Statistics 492, Problem Set 1

Wellner; 1/07/14

Reading: Karlin and Taylor, chapter 7, pages 340-365
Karlin and Taylor, chapter 6, section 8, pages 313-325.

Due: Tuesday, 14 January 2014.

1. Let $\{B(t) : t \geq 0\}$ be Brownian motion started at x. We showed in class that

$$P_x(B(t_1) \leq x_1, B(t_2) \leq x_2) = \int_{-\infty}^{x_1} \left(\int_{-\infty}^{x_2} p_{t_2 - t_1}(y_1, y_2) dy_2 \right) p_{t_1}(x, y_1) dy_1$$

   where $p_t(x, y)$ is the transition probability density for Brownian motion. Generalize this formula to $m \geq 3$ time points with $0 \leq t_1 \leq \cdots \leq t_m$ and levels $x_1, \ldots, x_m$.

2. Let B denote standard Brownian motion starting from 0 at time 0, and let $\alpha \in \mathbb{R}$. Show that $Y_\alpha(t) \equiv \exp(-\alpha t) B(e^{2\alpha t})$ is a Gaussian process and compute its covariance function.

3. Let B denote standard Brownian motion. Show that $\{U(t) = B(t) - tB(1) : \ 0 \leq t \leq 1\}$ is a Gaussian process. Find the covariance function of the process U.

4. Let B denote Brownian motion, and define a new process V by $V(t) = (1 - t)B(t/(1-t))$ for $0 \leq t \leq 1$. Show that V is a Gaussian process and compute the covariance function of the process V. (The process V is a *Brownian bridge process* on $[0, 1]$.)

5. Let $\{U(t) : \ 0 \leq t \leq 1\}$ be a Brownian bridge process on $[0, 1]$, and consider the process $W(t) = (1+t)U(\frac{t}{1+t})$ for $0 \leq t < \infty$. Show that W is a Brownian motion process on $[0, \infty)$. (This is called *Doob's transformation*.)

6. Let $B(t)$ denote standard Brown motion, let $t_j = j/m$ for $j = 0, \ldots, m$, and consider the Riemann sums $Y_m = \sum_{j=0}^m B(t_j)(t_{j+1} - t_j)$ approximating $\int_0^1 B(t)dt$. Compute $Var(Y_m) \equiv \sigma_m^2$ and show that $\lim_{m \to \infty} \sigma_m^2 = 1/3$.

7. **Optional bonus problem:** Karlin and Taylor, page 384, problem 9: If B denotes standard Brownian motion, derive the conditional distribution of $W = \int_0^t B(s)ds$ given that $B(t) = x$.