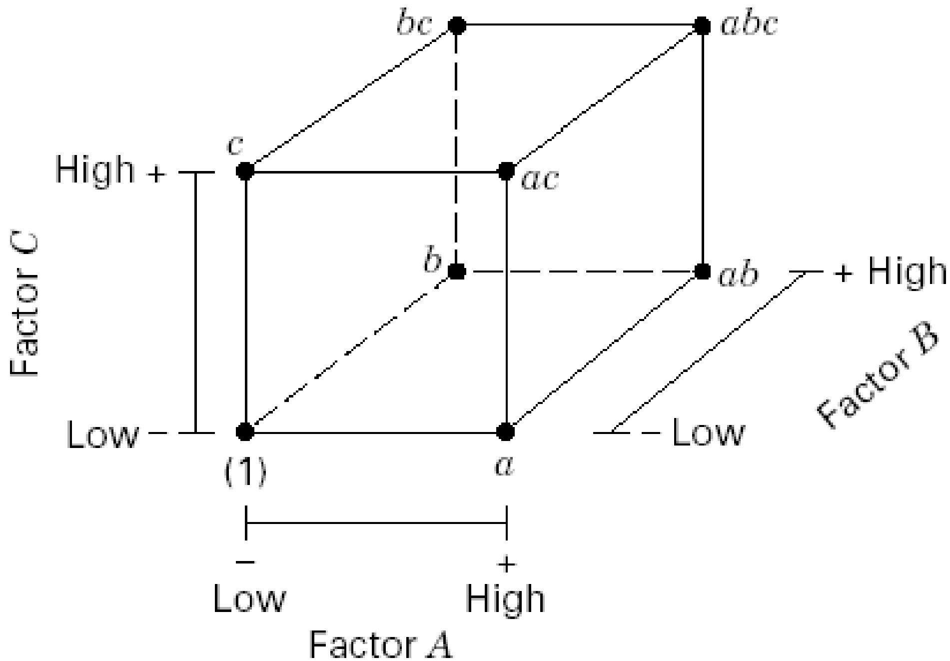
The Response Surface
(for the additive model)



(*a*) Response surface (*b*) Contour plot

Figure 6-3 Response surface plot and contour plot of yield from the chemical process experiment.

The 2^3 Factorial Design



(a) Geometric view

Run	Factor A	B	C
1	−	−	−
2	+	−	−
3	−	+	−
4	+	+	−
5	−	−	+
6	+	−	+
7	−	+	+
8	+	+	+

(b) The design matrix

Figure 6-4 The 2^3 factorial design.

Effects in The 2^3 Factorial Design

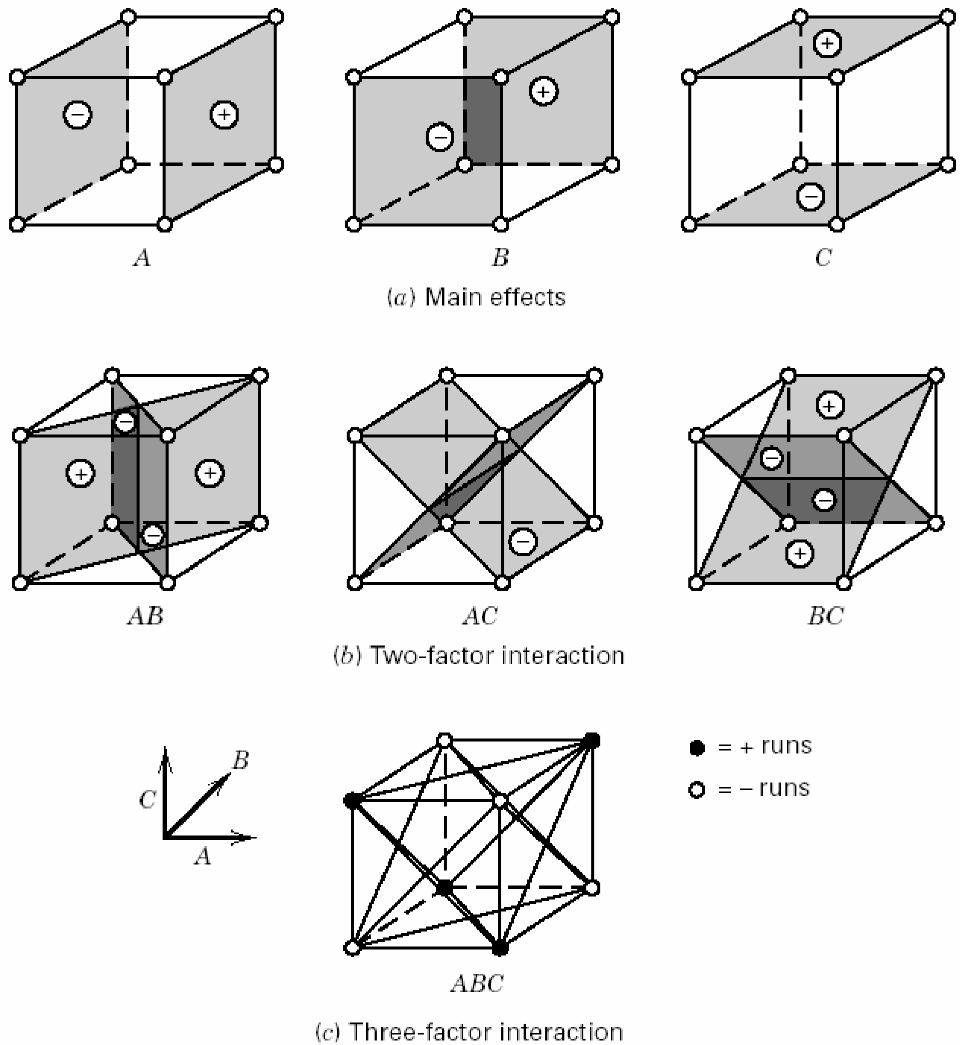


Figure 6-5 Geometric presentation of contrasts corresponding to the main effects and interactions in the 2^3 design.

$$A = \bar{y}_{A^+} - \bar{y}_{A^-}$$

$$B = \bar{y}_{B^+} - \bar{y}_{B^-}$$

$$C = \bar{y}_{C^+} - \bar{y}_{C^-}$$

etc, etc, ...

Interaction effects are also differences between averages of 4 runs.

