Design of Engineering Experiments Part 9 – Experiments with Random Factors

- Text reference, Chapter 13, Pg. 484
- Previous chapters have considered **fixed** factors
  - A specific set of factor levels is chosen for the experiment
  - Inference confined to those levels
  - Often **quantitative** factors are fixed (why?)
- When factor levels are chosen at random from a larger population of potential levels, the factor is **random**
  - Inference is about the entire population of levels
  - Industrial applications include measurement system studies

Random Effects Models

- Example 13-1 (pg. 487) – weaving fabric on looms
- Response variable is strength
- Interest focuses on determining if there is difference in strength due to the different looms
- However, the weave room contains many (100s) looms
- Solution – select a (random) **sample** of the looms, obtain fabric from each
- Consequently, "looms" is a **random factor**
- See data, Table 13-1; looks like standard single-factor experiment with $a = 4$ & $n = 4$

Random Effects Models

- The usual single factor ANOVA model is

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, n \end{cases}$$

- Now both the error term and the treatment effects are random variables:

$$\varepsilon_{ij} \text{ is } NID(0, \sigma^2) \text{ and } \tau_i \text{ is } NID(0, \sigma_\tau^2)$$

- **Variance components**: $V(y_{ij}) = \sigma^2 + \sigma_\tau^2$

Relevant Hypotheses in the Random Effects (or Components of Variance) Model

- In the fixed effects model we test equality of treatment means
- This is no longer appropriate because the treatments are randomly selected
  - the individual ones we happen to have are not of specific interest
  - we are interested in the **population** of treatments
- The appropriate hypotheses are

$$H_0 : \sigma_\tau^2 = 0$$
$$H_1 : \sigma_\tau^2 > 0$$

Testing Hypotheses - Random Effects Model

- The standard ANOVA partition of the total sum of squares still works; leads to usual ANOVA display
- Form of the hypothesis test depends on the **expected mean squares**

$$E(MS_E) = \sigma^2 \text{ and } E(MS_{Treatments}) = \sigma^2 + n\sigma_\tau^2$$

- Therefore, the appropriate test statistic is

$$F_0 = MS_{Treatments} / MS_E$$

Estimating the Variance Components

- Use the **ANOVA** method; equate expected mean squares to their observed values:

$$\hat{\sigma}^2 = MS_E \text{ and } \hat{\sigma}^2 + n\hat{\sigma}_\tau^2 = MS_{Treatments}$$

$$\hat{\sigma}_\tau^2 = \frac{MS_{Treatments} - MS_E}{n}$$

$$\hat{\sigma}^2 = MS_E$$

- Potential problems with these estimators
  - Negative estimates (woops!)
  - They are moment estimators & don't have best statistical properties

Minitab Solution (Balanced ANOVA)

```
Factor     Type Levels Values
Loom     random      4     1     2     3     4
Analysis of Variance for y


Source      DF          SS          MS       F      P
Loom         3      89.188      29.729   15.68  0.000
Error       12      22.750       1.896
Total       15     111.938


Source      Variance Error Expected Mean Square for Each Term
           component term (using unrestricted model)
 1 Loom        6.958   2   (2) + 4(1)
 2 Error       1.896       (2)
```

Confidence Intervals on the Variance Components

- Easy to find a 100(1-α)% CI on σ^2 :

$$\frac{(N-a)MS_E}{\chi^2_{\alpha/2,N-a}} \leq \sigma^2 \leq \frac{(N-a)MS_E}{\chi^2_{1-(\alpha/2),N-a}}$$

- Other confidence interval results are given in the book
- Sometimes the procedures are not simple

Extension to Factorial Treatment Structure

- Two factors, factorial experiment, both factors random (Section 13-2, pg. 490)

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}$$

$$V(\tau_i) = \sigma_\tau^2, V(\beta_j) = \sigma_\beta^2, V[(\tau\beta)_{ij}] = \sigma_{\tau\beta}^2, V(\varepsilon_{ijk}) = \sigma^2$$

$$V(y_{ijk}) = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$$

- The model parameters are *NID* random variables
- Random effects model

Testing Hypotheses - Random Effects Model

- Once again, the standard ANOVA partition is appropriate
- Relevant hypotheses:

$$H_0: \sigma_\tau^2 = 0 \qquad H_0: \sigma_\beta^2 = 0 \qquad H_0: \sigma_{\tau\beta}^2 = 0$$

$$H_1: \sigma_\tau^2 > 0 \qquad H_1: \sigma_\beta^2 > 0 \qquad H_1: \sigma_{\tau\beta}^2 > 0$$

