Design of Engineering Experiments
Part 9 – Experiments with Random Factors

• Text reference, Chapter 13, Pg. 484
• Previous chapters have considered **fixed** factors
  – A specific set of factor levels is chosen for the experiment
  – Inference confined to those levels
  – Often **quantitative** factors are fixed (why?)
• When factor levels are chosen at random from a larger population of potential levels, the factor is **random**
  – Inference is about the entire population of levels
  – Industrial applications include measurement system studies
Random Effects Models

- Example 13-1 (pg. 487) – weaving fabric on looms
- Response variable is strength
- Interest focuses on determining if there is difference in strength due to the different looms
- However, the weave room contains many (100s) looms
- Solution – select a (random) sample of the looms, obtain fabric from each
- Consequently, “looms” is a random factor
- See data, Table 13-1; looks like standard single-factor experiment with $a = 4$ & $n = 4$
Random Effects Models

• The usual single factor ANOVA model is

\[ y_{ij} = \mu + \tau_i + \varepsilon_{ij} \]

\[ i = 1, 2, \ldots, a \]

\[ j = 1, 2, \ldots, n \]

• Now both the error term and the treatment effects are random variables:

\[ \varepsilon_{ij} \text{ is } NID(0, \sigma^2) \text{ and } \tau_i \text{ is } NID(0, \sigma^2) \]

• Variance components: \[ V(y_{ij}) = \sigma^2 + \sigma^2 \]

\[ \tau \]
Relevant Hypotheses in the Random Effects (or Components of Variance) Model

• In the fixed effects model we test equality of treatment means

• This is no longer appropriate because the treatments are randomly selected
  – the individual ones we happen to have are not of specific interest
  – we are interested in the population of treatments

• The appropriate hypotheses are

\[ H_0 : \sigma^2_\tau = 0 \]
\[ H_1 : \sigma^2_\tau > 0 \]
Testing Hypotheses - Random Effects Model

• The standard ANOVA partition of the total sum of squares still works; leads to usual ANOVA display

• Form of the hypothesis test depends on the expected mean squares

\[ E(\text{MS}_E) = \sigma^2 \quad \text{and} \quad E(\text{MS}_{\text{Treatments}}) = \sigma^2 + n\sigma_{\tau}^2 \]

• Therefore, the appropriate test statistic is

\[ F_0 = \frac{\text{MS}_{\text{Treatments}}}{\text{MS}_E} \]
Estimating the Variance Components

• Use the ANOVA method; equate expected mean squares to their observed values:

\[
\hat{\sigma}^2 = MS_E \quad \text{and} \quad \hat{\sigma}^2 + n\hat{\sigma}^2_\tau = MS_{\text{Treatments}}
\]

\[
\hat{\sigma}^2_\tau = \frac{MS_{\text{Treatments}} - MS_E}{n}
\]

\[
\hat{\sigma}^2 = MS_E
\]

• Potential problems with these estimators
  – Negative estimates (woops!)
  – They are moment estimators & don’t have best statistical properties
# Minitab Solution (Balanced ANOVA)

<table>
<thead>
<tr>
<th>Factor</th>
<th>Type</th>
<th>Levels</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loom</td>
<td>random</td>
<td>4</td>
<td>1 2 3 4</td>
</tr>
</tbody>
</table>

## Analysis of Variance for y

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loom</td>
<td>3</td>
<td>89.188</td>
<td>29.729</td>
<td>15.68</td>
<td>0.000</td>
</tr>
<tr>
<td>Error</td>
<td>12</td>
<td>22.750</td>
<td>1.896</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>15</td>
<td>111.938</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source | Variance | Error | Expected Mean Square for Each Term component term (using unrestricted model)

1 Loom | 6.958 | 2 | (2) + 4(1)
2 Error | 1.896 | | (2)
Confidence Intervals on the Variance Components

• Easy to find a $100(1-\alpha)\%$ CI on $\sigma^2$:

$$\frac{(N - a)MS_E}{\chi^2_{\alpha/2,N-a}} \leq \sigma^2 \leq \frac{(N - a)MS_E}{\chi^2_{1-(\alpha/2),N-a}}$$

• Other confidence interval results are given in the book

• Sometimes the procedures are not simple
Extension to Factorial Treatment Structure