An Example of a 2^3 Factorial Design

Table 6-4 The Plasma Etch Experiment, Example 6-1

	Coded Factors			Etch Rate			Factor Levels		
Run	A	B	C	Replicate 1	Replicate 2	Total		Low (−1)	High (+1)
1	−1	−1	−1	550	604	(1) = 1154	A (Gap, cm)	0.80	1.20
2	1	−1	−1	669	650	a = 1319	B (C_2F_6 flow, SCCM)	125	200
3	−1	1	−1	633	601	b = 1234	C (Power, W)	275	325
4	1	1	−1	642	635	ab = 1277			
5	−1	−1	1	1037	1052	c = 2089			
6	1	−1	1	749	868	ac = 1617			
7	−1	1	1	1075	1063	bc = 2178			
8	1	1	1	729	860	abc = 1589			

A = gap, B = Flow, C = Power, y = Etch Rate

Table of – and + Signs for the 2^3 Factorial Design (pg. 214)

Table 6-3 Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	Factorial Effect							
	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

Properties of the Table

- Except for column I, every column has an equal number of + and – signs
- The sum of the product of signs in any two columns is zero: **orthogonal** design
- Multiplying any column by I leaves that column unchanged (identity element)
- The product of any two columns yields a column in the table:

$$A \times B = AB$$

$$AB \times BC = AB^2C = AC$$

- Orthogonality is an important property shared by all factorial designs

R computation

```
> etch.rate <- matrix(c(550,604,669,650,633,601,642,635,
+ 1037,1052,749,868,1075,1063,729,860),byrow=T,ncol=2)
> dimnames(etch.rate) <- list(
+ c("(1)","a","b","ab","c","ac","bc","abc"),c("Rep1","Rep2"))
>
> A <- rep(c(-1,1),4)
> B <- rep(c(-1,-1,1,1),2)
> C <- c(rep(-1,4),rep(1,4))
>
> Total <- apply(etch.rate,1,sum)
>
> cbind(A,B,C,etch.rate,Total)
     A  B  C Rep1 Rep2 Total
(1) -1 -1 -1  550  604  1154
a    1 -1 -1  669  650  1319
b   -1  1 -1  633  601  1234
ab   1  1 -1  642  635  1277
c   -1 -1  1 1037 1052  2089
ac   1 -1  1  749  868  1617
bc  -1  1  1 1075 1063  2138
abc  1  1  1  729  860  1589
```

R computation (cont)

```
> # #reps: n=2
> n <- 2
> # Effect estimates are differences of averages of 4 means ("runs")
> # Effect estimates:
> Aeff <- (Total %*% A)/(4*n)
> Beff <- (Total %*% B)/(4*n)
> Ceff <- (Total %*% C)/(4*n)
>
> # Interaction effects
> AB <- A*B
> AC <- A*C
> BC <- B*C
> ABC <- A*B*C
> cbind(A,B,C,AB,AC,BC,ABC,Total)
     A  B  C AB AC BC ABC Total
(1) -1 -1 -1  1  1  1  -1  1154
a    1 -1 -1 -1 -1  1   1  1319
b   -1  1 -1 -1  1 -1   1  1234
ab   1  1 -1  1 -1 -1  -1  1277
c   -1 -1  1  1 -1 -1   1  2089
ac   1 -1  1 -1  1 -1  -1  1617
bc  -1  1  1 -1 -1  1  -1  2138
abc  1  1  1  1  1  1   1  1589
>
> ABeff <- (Total %*% AB)/(4*n)
> ACeff <- (Total %*% AC)/(4*n)
> BCeff <- (Total %*% BC)/(4*n)
> ABCeff <- (Total %*% ABC)/(4*n)
```

R computation (cont)

```
> # Summary
> Effects <- t(Total) %*% cbind(A,B,C,AB,AC,BC,ABC)/(4*n)
> Summary <- rbind( cbind(A,B,C,AB,AC,BC,ABC),Effects )
> dimnames(Summary)[[1]] <- c(dimnames(etch.rate)[[1]],"Effect")
> Summary
             A      B       C      AB       AC     BC    ABC
(1)     -1.000 -1.000  -1.000   1.000    1.000  1.000 -1.000
a        1.000 -1.000  -1.000  -1.000   -1.000  1.000  1.000
b       -1.000  1.000  -1.000  -1.000    1.000 -1.000  1.000
ab       1.000  1.000  -1.000   1.000   -1.000 -1.000 -1.000
c       -1.000 -1.000   1.000   1.000   -1.000 -1.000  1.000
ac       1.000 -1.000   1.000  -1.000    1.000 -1.000 -1.000
bc      -1.000  1.000   1.000  -1.000   -1.000  1.000 -1.000
abc      1.000  1.000   1.000   1.000    1.000  1.000  1.000
Effect -101.625  7.375 306.125 -24.875 -153.625 -2.125  5.625
>
> # Fit as an ANOVA model
> etch.vec <- c(t(etch.rate))
> Af <- rep(as.factor(A),rep(2,8))
> Bf <- rep(as.factor(B),rep(2,8))
> Cf <- rep(as.factor(C),rep(2,8))
> options(contrasts=c("contr.sum","contr.poly"))
> etch.lm <- lm(etch.vec ~ Af*Bf*Cf)
```

Estimation of Factor Effects

Table 6-5 Effect Estimate Summary for Example 6-1

Factor	Effect Estimate	Sum of Squares	Percent Contribution
A	−101.625	41,310.5625	7.7736
B	7.375	217.5625	0.0409
C	306.125	374,850.0625	70.5373
AB	−24.875	2475.0625	0.4657
AC	−153.625	94,402.5625	17.7642
BC	−2.125	18.0625	0.0034
ABC	5.625	126.5625	0.0238

Model Coefficients – Full Model

Factor	Coefficient Estimated	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	776.06	1	11.87	748.70	803.42	
A-Gap	−50.81	1	11.87	−78.17	−23.45	1.00
B-Gas flow	3.69	1	11.87	−23.67	31.05	1.00
C-Power	153.06	1	11.87	125.70	180.42	1.00
AB	−12.44	1	11.87	−39.80	14.92	1.00
AC	−76.81	1	11.87	−104.17	−49.45	1.00
BC	−1.06	1	11.87	−28.42	26.30	1.00
ABC	2.81	1	11.87	−24.55	30.17	1.00

R computation (cont)

```
> options(contrasts=c("contr.sum","contr.poly"))
> etch.lm <- lm(etch.vec ~ Af*Bf*Cf)
> summary(etch.lm)

Call:
lm(formula = etch.vec ~ Af * Bf * Cf)

Residuals:
       Min         1Q     Median         3Q        Max
-6.550e+01 -1.113e+01  8.882e-16  1.113e+01  6.550e+01

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  776.062     11.865  65.406 3.32e-12 ***
Af1           50.813     11.865   4.282 0.002679 **
Bf1           -3.687     11.865  -0.311 0.763911
Cf1         -153.062     11.865 -12.900 1.23e-06 ***
Af1:Bf1      -12.437     11.865  -1.048 0.325168
Af1:Cf1      -76.812     11.865  -6.474 0.000193 ***
Bf1:Cf1       -1.063     11.865  -0.090 0.930849
Af1:Bf1:Cf1   -2.812     11.865  -0.237 0.818586
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 47.46 on 8 degrees of freedom
Multiple R-Squared: 0.9661,     Adjusted R-squared: 0.9364
F-statistic: 32.56 on 7 and 8 DF,  p-value: 2.896e-05
```

Review question:

Why are the anova model coefficients ½ the "effect estimates"?

