STAT 583: Advanced Theory of Statistical Inference	Homework 2

Lecture 2: Homework

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- Please send the electronic copies of your HW to Yandi Shen's email box (*ydshen@uw.edu*).

2.1 Symmetrization

For any function class $\mathcal{F}$, we define the empirical Rademacher complexity of $\mathcal{F}$ to be:

$$\widehat{\mathcal{R}}_n(\mathcal{G}) := \mathbb{E}_\epsilon\Big(\sup_{f\in\mathcal{F}}\Big|\frac{1}{n}\sum_{i=1}^n \epsilon_i f(x_i)\Big|\Big)$$

where the expectation is on the Rademacher sequence $\{\epsilon_i, i \in [n]\}$ conditionally on the data $\{x_1, \ldots, x_n\}$.

Problem 1

Consider the functional class

$$\mathcal{F} = \{x \to \mathrm{sign}(\langle\theta, x\rangle) \mid \theta \in \mathbb{R}^d, \|\theta\|_2 = 1\},$$

corresponding to the $\{-1, +1\}$-valued classification rules defined by linear functions in $\mathbb{R}^d$. Suppose $d \geq n$ so that it is possible to have $x_1^n = \{x_1, \ldots, x_n\}$ that is a collection of vectors in $\mathbb{R}^d$ that are linearly independent. For such an x_1^n, show that

$$\mathbb{E}_\epsilon\Big(\sup_{f\in\mathcal{F}}\Big|\frac{1}{n}\sum_{i=1}^n \epsilon_i f(x_i)\Big|\Big) = 1.$$

2.2 VC dimension

Problem 2

Let $\mathcal{A}$ be a finite class of sets (i.e., $|\mathcal{A}| < \infty$). Determine upper bounds of $\Pi_{\mathcal{A}}(n)$ and $\nu(\mathcal{A})$. Provide an example for which your upper bounds are tight.

Problem 3

Determine the VC dimensions of the following classes of sets:

(a) The class of sets in $\mathbb{R}^d$:

$$\mathcal{A} := \{(-\infty, a_1] \times (-\infty, a_n] \times \cdots \times (-\infty, a_d] \mid (a_1, \ldots, a_d) \in \mathbb{R}^d\}.$$

(b) The class of sets in $\mathbb{R}^d$:

$$\mathcal{A} := \{(b_1, a_1] \times (b_2, a_2] \times \cdots \times (b_d, a_d] \mid (a_1, \ldots, a_d), (b_1, \ldots, b_d) \in \mathbb{R}^d\}.$$

Problem 4

Determine the VC dimensions of the following classes of sets:

(a) A half space is defined to be a set of the form $\{x \in \mathbb{R}^d : \langle x, u \rangle \leq c\}$ for some fixed $u \in \mathbb{R}^d$ and $c \in \mathbb{R}$. Show that the collection of all half-spaces in $\mathbb{R}^d$ is a VC-class of index $d+1$.

(b) The class of all closes balls in $\mathbb{R}^2$, that is, $\mathcal{A}$ is the class of all subsets of the form

$$\left\{ x \in \mathbb{R}^2 \mid \sum_{i=1}^{2} (x_i - a_i)^2 \leq R, \quad \text{for some } (a_1, a_2) \in \mathbb{R}^2 \text{ and } R > 0 \right\}.$$

Hint: You might consider Radon's Theorem (`https://en.wikipedia.org/wiki/Radon%27s_theorem`) useful.

2.3 VC-subgraph

Problem 5

Prove the following example in Lecture note 2: Suppose $\mathcal{C}$ is a VC class of index $\nu(\mathcal{C})$, then by definition the class of functions $\mathcal{F} := \{\mathbb{1}_C; C \in \mathcal{C}\}$ is VC subgraph of index $\nu(\mathcal{C})$.

Problem 6 (Problem 10 on Page 152 in VW1996)

For a set $\mathcal{F}$ of measurable functions, define "closed" and "open" subgraphs by $\{(x,t) : t \leq f(x)\}$ and $\{(x,t) : t < f(x)\}$ respectively. Show that the collections of "closed" and "open" subgraphs have the same VC-index.

Problem 7 (Problem 20 on Page 153 in VW1996)

The class of functions of the form $x \to c\mathbb{1}_{(a,b]}(x)$ with $a, b, c > 0$ ranging over $\mathbb{R}$ is VC-subgraph. Determine the index.

Problem 8 (Problem 21 on Page 153 in VW1996)

The "Box-Cox family of transformations" $\mathcal{F} := \{f_\lambda : (0, \infty) \to \mathbb{R} : \lambda \in \mathbb{R} - \{0\}\}$, with $f_\lambda(x) := (x^\lambda - 1)/\lambda$ is a VC-subgraph class.

Problem 9

Prove Lemma 23 in the lecture note 2.