• Two factors, factorial experiment, both factors random (Section 13-2, pg. 490)

\[ y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \]

\[
\begin{align*}
  i &= 1, 2, \ldots, a \\
  j &= 1, 2, \ldots, b \\
  k &= 1, 2, \ldots, n
\end{align*}
\]

\[
V(\tau_i) = \sigma^2_\tau, V(\beta_j) = \sigma^2_\beta, V[(\tau\beta)_{ij}] = \sigma^2_{\tau\beta}, V(\varepsilon_{ijk}) = \sigma^2
\]

\[
V(y_{ijk}) = \sigma^2_\tau + \sigma^2_\beta + \sigma^2_{\tau\beta} + \sigma^2
\]

• The model parameters are NID random variables
• Random effects model
Testing Hypotheses - Random Effects Model

• Once again, the standard ANOVA partition is appropriate

• Relevant hypotheses:

\[
H_0 : \sigma_r^2 = 0 \quad H_0 : \sigma_\beta^2 = 0 \quad H_0 : \sigma_{r\beta}^2 = 0
\]

\[
H_1 : \sigma_r^2 > 0 \quad H_1 : \sigma_\beta^2 > 0 \quad H_1 : \sigma_{r\beta}^2 > 0
\]

• Form of the test statistics depend on the expected mean squares:

\[
E(MS_A) = \sigma^2 + n\sigma_{r\beta}^2 + bn\sigma_r^2 \quad \Rightarrow \quad F_0 = \frac{MS_A}{MS_{AB}}
\]

\[
E(MS_B) = \sigma^2 + n\sigma_{r\beta}^2 + an\sigma_\beta^2 \quad \Rightarrow \quad F_0 = \frac{MS_B}{MS_{AB}}
\]

\[
E(MS_{AB}) = \sigma^2 + n\sigma_{r\beta}^2 \quad \Rightarrow \quad F_0 = \frac{MS_{AB}}{MS_E}
\]

\[
E(MS_E) = \sigma^2
\]
Estimating the Variance Components – Two Factor Random model

• As before, use the ANOVA method; equate expected mean squares to their observed values:

\[
\hat{\sigma}_\tau^2 = \frac{MS_A - MS_{AB}}{bn}
\]

\[
\hat{\sigma}_\beta^2 = \frac{MS_B - MS_{AB}}{an}
\]

\[
\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n}
\]

\[
\hat{\sigma}^2 = MS_E
\]

• Potential problems with these estimators
Example 13-2 (pg. 492)
A Measurement Systems Capability Study

• Gauge capability (or R&R) is of interest
• The gauge is used by an operator to measure a critical dimension on a part
• **Repeatability** is a measure of the variability due only to the gauge
• **Reproducibility** is a measure of the variability due to the operator
• See experimental layout, Table 13-3. This is a two-factor factorial (completely randomized) with both factors (operators, parts) **random – a random effects model**
Example 13-2 (pg. 493)
Minitab Solution – Using Balanced ANOVA

Table 13-4  Analysis of Variance (Minitab Balanced ANOVA) for Example 13-2

<table>
<thead>
<tr>
<th>Analysis of Variance (Balanced Designs)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factor</strong></td>
</tr>
<tr>
<td>part</td>
</tr>
<tr>
<td>operator</td>
</tr>
</tbody>
</table>

Analysis of Variance for $y$

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>part</td>
<td>19</td>
<td>1185.425</td>
<td>62.391</td>
<td>87.65</td>
<td>0.000</td>
</tr>
<tr>
<td>operator</td>
<td>2</td>
<td>2.617</td>
<td>1.308</td>
<td>1.84</td>
<td>0.173</td>
</tr>
<tr>
<td>part*operator</td>
<td>38</td>
<td>27.050</td>
<td>0.712</td>
<td>0.72</td>
<td>0.861</td>
</tr>
<tr>
<td>Error</td>
<td>60</td>
<td>59.500</td>
<td>0.992</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>119</td>
<td>1274.592</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>Variance component</th>
<th>Error term</th>
<th>Expected Mean Square for Each Term (using unrestricted model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 part</td>
<td>10.2798</td>
<td>3</td>
<td>(4) + 2(3) + 6(1)</td>
</tr>
<tr>
<td>2 operator</td>
<td>0.0149</td>
<td>3</td>
<td>(4) + 2(3) + 40(2)</td>
</tr>
<tr>
<td>3 part*operator</td>
<td>-0.1399</td>
<td>4</td>
<td>(4) + 2(3)</td>
</tr>
<tr>
<td>4 Error</td>
<td>0.9917</td>
<td></td>
<td>(4)</td>
</tr>
</tbody>
</table>
Example 13-2 (pg. 493)  
Minitab Solution – Balanced ANOVA