ANOVA Summary – Full Model

Table 6-6 Analysis of Variance for the Plasma Etching Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	P-Value
Gap (A)	41,310.5625	1	41,310.5625	18.34	0.0027
Gas flow (B)	217.5625	1	217.5625	0.10	0.7639
Power (C)	374,850.0625	1	374,850.0625	166.41	0.0001
AB	2475.0625	1	2475.0625	1.10	0.3252
AC	94,402.5625	1	94,402.5625	41.91	0.0002
BC	18.0625	1	18.0625	0.01	0.9308
ABC	126.5625	1	126.5625	0.06	0.8186
Error	18,020.5000	8	2252.5625		
Total	531,420.9375	15			

R computation (cont)

```
> anova(etch.lm)
Analysis of Variance Table

Response: etch.vec
          Df Sum Sq Mean Sq  F value    Pr(>F)
Af         1  41311   41311  18.3394 0.0026786 **
Bf         1    218     218   0.0966 0.7639107
Cf         1 374850  374850 166.4105 1.233e-06 ***
Af:Bf      1   2475    2475   1.0988 0.3251679
Af:Cf      1  94403   94403  41.9090 0.0001934 ***
Bf:Cf      1     18      18   0.0080 0.9308486
Af:Bf:Cf   1    127     127   0.0562 0.8185861
Residuals  8  18020    2253
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
> model.matrix(etch.lm)
   (Intercept) Af1 Bf1 Cf1 Af1:Bf1 Af1:Cf1 Bf1:Cf1 Af1:Bf1:Cf1
1            1   1   1   1       1       1       1           1
2            1   1   1   1       1       1       1           1
3            1  -1   1   1      -1      -1       1          -1
4            1  -1   1   1      -1      -1       1          -1
5            1   1  -1   1      -1       1      -1          -1
6            1   1  -1   1      -1       1      -1          -1
7            1  -1  -1   1       1      -1      -1           1
8            1  -1  -1   1       1      -1      -1           1
9            1   1   1  -1       1      -1      -1          -1
10           1   1   1  -1       1      -1      -1          -1
11 . . .
```

BHH sect 5.10: "Misuse of the ANOVA for 2^k Factorial Experiments"

- For 2^k designs, the use of the ANOVA is confusing and makes little sense. $N=n\times 2^k$ observations. 2^k-1 d.f. partitioned into individual "SS" for effects, each equal to $N(\text{effect})^2/4$, divided by df=1, and turned into an F-ratio. Experimenter wants magnitude of effect, $\bar{y}_+ - \bar{y}_-$, and t ratio = effect/se(effect).
- P-values should not be used mechanically for yes-or-no decisions on what effects are real. Information about the size of an effect and its possible error must be allowed to interact with experimenter's subject matter knowledge. Graphical methods (coming) provide a valuable means of allowing information in the data and in the mind of the experimenter to interact properly.

Refine Model – Remove Nonsignificant Factors

Table 6-7 (continued)

Response: Etch rate
ANOVA for Selected Factorial Model
Analysis of variance table [Partial sum of squares]

Source	Sum of Squares	DF	Mean Square	*F* Value	Prob > *F*
Model	5.106E+005	3	1.702E+005	97.91	<0.0001
A	*41310.56*	*1*	*41310.56*	*23.77*	*0.0004*
C	*3.749E+005*	*1*	*3.749E+005*	*215.66*	*<0.0001*
AC	*94402.56*	*1*	*94402.56*	*54.31*	*<0.0001*
Residual	20857.75	12	1738.15		
Lack of Fit	*2837.25*	*4*	*709.31*	*0.31*	*0.8604*
Pure Error	*18020.50*	*8*	*2252.56*		
Cor Total	5.314E+005	15			

Std. Dev.	41.69	R-Squared	0.9608
Mean	776.06	Adj R-Squared	0.9509
C.V.	5.37	Pred R-Squared	0.9302
PRESS	37080.44	Adeq Precision	22.055

Note that Sums of Squares for A, C, AC did not change.

Model Coefficients – Reduced Model

Factor	Coefficient Estimate	DF	Standard Error	95% CI Low	95% CI High	VIF
Intercept	776.06	1	10.42	753.35	798.77	
A-Gap	−50.81	1	10.42	−73.52	28.10	1.00
C-Power	153.06	1	10.42	130.35	175.77	1.00
AC	−76.81	1	10.42	−99.52	−54.10	1.00

What has changed from the previous larger table of coefficient estimates?

Model Summary Statistics for Reduced Model (pg. 222)

- R^2 and adjusted R^2

$$R^2 = \frac{SS_{Model}}{SS_T} = \frac{5.106\times10^5}{5.314\times10^5} = 0.9608$$

$$R^2_{Adj} = 1 - \frac{SS_E / df_E}{SS_T / df_T} = 1 - \frac{20857.75/12}{5.314\times10^5/15} = 0.9509$$

- R^2 for prediction (based on PRESS)

$$R^2_{\text{Pred}} = 1 - \frac{PRESS}{SS_T} = 1 - \frac{37080.44}{5.314\times10^5} = 0.9302$$

Model Summary Statistics (pg. 222)

- **Standard error** of model coefficients (full model)

$$se(\hat{\beta}) = \sqrt{V(\hat{\beta})} = \sqrt{\frac{\sigma^2}{n2^k}} \stackrel{\hat{\beta}}{=} \sqrt{\frac{MS_E}{n2^k}} = \sqrt{\frac{2252.56}{2(8)}} = 11.87$$

- **Confidence interval** on model coefficients

$$\hat{\beta} - t_{\alpha/2, df_E} se(\hat{\beta}) \le \beta \le \hat{\beta} + t_{\alpha/2, df_E} se(\hat{\beta})$$

