Statistics 492, Problem Set 2

Wellner; 1/14/14

Reading:	Karlin and Taylor, chapter 7, pages 340-365 Karlin and Taylor, chapter 6, section 8, pages 313-325. Klebaner, chapter 3, pages 55-86, Klebaner, chapter 8, pages 169-177.
Reminder:	Preliminary project titles and descriptions due Thursday, 23 January
Due:	Tuesday, 21 January 2014.

1. Show that the transition density for Brownian motion, $p_t(x, y) = (2\pi t)^{-1/2} \exp(-(y-x)^2/(2t))$, satisfies the "heat equation"
$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial t^2}.$$

2. Klebaner, Exercise 3.12, page 86: Formulate the law of large numbers and the law of the iterated logarithm for Brownian motion near zero. [Hint: Use the fact that if B is standard Brownian motion, then $W(t) = tB(1/t)$, $t > 0$, and $W(0) = 0$ is also Brownian motion.]

3. (i) Let $f(x, t) \equiv x^4 - 6x^2 t + 3t^2$ and let B denote standard Brownian motion. Show that $\{f(B(t), t) : t \geq 0\}$ is a martingale. [Hint: Use the exponential martingale $Y_c(t) \equiv \exp(cB(t) - c^2 t/2)$ and compute $(\partial^4/\partial c^4)(Y_c)|_{c=0}$.]
(ii) Show that $f(x, t)$ given in (i) satisfies the "backwards heat equation"
$$\frac{\partial f}{\partial t} = -\frac{1}{2}\frac{\partial^2 f}{\partial t^2}.$$
It turns out that for any polynomial function f in x and t which satisfies the backwards heat equation, $f(B(t), t)$ is a martingale.

4. Derive the joint distribution of $B(t)$ and $m(t) \equiv \min_{0\leq s\leq t} B(s)$. [Hint: Consider the process $-B$ and the joint distribution of $(B(t), M(t))$ where $M(t) \equiv \max_{0\leq s\leq t} B(s)$.]

5. **Optional bonus problem:** Let $T_a \equiv \inf\{t > 0 : B_t \notin (-a, a)\}$. Show that $E(T_a) = a^2$ and that $E(T_a^2) = 5a^4/3$. Conclude that $Var(T_a) = 2a^4/3$. [Hint: Use the martingale in problem 3 above.