Markov Chain Review – and questions
Math/Stat 491: Introduction to Stochastic Processes
Wellner; 11/15/2013

Terminology, Definitions, Basic Results

- A state $j \in \mathbb{E}$ is an *absorbing state* if $P(j, j) = 1$
- A set of states $A \subset \mathbb{E}$ is *closed* if for any $i \in A$ and $j \in A^c$, the transition probability $P(i, j) = 0$.
- A set $B \subset \mathbb{E}$ is *irreducible* if for any $i, j \in B$ we have $i \to j$ (and $j \to i$); that is, $\rho_{ij} \equiv P_i(T_j < \infty) > 0$; or, equivalently $P^n(i, j) > 0$ for some $n \geq 1$.
- A state $j \in \mathbb{E}$ is *transient* if $\rho_{jj} = P_j(T_j < \infty) < 1$.
- Durrett, Theorem 1.5: If $\rho_{i,j} > 0$ but $\rho_{j,i} < 1$, then i is transient.
- Proposition: A set closed set C is irreducible if and only if no proper subset of it is closed.
- Durrett, Theorem 1.7: If C is a finite closed and irreducible set, then all states in C are recurrent.

Questions:

- **Q1:** (Jake McAferty). How are absorbing states treated when finding irreducible closed sets of a Markov chain?
  A1: An absorbing state j yields a closed set $A = \{j\}$ by definition of a closed set. By the next to last item in the list above, $A = \{j\}$ is irreducible and closed.
- **Q2:** (Yuzhe Zhou). Does there exist a Markov chain with state space $\mathbb{E}$ that is irreducible but not all states recurrent?
  A2: Yes. Consider a Galton-Watson process as discussed in class on 13 November with offspring distribution $\{p_k\}$ satisfying $p_k > 0$ for all $k \geq 0$. Then $\mathbb{E} = \{0, 1, 2, \ldots\}$ and $P(i, j) > 0$ for all $i \geq 1$ and $j > 0$, while $P(0, 0) = 1$. Thus the whole state space is irreducible, and 0 is a

closed, irreducible, absorbing state. But the set of states $A = \{1, 2, \ldots\}$ is a set of transient states if $\mu < 1$ or if $\mu \geq 1$ by Durrett, Theorem 1.5, page 16.

- **Q3:** (Yuzhe Zhou). Does there exist a Markov chain with finite state space that is irreducible but not R (all states recurrent)?
  A3: No. This follows from Durrett's Theorem 1.7, page 17 (the last last item in the list above).