<u>Exercise:</u> derive the above expression for $se(\hat{\beta})$

Model Interpretation

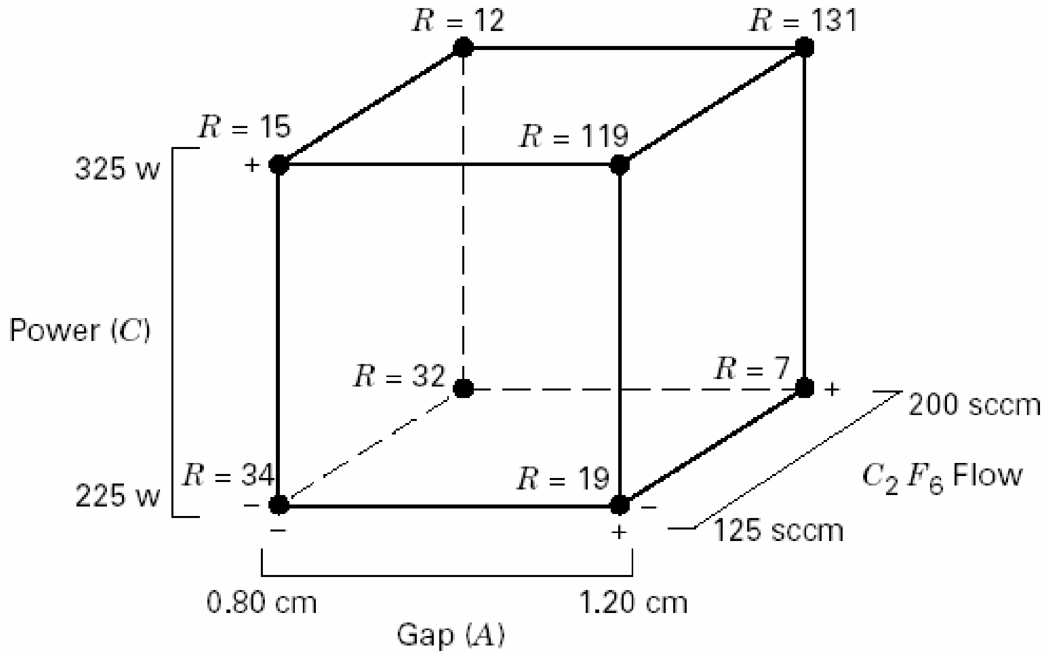


Figure 6-8 Ranges of etch rates for Example 6-1.

Cube plots are often useful visual displays of experimental results

Assessing "error" or residual variation

Often there are more factors to be investigated that can conveniently be accommodated with the time and budget available. Rather than make 16 runs for a replicated 2^3 factorial, it might be preferable to introduce a 4^{th} factor and run an *un*-replicated 2^4 design.

Options:

1. With replication, use the usual pooled variance computed from the replicates.
2. Assume that higher order interaction effects are noise and construct and internal reference set.
3. Assess meaningful effects, including possibly meaningful higher order interactions, using Normal and "Lenth" plots.

Example: Process development experiment.

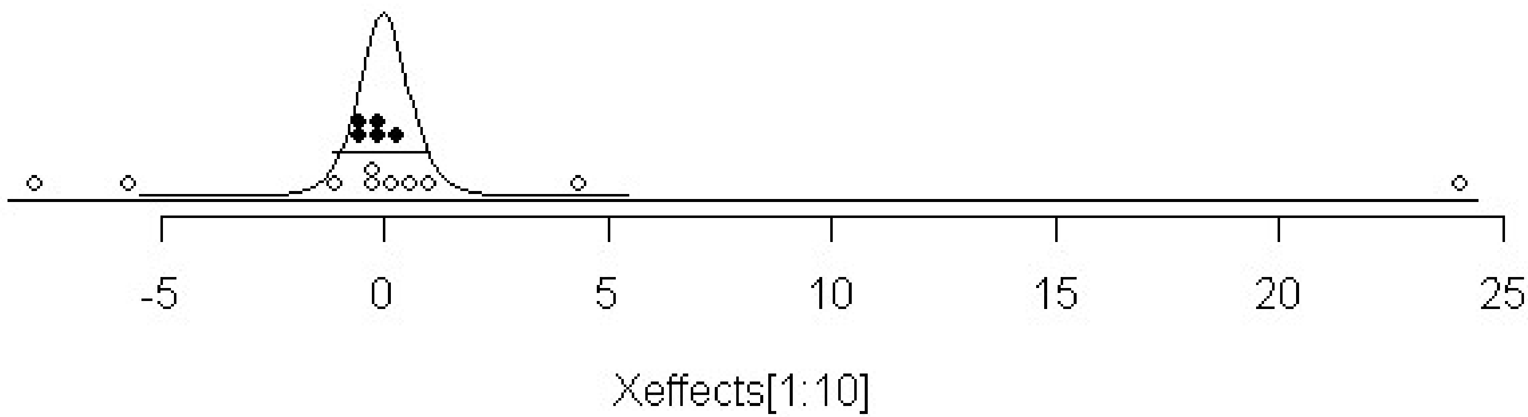
Factor	Level 1	Level 2
Catalyst charge (lb)	10	15
Temperature ©	220	240
Pressure (psi)	50	80
Reactant concentration (%)	10	12

Response: “percent conversion”

```
> # Read in process development data of BHH2 Table 5.10a
> tab5.10.dat <- read.table(file.choose(),header=T)
> dimnames(tab5.10.dat)[[2]][2:5] <- c("A","B","C","D")
> tab5.10.dat
   yatesOrd  A  B  C  D conversion randomOrd
1         1 -1 -1 -1 -1         70         8
2         2  1 -1 -1 -1         60         2
3         3 -1  1 -1 -1         89        10
4         4  1  1 -1 -1         81         4
5         5 -1 -1  1 -1         69        15
6         6  1 -1  1 -1         62         9
7         7 -1  1  1 -1         88         1
8         8  1  1  1 -1         81        13
9         9 -1 -1 -1  1         60        16
10       10  1 -1 -1  1         49         5
11       11 -1  1 -1  1         88        11
12       12  1  1 -1  1         82        14
13       13 -1 -1  1  1         60         3
14       14  1 -1  1  1         52        12
15       15 -1  1  1  1         86         6
16       16  1  1  1  1         79         7
```