- There is a large effect of parts (not unexpected)
- Small operator effect
- No Part – Operator interaction
- Negative estimate of the Part – Operator interaction variance component
- Fit a reduced model with the Part – Operator interaction deleted
Example 13-2 (pg. 493)  
Minitab Solution – Reduced Model

Table 13-5  Analysis of Variance for the Reduced Model, Example 13-2

Analysis of Variance (Balanced Designs)

<table>
<thead>
<tr>
<th>Factor</th>
<th>Type</th>
<th>Levels</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>part</td>
<td>random</td>
<td>20</td>
<td>1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20</td>
</tr>
<tr>
<td>operator</td>
<td>random</td>
<td>3</td>
<td>1 2 3</td>
</tr>
</tbody>
</table>

Analysis of Variance for $y$

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>part</td>
<td>19</td>
<td>1185.425</td>
<td>62.391</td>
<td>70.64</td>
<td>0.000</td>
</tr>
<tr>
<td>operator</td>
<td>2</td>
<td>2.617</td>
<td>1.308</td>
<td>1.48</td>
<td>0.232</td>
</tr>
<tr>
<td>Error</td>
<td>98</td>
<td>86.550</td>
<td>0.883</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>119</td>
<td>1274.592</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Source</th>
<th>Variance component Error term</th>
<th>Expected Mean Square for Each Term (using unrestricted model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 part</td>
<td>10.2513 3</td>
<td>(3) + 6(1)</td>
</tr>
<tr>
<td>2 operator</td>
<td>0.0106 3</td>
<td>(3) + 40(2)</td>
</tr>
<tr>
<td>3 Error</td>
<td>0.8832</td>
<td>(3)</td>
</tr>
</tbody>
</table>
Example 13-2 (pg. 493)
Minitab Solution – Reduced Model

• Estimating gauge capability:

\[ \hat{\sigma}_{\text{gauge}}^2 = \hat{\sigma}^2 + \hat{\sigma}_\beta^2 \]

\[ = 0.88 + 0.01 \]

\[ = 0.89 \]

• If interaction had been significant?
The Two-Factor Mixed Model

• Two factors, factorial experiment, factor \( A \) fixed, factor \( B \) random (Section 12-3, pg. 522)

\[
y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \quad \begin{cases} 
  i = 1, 2, \ldots, a \\
  j = 1, 2, \ldots, b \\
  k = 1, 2, \ldots, n
\end{cases}
\]

\[
V(\beta_j) = \sigma^2_{\beta}, V[(\tau\beta)_{ij}] = [(a - 1)/a]\sigma^2_{\tau\beta}, V(\varepsilon_{ijk}) = \sigma^2
\]

\[
\sum_{i=1}^{a} \tau_i = 0, \sum_{i=1}^{a} (\tau\beta)_{ij} = 0
\]

• The model parameters \( \beta_j \) and \( \varepsilon_{ijk} \) are \textit{NID} random variables, the interaction effect is normal, but not independent

• This is called the \textbf{restricted model}
Testing Hypotheses - Mixed Model

• Once again, the standard ANOVA partition is appropriate
• Relevant hypotheses:

\[\begin{align*}
H_0 : \tau_i &= 0 \quad & H_0 : \sigma^2_{\beta} &= 0 \quad & H_0 : \sigma^2_{\tau\beta} &= 0 \\
H_1 : \tau_i &\neq 0 \quad & H_1 : \sigma^2_{\beta} &> 0 \quad & H_1 : \sigma^2_{\tau\beta} &> 0
\end{align*}\]