- Form of the test statistics depend on the **expected mean squares**:

$$E(MS_A) = \sigma^2 + n\sigma_{\tau\beta}^2 + bn\sigma_\tau^2 \quad \Rightarrow \quad F_0 = \frac{MS_A}{MS_{AB}}$$

$$E(MS_B) = \sigma^2 + n\sigma_{\tau\beta}^2 + an\sigma_\beta^2 \quad \Rightarrow \quad F_0 = \frac{MS_B}{MS_{AB}}$$

$$E(MS_{AB}) = \sigma^2 + n\sigma_{\tau\beta}^2 \quad \Rightarrow \quad F_0 = \frac{MS_{AB}}{MS_E}$$

$$E(MS_E) = \sigma^2$$

Estimating the Variance Components – Two Factor Random model

- As before, use the **ANOVA** method; equate expected mean squares to their observed values:

$$\hat{\sigma}_{\tau}^2 = \frac{MS_A - MS_{AB}}{bn}$$

$$\hat{\sigma}_{\beta}^2 = \frac{MS_B - MS_{AB}}{an}$$

$$\hat{\sigma}_{\tau\beta}^2 = \frac{MS_{AB} - MS_E}{n}$$

$$\hat{\sigma}^2 = MS_E$$

- Potential problems with these estimators

Example 13-2 (pg. 492)
A Measurement Systems Capability Study

- Gauge capability (or R&R) is of interest
- The gauge is used by an operator to measure a critical dimension on a part
- **Repeatability** is a measure of the variability due only to the gauge
- **Reproducibility** is a measure of the variability due to the operator
- See experimental layout, Table 13-3. This is a two-factor factorial (completely randomized) with both factors (operators, parts) **random – a random effects model**

Example 13-2 (pg. 493)
Minitab Solution – Using Balanced ANOVA

Table 13-4 Analysis of Variance (Minitab Balanced ANOVA) for Example 13-2

```
Analysis of Variance (Balanced Designs)

Factor     Type Levels Values
part     random     20     1     2     3     4     5     6     7
                           8     9    10    11    12    13    14
                          15    16    17    18    19    20
operator random      3     1     2     3

Analysis of Variance for y

Source             DF         SS        MS        F        P
part               19   1185.425    62.391    87.65    0.000
operator            2      2.617     1.308     1.84    0.173
part*operator      38     27.050     0.712     0.72    0.861
Error              60     59.500     0.992
Total             119   1274.592

Source              Variance   Error  Expected Mean Square for Each Term
                    component  term   (using unrestricted model)
1 part               10.2798     3    (4) + 2(3) +  6(1)
2 operator            0.0149     3    (4) + 2(3) + 40(2)
3 part*operator      -0.1399     4    (4) + 2(3)
4 Error               0.9917          (4)
```

Example 13-2 (pg. 493)
Minitab Solution – Balanced ANOVA

- There is a large effect of parts (not unexpected)
- Small operator effect
- No Part – Operator interaction
- Negative estimate of the Part – Operator interaction variance component
- Fit a reduced model with the Part – Operator interaction deleted

Example 13-2 (pg. 493)
Minitab Solution – Reduced Model

Table 13-5 Analysis of Variance for the Reduced Model, Example 13-2

```
Analysis of Variance (Balanced Designs)

Factor    Type   Levels Values
part      random     20      1      2      3      4      5      6      7
                             8      9     10     11     12     13     14
                            15     16     17     18     19     20
operator  random      3      1      2      3

Analysis of Variance for y

Source        DF        SS         MS         F        P
part          19   1185.425    62.391     70.64    0.000
operator       2      2.617     1.308      1.48    0.232
Error         98     86.550     0.883
Total        119   1274.592

Source       Variance    Error   Expected Mean Square for Each Term
             component   term    (using unrestricted model)
1 part        10.2513      3     (3) + 6(1)
2 operator     0.0106      3     (3) + 40(2)
3 Error        0.8832            (3)
```

Example 13-2 (pg. 493)
Minitab Solution – Reduced Model

- Estimating gauge capability:

$$
\begin{aligned}
\hat{\sigma}^2_{gauge} &= \hat{\sigma}^2 + \hat{\sigma}^2_{\beta} \\
&= 0.88 + 0.01 \\
&= 0.89
\end{aligned}
$$

- If interaction had been significant?