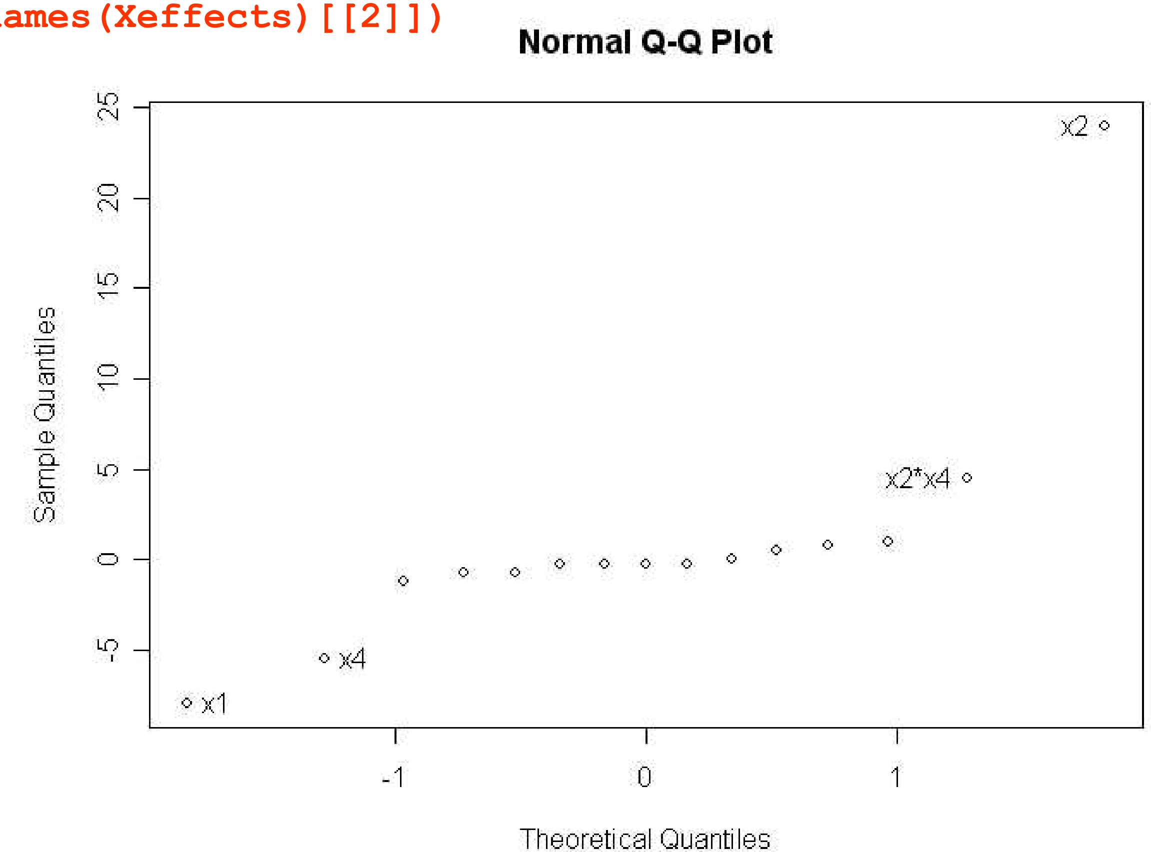
```
> # Full design matrix with interactions
> des4 <- ffFullMatrix(X,x=c(1,2,3,4),maxInt=4)
> des4
$Xa
   one x1 x2 x3 x4 x1*x2 x1*x3 x1*x4 x2*x3 x2*x4 x3*x4 x1*x2*x3 x1*x2*x4
1    1 -1 -1 -1 -1     1     1     1     1     1     1       -1       -1
2    1  1 -1 -1 -1    -1    -1    -1     1     1     1        1        1
3    1 -1  1 -1 -1    -1     1     1    -1    -1     1        1        1
4    1  1  1 -1 -1     1    -1    -1    -1    -1     1       -1       -1
5    1 -1 -1  1 -1     1    -1     1    -1     1    -1        1       -1
6    1  1 -1  1 -1    -1     1    -1    -1     1    -1       -1        1
7    1 -1  1  1 -1    -1    -1     1     1    -1    -1       -1        1
8    1  1  1  1 -1     1     1    -1     1    -1    -1        1       -1
9    1 -1 -1 -1  1     1     1    -1     1    -1    -1       -1        1
10   1  1 -1 -1  1    -1    -1     1     1    -1    -1        1       -1
11   1 -1  1 -1  1    -1     1    -1    -1     1    -1        1       -1
12   1  1  1 -1  1     1    -1     1    -1     1    -1       -1        1
13   1 -1 -1  1  1     1    -1    -1    -1    -1     1        1        1
14   1  1 -1  1  1    -1     1     1    -1    -1     1       -1       -1
15   1 -1  1  1  1    -1    -1    -1     1     1     1       -1       -1
16   1  1  1  1  1     1     1     1     1     1     1        1        1
[. . . additional columns of 1's and -1's . . . ]

$x
[1] 1 2 3 4
$maxInt
[1] 4
$nTerms
  blk  main int.2 int.3 int.4
    0     4     6     4     1
```

```
> # Use the higher order interaction effects as the reference set of
> # (independent) effects that represent noise.  The standard
> # deviation of these (about zero) provides a relevant se for
> # the rest of the effects.
>
> Xeffects <- matrix(tab5.10.dat$conversion,nrow=1) %*% des4$Xa[,-1]/8
> dotPlot(Xeffects[1:10])
> dots(Xeffects[11:15],y=0.1,stacked=T,pch=19)   # add the higher order effects
> SEeffect <- sqrt(sum(Xeffects[11:15]^2)/5)
> SEeffect
[1] 0.5477226
> lines(SEeffect*seq(-10,10,.11),dt(seq(-10,10,.11),df=5))  # add t(df=5)
reference density
> t.ratios <- Xeffects[11:15]/SEeffect
> round(t.ratios,2)
[1]
```

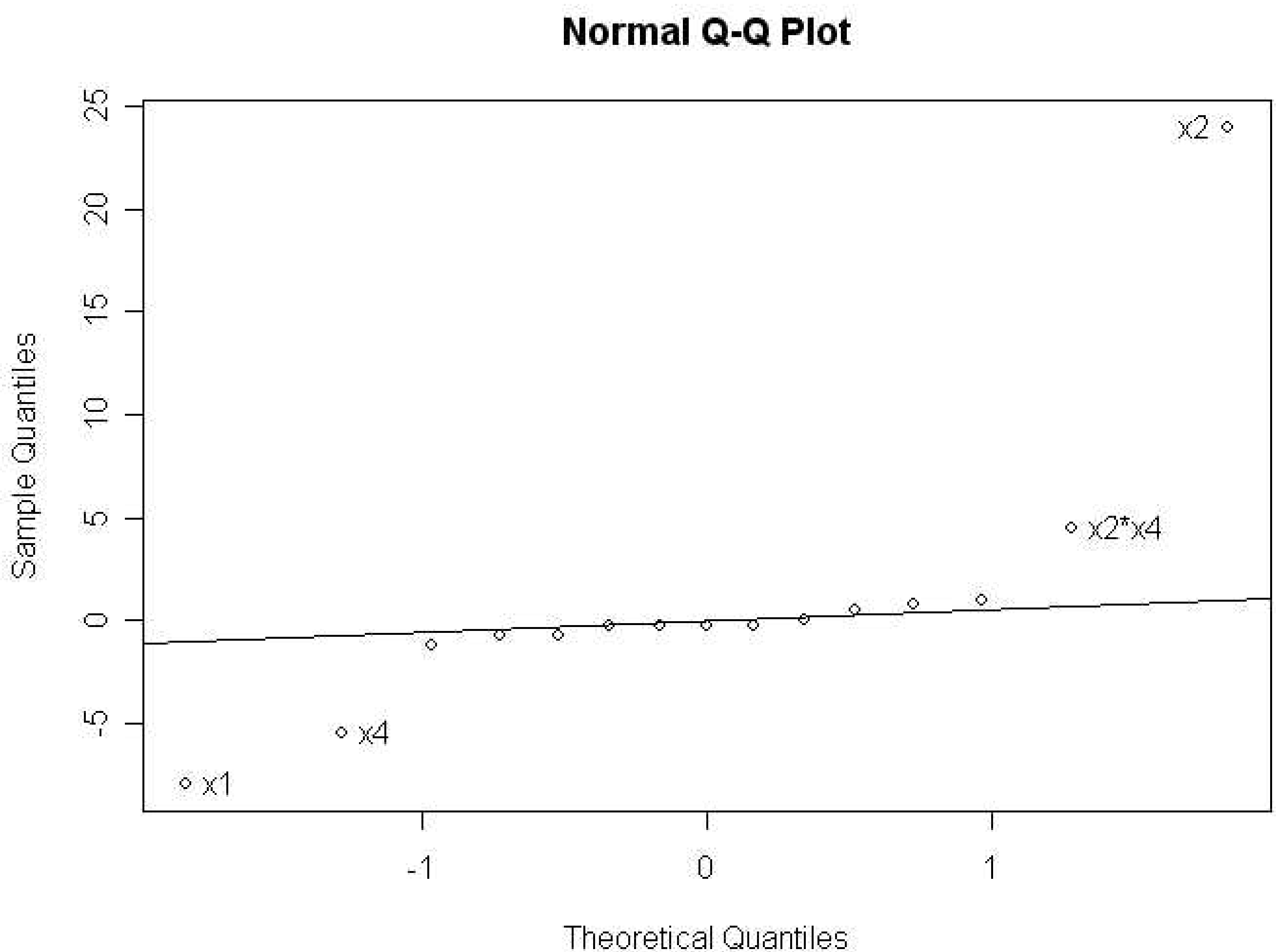



```
> # The "significant" design effects relative to the higher
> # order interactions as a reference set are clear.
```