• Test statistics depend on the expected mean squares:

\[
E(\text{MS}_A) = \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a - 1} \quad \Rightarrow \quad F_0 = \frac{\text{MS}_A}{\text{MS}_{AB}}
\]

\[
E(\text{MS}_B) = \sigma^2 + an\sigma^2_{\beta} \quad \Rightarrow \quad F_0 = \frac{\text{MS}_B}{\text{MS}_E}
\]

\[
E(\text{MS}_{AB}) = \sigma^2 + n\sigma^2_{\tau\beta} \quad \Rightarrow \quad F_0 = \frac{\text{MS}_{AB}}{\text{MS}_E}
\]

\[
E(\text{MS}_E) = \sigma^2
\]
Estimating the Variance Components – Two Factor Mixed model

- Use the ANOVA method; equate expected mean squares to their observed values:

\[
\hat{\sigma}_\beta^2 = \frac{MS_B - MS_E}{an}
\]

\[
\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n}
\]

\[
\hat{\sigma}^2 = MS_E
\]

- Estimate the fixed effects (treatment means) as usual
Example 13-3 (pg. 497)
The Measurement Systems Capability Study Revisited

• Same experimental setting as in example 13-2
• Parts are a random factor, but Operators are fixed
• Assume the restricted form of the mixed model
• Minitab can analyze the mixed model
Example 13-3 (pg. 497)
Minitab Solution – Balanced ANOVA

Table 13-6  Analysis of Variance (Minitab) for the Mixed Model in Example 13-3.  The Restricted Model is Assumed.

Analysis of Variance (Balanced Designs)

<table>
<thead>
<tr>
<th>Factor</th>
<th>Type</th>
<th>Levels</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>part</td>
<td>random</td>
<td>20</td>
<td>1  2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>15  16</td>
<td>17  18</td>
</tr>
<tr>
<td>operator</td>
<td>fixed</td>
<td>3</td>
<td>1  2</td>
</tr>
</tbody>
</table>

Analysis of Variance for y

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>part</td>
<td>19</td>
<td>1185.425</td>
<td>62.391</td>
<td>62.92</td>
<td>0.000</td>
</tr>
<tr>
<td>operator</td>
<td>2</td>
<td>2.617</td>
<td>1.308</td>
<td>1.84</td>
<td>0.173</td>
</tr>
<tr>
<td>part*operator</td>
<td>38</td>
<td>27.050</td>
<td>0.712</td>
<td>0.72</td>
<td>0.861</td>
</tr>
<tr>
<td>Error</td>
<td>60</td>
<td>59.500</td>
<td>0.992</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>119</td>
<td>1274.592</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Source          Variance component Error term Expected Mean Square for Each Term (using restricted model)
1 part          10.2332  4  (4) + 6(1)
2 operator      3  (4) + 2(3) + 40Q[2]
3 part*operator -0.1399  4  (4) + 2(3)
4 Error         0.9917  (4)
Example 13-3
Minitab Solution – Balanced ANOVA

- There is a large effect of parts (not unexpected)
- Small operator effect
- No Part – Operator interaction
- Negative estimate of the Part – Operator interaction variance component
- Fit a reduced model with the Part – Operator interaction deleted
- This leads to the same solution that we found previously for the two-factor random model
The Unrestricted Mixed Model

• Two factors, factorial experiment, factor $A$ fixed, factor $B$ random (pg. 526)

$$y_{ijk} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \varepsilon_{ijk} \left\{ \begin{array}{l} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{array} \right.$$

$$V(\gamma_j) = \sigma^2_\beta, V[(\alpha\gamma)_{ij}] = \sigma^2_{\alpha\gamma}, V(\varepsilon_{ijk}) = \sigma^2$$

$$\sum_{i=1}^{a} \alpha_i = 0$$

• The random model parameters are now all assumed to be $NID$
Testing Hypotheses – Unrestricted Mixed Model

• The standard ANOVA partition is appropriate
• Relevant hypotheses:
  \[ H_0 : \alpha_i = 0 \quad H_0 : \sigma^2_\gamma = 0 \quad H_0 : \sigma^2_{\alpha\gamma} = 0 \]
  \[ H_1 : \alpha_i \neq 0 \quad H_1 : \sigma^2_\gamma > 0 \quad H_1 : \sigma^2_{\alpha\gamma} > 0 \]