The Two-Factor Mixed Model

- Two factors, factorial experiment, factor *A* fixed, factor *B* random (Section 12-3, pg. 522)

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1, 2, ..., a \\ j = 1, 2, ..., b \\ k = 1, 2, ..., n \end{cases}$$

$$V(\beta_j) = \sigma_\beta^2, V[(\tau\beta)_{ij}] = [(a-1)/a]\sigma_{\tau\beta}^2, V(\varepsilon_{ijk}) = \sigma^2$$

$$\sum_{i=1}^{a} \tau_i = 0, \sum_{i=1}^{a} (\tau\beta)_{ij} = 0$$

- The model parameters β_j and ε_{ijk} are *NID* random variables, the interaction effect is normal, but not independent
- This is called the **restricted model**

Testing Hypotheses - Mixed Model

- Once again, the standard ANOVA partition is appropriate
- Relevant hypotheses:

$$H_0 : \tau_i = 0 \qquad H_0 : \sigma_\beta^2 = 0 \qquad H_0 : \sigma_{\tau\beta}^2 = 0$$

$$H_1 : \tau_i \neq 0 \qquad H_1 : \sigma_\beta^2 > 0 \qquad H_1 : \sigma_{\tau\beta}^2 > 0$$

- Test statistics depend on the **expected mean squares**:

$$E(MS_A) = \sigma^2 + n\sigma_{\tau\beta}^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \quad \Rightarrow \quad F_0 = \frac{MS_A}{MS_{AB}}$$

$$E(MS_B) = \sigma^2 + an\sigma_\beta^2 \quad \Rightarrow \quad F_0 = \frac{MS_B}{MS_E}$$

$$E(MS_{AB}) = \sigma^2 + n\sigma_{\tau\beta}^2 \quad \Rightarrow \quad F_0 = \frac{MS_{AB}}{MS_E}$$

$$E(MS_E) = \sigma^2$$

Estimating the Variance Components – Two Factor Mixed model

- Use the **ANOVA** method; equate expected mean squares to their observed values:

$$\hat{\sigma}_{\beta}^{2} = \frac{MS_B - MS_E}{an}$$

$$\hat{\sigma}_{\tau\beta}^{2} = \frac{MS_{AB} - MS_E}{n}$$

$$\hat{\sigma}^{2} = MS_E$$

- Estimate the fixed effects (treatment means) as usual

Example 13-3 (pg. 497)
The Measurement Systems Capability Study Revisited

- Same experimental setting as in example 13-2
- Parts are a random factor, but Operators are fixed
- Assume the restricted form of the mixed model
- Minitab can analyze the mixed model

Example 13-3 (pg. 497)
Minitab Solution – Balanced ANOVA

Table 13-6 Analysis of Variance (Minitab) for the Mixed Model in Example 13-3. The Restricted Model is Assumed.

```
Analysis of Variance (Balanced Designs)

Factor     Type Levels Values
part     random     20      1      2      3      4      5      6      7
                            8      9     10     11     12     13     14
                           15     16     17     18     19     20
operator fixed       3      1      2      3

Analysis of Variance for y

Source            DF         SS        MS        F        P
part              19   1185.425    62.391    62.92    0.000
operator           2      2.617     1.308     1.84    0.173
part*operator     38     27.050     0.712     0.72    0.861
Error             60     59.500     0.992
Total            119   1274.592

Source             Variance   Error  Expected Mean Square for Each Term
                   component  term   (using restricted model)
1 part              10.2332     4    (4) + 6(1)
2 operator                      3    (4) + 2(3) + 40Q[2]
3 part*operator    -0.1399      4    (4) + 2(3)
4 Error             0.9917           (4)
```

Example 13-3
Minitab Solution – Balanced ANOVA

- There is a large effect of parts (not unexpected)
- Small operator effect
- No Part – Operator interaction
- Negative estimate of the Part – Operator interaction variance component
- Fit a reduced model with the Part – Operator interaction deleted
- This leads to the same solution that we found previously for the two-factor random model