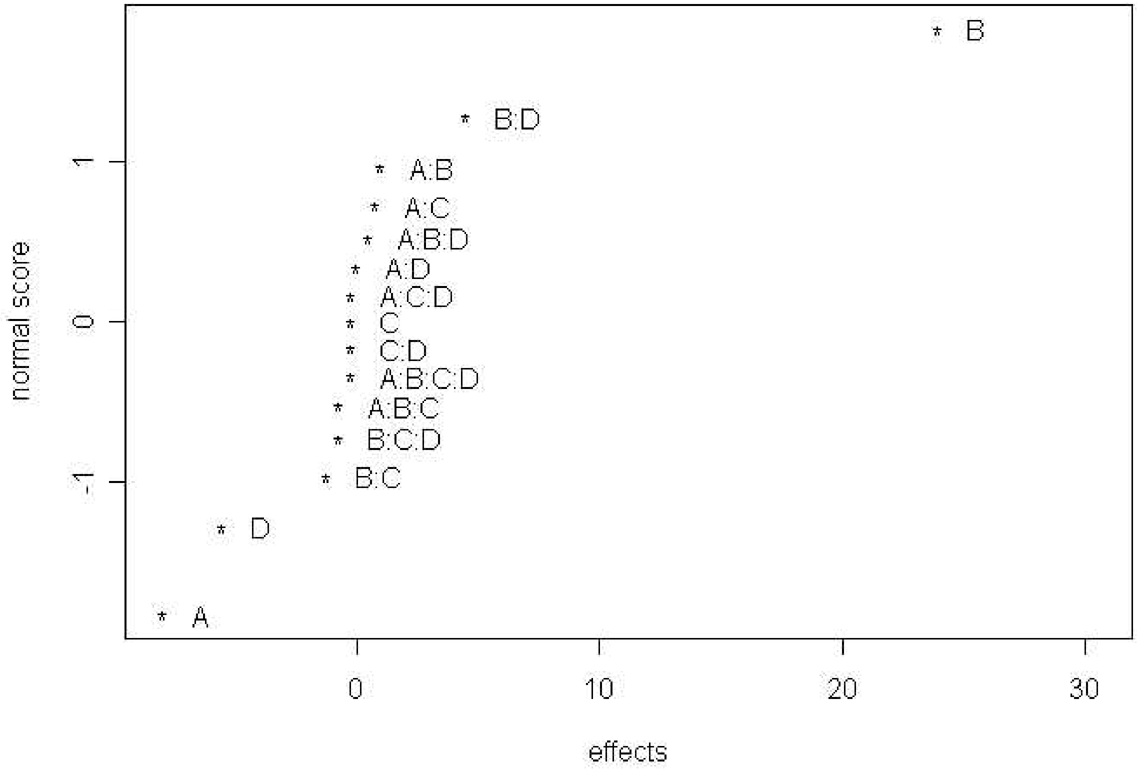
```
> # Two problems arise in the assessment of effects from unreplicated
> # factorials:
> # (a) occasionally meaningful high-order interactions do occur,
> # (b) it is necessary to allow for selection.
> # Daniel (1959) suggested "normal probability" (or, effectively, QQ) plots.
> # Idea:  if none of the effects are "real", the estimated effects, which all
> # have the same std error, should look like a sample from a normal distr.
> # There will always be a largest computed effect, so the question is:
> # Are the largest (smallest) effects bigger (smaller) than expected for a
> # normal distribution?
> temp <- qqnorm(Xeffects)
> identify(temp$x,temp$y,dimnames(Xeffects)[[2]])
[1] 1 2 4 9
>
```



```
> # If we were correct in assessing the standard error of the effects from the
> # higher order interactions, as above, then the a line with slop SEeffect
> # should characterize the appropriate std dev (slope of the qqplot)
> # for the majority of the effects.
> abline(0,.55)
```