• Expected mean squares determine the test statistics:

  \[ E(MS_A) = \sigma^2 + n\sigma^2_{\alpha\gamma} + \frac{bn\sum_{i=1}^{a} \alpha^2_i}{a - 1} \quad \Rightarrow \quad F_0 = \frac{MS_A}{MS_{AB}} \]

  \[ E(MS_B) = \sigma^2 + n\sigma^2_{\alpha\gamma} + an\sigma^2_\gamma \quad \Rightarrow \quad F_0 = \frac{MS_B}{MS_{AB}} \]

  \[ E(MS_{AB}) = \sigma^2 + n\sigma^2_{\alpha\gamma} \quad \Rightarrow \quad F_0 = \frac{MS_{AB}}{MS_E} \]

  \[ E(MS_E) = \sigma^2 \]
Estimating the Variance Components – Unrestricted Mixed Model

• Use the ANOVA method; equate expected mean squares to their observed values:

\[
\hat{\sigma}_\gamma^2 = \frac{MS_B - MS_{AB}}{an}
\]

\[
\hat{\sigma}_{\alpha\gamma}^2 = \frac{MS_{AB} - MS_E}{n}
\]

\[
\hat{\sigma}^2 = MS_E
\]

• The only change compared to the restricted mixed model is in the estimate of the random effect variance component
### Example 13-4 (pg. 499)
**Minitab Solution – Unrestricted Model**

Table 13-7  Analysis of the Experiment in Example 13-3 Using the Unrestricted Model

<table>
<thead>
<tr>
<th>Analysis of Variance (Balanced Designs)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Factor</strong></td>
</tr>
<tr>
<td>part</td>
</tr>
<tr>
<td>operator</td>
</tr>
</tbody>
</table>

#### Analysis of Variance for \( y \)

<table>
<thead>
<tr>
<th>Source</th>
<th>DF</th>
<th>SS</th>
<th>MS</th>
<th>F</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>part</td>
<td>19</td>
<td>1185.425</td>
<td>62.391</td>
<td>87.65</td>
<td>0.000</td>
</tr>
<tr>
<td>operator</td>
<td>2</td>
<td>2.617</td>
<td>1.308</td>
<td>1.84</td>
<td>0.173</td>
</tr>
<tr>
<td>part*operator</td>
<td>38</td>
<td>27.050</td>
<td>0.712</td>
<td>0.72</td>
<td>0.861</td>
</tr>
<tr>
<td>Error</td>
<td>60</td>
<td>59.500</td>
<td>0.992</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>119</td>
<td>1274.592</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Source

<table>
<thead>
<tr>
<th>Source</th>
<th>Variance component</th>
<th>Error term</th>
<th>Expected Mean Square for Each Term (using unrestricted model)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 part</td>
<td>10.2798</td>
<td>3</td>
<td>( (4) + 2(3) + 6(1) )</td>
</tr>
<tr>
<td>2 operator</td>
<td></td>
<td>3</td>
<td>( (4) + 2(3) + Q[2] )</td>
</tr>
<tr>
<td>3 part*operator</td>
<td>-0.1399</td>
<td>4</td>
<td>( (4) + 2(3) )</td>
</tr>
<tr>
<td>4 Error</td>
<td>0.9917</td>
<td></td>
<td>( (4) )</td>
</tr>
</tbody>
</table>
Finding Expected Mean Squares

• Obviously important in determining the form of the test statistic
• In fixed models, it’s easy:

\[ E(\text{MS}) = \sigma^2 + f(\text{fixed factor}) \]

• Can always use the “brute force” approach – just apply the expectation operator
• Straightforward but tedious
• Rules on page 502-504 work for any balanced model
• Rules are consistent with the restricted mixed model – can be modified to incorporate the unrestricted model assumptions
Approximate $F$ Tests

- Sometimes we find that there are no exact tests for certain effects (page 505)
- Leads to an approximate $F$ test (“pseudo” $F$ test)
- Test procedure is due to Satterthwaite (1946), and uses linear combinations of the original mean squares to form the $F$-ratio
- The linear combinations of the original mean squares are sometimes called “synthetic” mean squares
- Adjustments are required to the degrees of freedom
- Refer to Example 13-7, page 507
- Minitab will analyze these experiments, although their “synthetic” mean squares are not always the best choice