Handout 4: THE POISSON PROCESS

Math/Stat 394: Probability I

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$N(t) \equiv$	(the total *count* at time t) $\sim Poisson(\nu t)$
ν	is the *intensity*, or mean number of counts per unit of time
$\theta \equiv$	$1/\nu$ will be seen to be the mean time between counts.
Examples:	traffic accidents, telephone calls, defects per foot, ...
$Y_i \equiv$	(the i-th interarrival time) $\sim Exponential(\nu)$ $f_{Y_i}(t) = \nu \exp(-\nu t) 1_{(0,\infty)}(t)$ with $E(Y_i) = 1/\nu = \theta$ and $Var(Y) = 1/\nu^2 = \theta^2$.
$W_r \equiv$	$Y_1 + \cdots + Y_r$ (the waiting time until the r-th event) $\sim Gamma(r, \nu)$; $f_{W_r}(t) = (\nu t)^{r-1}/\Gamma(r))\nu \exp(-\nu t) 1_{(0,\infty)}(t)$, with $E(W_r) = r/\nu$ and $Var(W_r) = r/\nu^2$.

- Key facts: $[Y_1 > t] = [N(t) = 0]$ and $[W_r > t] = [N(t) < r]$.
- Poisson facts:

  (a) The rv $N(t) \sim$ Poisson(νt).
  $E(N(t)) = \nu t$ and $Var(N(t)) = \nu t$.

  (b) The rv
  $$Z_t \equiv \frac{N(t) - \mu_t}{\sigma_t} = \frac{N(t) - \nu t}{\sqrt{\nu t}}$$
  is approximately $N(0,1)$ for large values of νt.

  (c) $N(s) \sim Poisson(\nu s)$ and $N(t) - N(s) \sim Poisson(\nu(t - s))$ are independent, and their sum, $N(t) \sim Poisson(\nu t)$.

  (e) Given that $N(t) = m$, for an integer $m \geq 1$, for $0 < s < t$ $(N(s)|N(t) = m) \sim Binomial(m, s/t)$.

- Gamma facts:

  (a) The first waiting time rv $Y \equiv Y_1$ has mean $E(Y) = 1/\nu = \theta$ and variance $Var(Y) = 1/\nu^2 = \theta^2$.

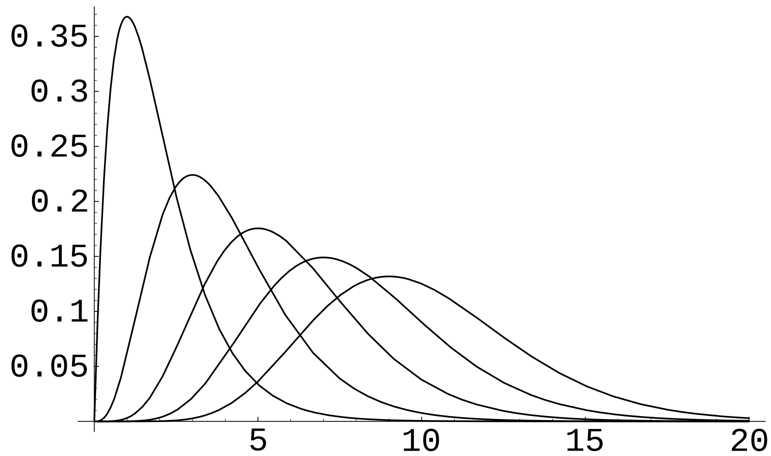

Figure 1: Plot of Gamma$(k, 1)$ densities $k = 2, 4, 6, 8, 10$.

(b) The rv W_r has mean $E(W_r) = r/\nu = r\theta$ and variance $Var(W_r) = r/\nu^2 = r\theta^2$.

(c) The rv

$$Z_r \equiv \frac{W_r - E(W_r)}{\sigma_r} = \frac{W_r - r\theta}{\sqrt{r\theta^2}}$$

is approximately $N(0, 1)$.

(d) If $W_r \sim Gamma(r, \nu)$ and $W_s \sim Gamma(s, \nu)$ are independent, then $W_r + W_s \sim Gamma(r + s, \nu)$.

- Exponential facts:

(a) If $Y \equiv Y_1$, $P(Y > t) = P(N(t) = 0 = e^{-\nu t}$ for $t \geq 0$.
Hence $f_Y(t) = (d/dt)F_Y(t) = (d/dt)(1 - e^{-\nu t}) = \nu e^{-\nu t}$ for $t \geq 0$.

(b) $P(Y > t + s|Y > s) = P(Y > t + s)/P(Y > s) = e^{-\nu(t+s)}/e^{-\nu s} = e^{-\nu t} = P(Y > t)$. This is the *memoryless* property of the Exponential distribution.

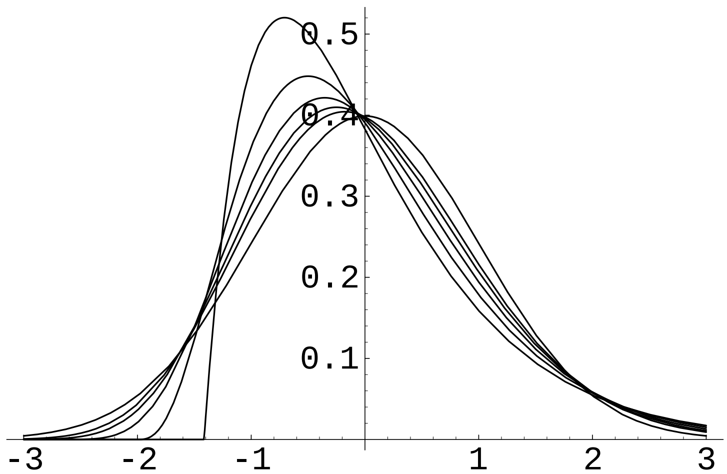

Figure 2: Plots of standardized Gamma$(k, 1)$ densities $k = 2, 4, 8, 16, 32$.

- Conditional on either $N(t_0) = m$ or $W_{m+1} = t_0$, there are m arrivals uniformly distributed over $[0, t_0]$. That is, given the number m of events on or before a given time, the location is random and does not depend on ν.