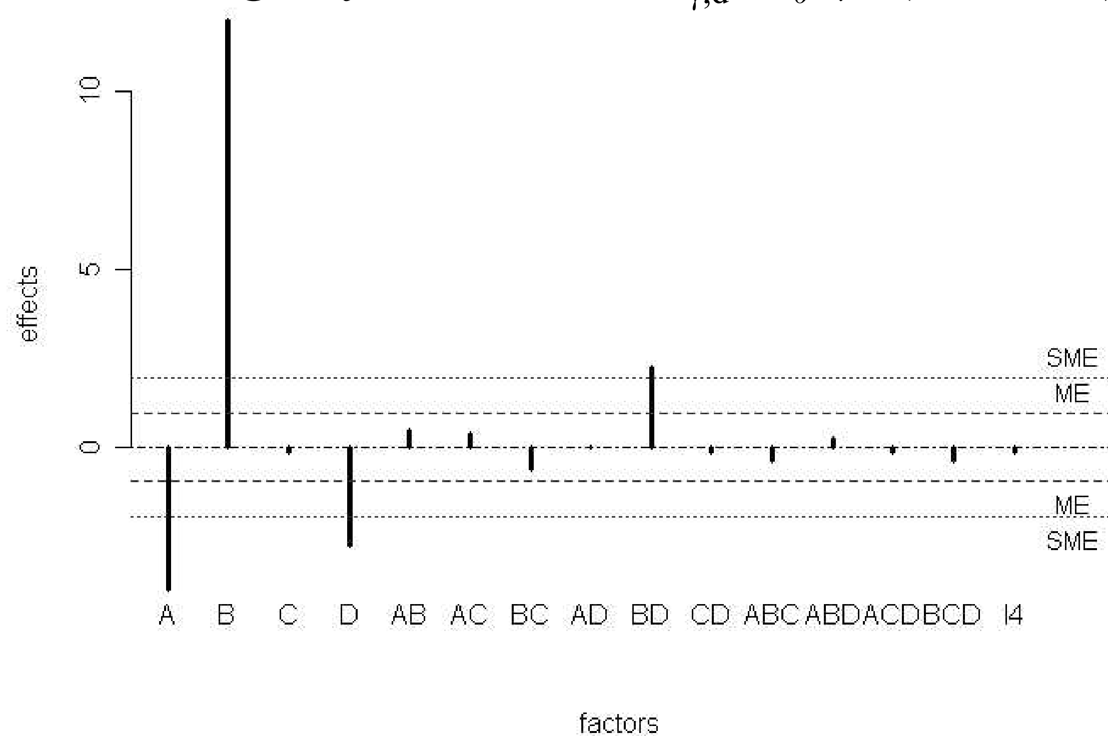


```
> # Or try the DanielPlot function in the BHH2 library
> # Ref: C. Daniel (1976).  Application of Statistics to
> # Industrial Experimentation.  Wiley.

> attach(tab5.10.dat)
> options(contrasts=c("contr.sum","contr.poly"))
> A <- as.factor(-X[,1])
> B <- as.factor(-X[,2])
> C <- as.factor(-X[,3])
> D <- as.factor(-X[,4])
> lm.conversion <- lm( conversion ~ A*B*C*D )
> DanielPlot(lm.conversion)
>
```

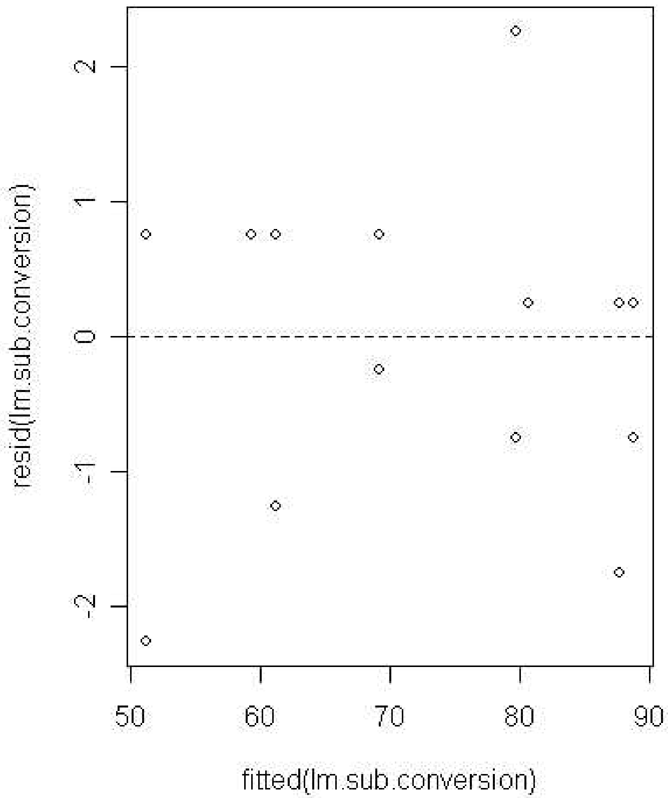


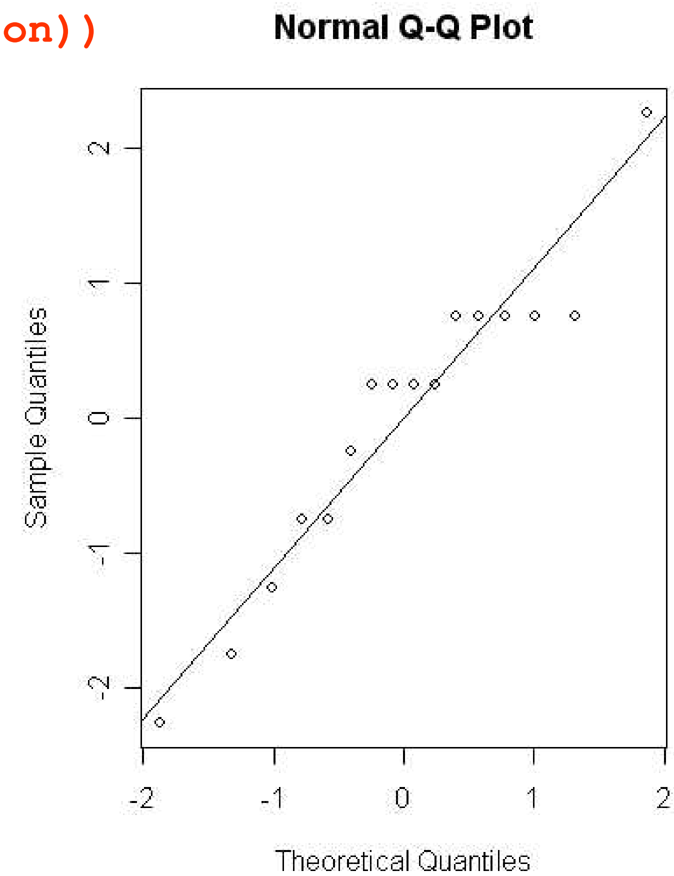
Lenth plots

- Lenth (1989) defined an alternative (“robust”) procedure that identifies “significant” effects.
- m is median of k effects.
- *pseudo s.e* is $s_0 = 1.5m$. Exclude effects exceeding $2.5s_0$ and recompute m and s_0.
- *Margin of error,* ME = $t_{0.975,d} \times s_0$, $d=k/3$ (approx 95% CI).
- *Simultaneous margin of error,* SME= $t_{\gamma,d} \times s_0$, $\gamma=(1+0.95^{1/k})/2$.



```
> # Diagnostic plotting of residuals
> # Fit without identified "significant" effects

> lm.sub.conversion <- lm(conversion ~
     des4$Xa[,c("x1","x2","x4","x2*x4")])
> par(mfrow=c(1,2))
> plot(fitted(lm.sub.conversion),resid(lm.sub.conversion))
> abline(h=0,lty=2)
> qqnorm(resid(lm.sub.conversion))
> qqline(resid(lm.sub.conversion))
```





Blocking the 2^k factorial design

- May be interested in a 2^3 design, but batches of raw material (or periods of time) only large enough to make 4 runs.
- Define blocks so that all runs in which 3-factor interaction "123" is minus are in one block and all other runs in the other block.
- Note: due if all observations in 2nd block were increased by some value d, this would affect only the 123 interaction; because of orthogonality it would *sum out* in the calculation of the main and 2-way effects: 1, 2, 3, 12, 13, 23. *Systematic differences* between blocks are eliminated from main effects and 2-factor interactions.
- Think of block as a 4th factor. We are considering a half fraction of a 2^4 design for all 4 factors.