The Unrestricted Mixed Model

- Two factors, factorial experiment, factor *A* fixed, factor *B* random (pg. 526)

$$y_{ijk} = \mu + \alpha_i + \gamma_j + (\alpha\gamma)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1,2,...,a \\ j = 1,2,...,b \\ k = 1,2,...,n \end{cases}$$

$$V(\gamma_j) = \sigma_\beta^2, V[(\alpha\gamma)_{ij}] = \sigma_{\alpha\gamma}^2, V(\varepsilon_{ijk}) = \sigma^2$$

$$\sum_{i=1}^{a} \alpha_i = 0$$

- The random model parameters are now **all** assumed to be *NID*

Testing Hypotheses – Unrestricted Mixed Model

- The standard ANOVA partition is appropriate
- Relevant hypotheses:

$$H_0: \alpha_i = 0 \qquad H_0: \sigma_\gamma^2 = 0 \qquad H_0: \sigma_{\alpha\gamma}^2 = 0$$

$$H_1: \alpha_i \neq 0 \qquad H_1: \sigma_\gamma^2 > 0 \qquad H_1: \sigma_{\alpha\gamma}^2 > 0$$

- **Expected mean squares** determine the test statistics:

$$E(MS_A) = \sigma^2 + n\sigma_{\alpha\gamma}^2 + \frac{bn\sum_{i=1}^{a}\alpha_i^2}{a-1} \quad \Rightarrow \quad F_0 = \frac{MS_A}{MS_{AB}}$$

$$E(MS_B) = \sigma^2 + n\sigma_{\alpha\gamma}^2 + an\sigma_\gamma^2 \quad \Rightarrow \quad F_0 = \frac{MS_B}{MS_{AB}}$$

$$E(MS_{AB}) = \sigma^2 + n\sigma_{\alpha\gamma}^2 \quad \Rightarrow \quad F_0 = \frac{MS_{AB}}{MS_E}$$

$$E(MS_E) = \sigma^2$$

Estimating the Variance Components – Unrestricted Mixed Model

- Use the **ANOVA** method; equate expected mean squares to their observed values:

$$\hat{\sigma}_{\gamma}^{2} = \frac{MS_{B} - MS_{AB}}{an}$$

$$\hat{\sigma}_{\alpha\gamma}^{2} = \frac{MS_{AB} - MS_{E}}{n}$$

$$\hat{\sigma}^{2} = MS_{E}$$

- The only change compared to the restricted mixed model is in the estimate of the random effect variance component

Example 13-4 (pg. 499)
Minitab Solution – Unrestricted Model

Table 13-7 Analysis of the Experiment in Example 13-3 Using the Unrestricted Model

```
Analysis of Variance (Balanced Designs)

Factor     Type Levels Values
part     random     20      1      2      3      4      5      6      7
                            8      9     10     11     12     13     14
                           15     16     17     18     19     20
operator fixed       3      1      2      3

Analysis of Variance for y

Source             DF         SS        MS        F        P
part               19   1185.425    62.391    87.65    0.000
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part*operator      38     27.050     0.712     0.72    0.861
Error              60     59.500     0.992
Total             119   1274.592

Source             Variance   Error   Expected Mean Square for Each Term
                   component  term    (using unrestricted model)
1 part              10.2798    3      (4) + 2(3) + 6(1)
2 operator                     3      (4) + 2(3) + Q[2]
3 part*operator     -0.1399    4      (4) + 2(3)
4 Error              0.9917           (4)
```

Finding Expected Mean Squares

- Obviously important in determining the form of the test statistic
- In fixed models, it's easy:

$$E(MS) = \sigma^2 + f(\text{fixed factor})$$

- Can always use the "brute force" approach – just apply the expectation operator
- Straightforward but tedious
- Rules on page 502-504 work for any balanced model
- Rules are consistent with the **restricted** mixed model – can be modified to incorporate the **unrestricted** model assumptions

Approximate *F* Tests

- Sometimes we find that there are no exact tests for certain effects (page 505)
- Leads to an approximate *F* test ("pseudo" *F* test)
- Test procedure is due to Satterthwaite (1946), and uses **linear combinations** of the original mean squares to form the *F*-ratio
- The linear combinations of the original mean squares are sometimes called "synthetic" mean squares
- Adjustments are required to the degrees of freedom
- Refer to Example 13-7, page 507
- Minitab will analyze these experiments, although their "synthetic" mean squares are not always the best choice