Run	1	2	3	12	13	23	123	Block
1	−	−	−	+	+	+	−	I
2	+	−	−	−	−	+	+	II
3	−	+	−	−	+	−	+	II
4	+	+	−	+	−	−	−	I
5	−	−	+	+	−	−	+	II
6	+	−	+	−	+	−	−	I
7	−	+	+	−	−	+	−	I
8	+	+	+	+	+	+	+	II

	1	2	3	Run
Block I	−	−	−	1
	+	+	−	4
	+	−	+	6
	−	+	+	7
Block II	+	−	−	2
	−	+	−	3
	−	−	+	5
	+	+	+	8

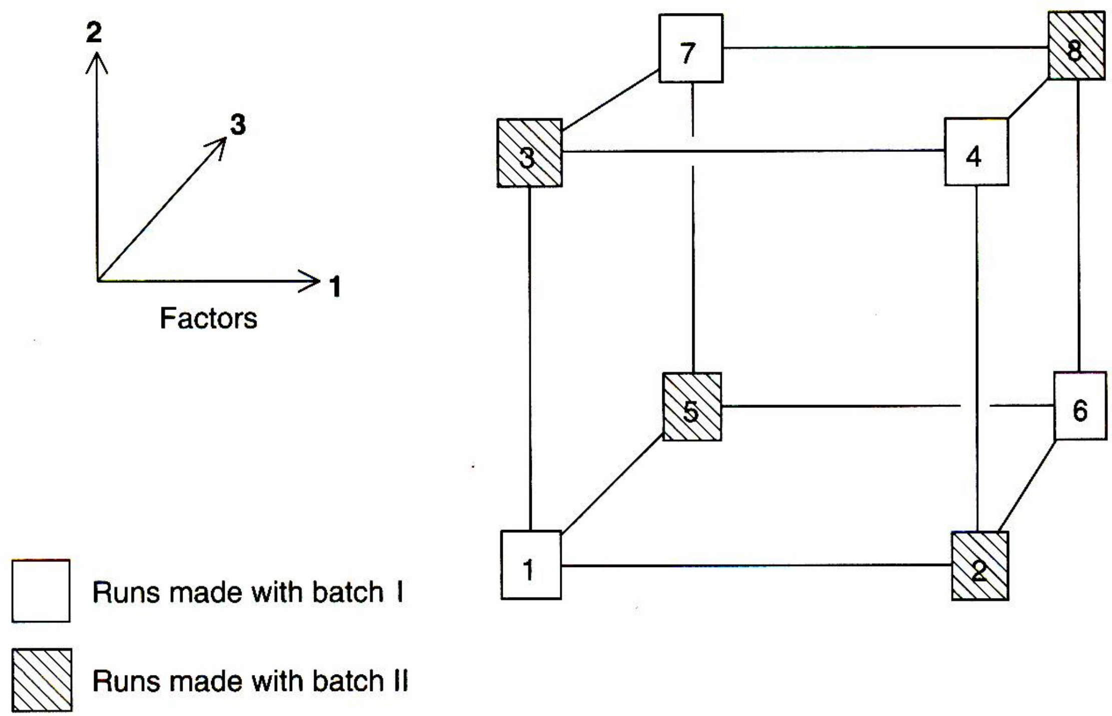


Figure 5.16. Arranging a 2^3 factorial design in two blocks of size 4.

Blocks of size 2

- Want to conduct experiment in blocks of size 2 so as to do no damage to estimates of main effects.
- Define 4 blocks of size 2 by the combinations of two blocking factors, which we may call 4 and 5.
- For example, we might start with “4” = “123”, as before, and confound some other expendible 2-factor interaction with the other, say “5” = “23”

Table 5.14. (*a*) An *Undesirable* Arrangement: A 2^3 Design in Blocks of Size 2
(*b*) Allocation to Blocks (Variables 4 & 5)

Run	Experimental Variable			Block Variable			Experiment Arranged in Four Blocks				
Number	1	2	3	4 = 123	5 = 23	45 = 1	Block	1	2	3	Run
				(a)							
1	−	−	−	−	+	−	I	+	+	−	4
2	+	−	−	+	+	+		+	−	+	6
3	−	+	−	+	−	−	II	−	+	−	3
4	+	+	−	−	−	+		−	−	+	5
5	−	−	+	+	−	−	III	−	−	−	1
6	+	−	+	−	−	+		−	+	+	7
7	−	+	+	−	+	−	IV	+	−	−	2
8	+	+	+	+	+	+		+	+	+	8

(b) Runs

		Variable 4 −	Variable 4 +
Variable 5	+	1,7	2,8
	−	4,6	3,5

Generators and defining relations

- We write **I** for the vector of 1's, and the product of any design column with itself is **I**=**11**=**22**=**33**=**44**=**55**
- Take the two specifications for the blocking variables, **4**=**123** and **5**=**23**. Multiply 1st expresion by **4** and 2nd by **5**: **I**=**1234** and **I**=**235.** These are called the *generators* of the blocking arrangements.
- Multiply these two together and to get 1223345=145 to complete the *defining relation* **I**=**1234**=**235**=**145.**
- The third generator shows that the main effect **1** is confounded with the **45** block effect, which we don't want.
- Better: confound the two block variables **4** and **5** with any two of the 2-factor interactions, say **4**=**12**, **5**=**13**

Table 5.15. A 2^3 Design in Blocks of Size 2, a Good Arrangement

Run	Experimental Variable			Block Variable		Experiment Arranged in Four Blocks				
Number	1	2	3	4 = 12	5 = 13	Block	1	2	3	Run
1	−	−	−	+	+	I	+	−	−	2
2	+	−	−	−	−		−	+	+	7
3	−	+	−	−	+	II	−	+	−	3
4	+	+	−	+	−		+	−	+	6
5	−	−	+	+	−	III	+	+	−	4
6	+	−	+	−	+		−	−	+	5
7	−	+	+	−	−	IV	−	−	−	1
8	+	+	+	+	+		+	+	+	8

5.17. LEARNING BY DOING

Fractional Factorial Designs

- Chapter 6 of BHH (2^{nd} ed) discusses fractional factorial designs.
- Example: full 2^5 factorial would require 32 runs. An experiment with only 8 runs is a $1/4^{th}$ (quarter) fraction. Because $¼=(½)^2=2^{-2}$, this is referred to as a 2^{5-2} design.
- In general, 2^{k-p} design is a $(½)^p$ fraction of a 2^k design using 2^{k-p} runs.
- Note that the first blocked design we considered was a half fraction: 2^{4-1} defined by the generating relation **I**=**1234**, which provides all the confounded ("aliased") relationships. E.g. **1**=**1I**=**11234**=**234**.