

Statistics 491, Problem Set 6

Wellner; 10/30/13

Reading: Ross; Chapter 4, pages 230 - 265
Durrett; Chapter 1, pages 20 - 74 .

Due: Wednesday, November 6, 2013.

1. Durrett, chapter 1, problem 1.12, page 76.

(a) Find the stationary distribution for the transition probability matrix P given by

$$\mathbf{P} = \begin{pmatrix} 0 & 2/3 & 0 & 1/3 \\ 1/3 & 0 & 2/3 & 0 \\ 0 & 1/6 & 0 & 5/6 \\ 2/5 & 0 & 3/5 & 0 \end{pmatrix}$$

and show that it does not satisfy the detailed balance condition (1.11).

(b) Consider the transition matrix of the form P given by

$$\mathbf{P} = \begin{pmatrix} 0 & a & 0 & 1-a \\ 1-b & 0 & b & 0 \\ 0 & 1-c & 0 & c \\ d & 0 & 1-d & 0 \end{pmatrix}$$

and show that there is a stationary distribution satisfying (1.11) if $0 < abcd = (1-a)(1-b)(1-c)(1-d)$.

2. Durrett, chapter 1, problem 1.31, page 80.

A plant species has red, pink or white flowers according to the genotypes RR , RW , and WW , respectively. If each of these genotypes is crossed with a pink (RW) plant then the offspring fractions are:

	RR	RW	WW
RR	0.5	0.5	0
RW	0.25	0.5	0.25
WW	0	0.5	0.5

What is the long run fraction of plants of the three types?

3. Durrett, chapter 1, problem 1.37, page 82.

An individual has three umbrellas, some at her office, and some at home. If she is leaving home in the morning (or leaving work at night) and it is raining,

she will take an umbrella, if one is there. Otherwise she gets wet. Assume that independent of the past, it rains on each trip with probability .2. To formulate a Markov chain, let X_n be the number of umbrellas at her current location. (a) Find the transition probability matrix for this Markov chain. (b) Calculate the limiting fraction of time she gets wet.

4. Durrett, chapter 1, problem 1.44, page 84.

5. (Sampling with and without replacement; Ross pages 54-56 and 100)

Suppose that an urn contains a total of N balls. W of the balls in the urn are white and B of the balls are black, so $W + B = N$.

(a) Consider sampling from the urn *with replacement*: letting $X_i = 1\{\text{ith ball drawn is White}\}$ for $i = 1, \dots, n$, the random variables in n draws from the urn, X_1, \dots, X_n , are independent and identically distributed Bernoulli(p) random variables with $p = W/N$. What is the distribution of $S_n = \sum_{i=1}^n X_i$ in this case? Calculate $E(S_n)$ and $Var(S_n)$, expressing both in terms of n , W , and N .

(b) Now consider sampling from the urn *without replacement*: letting $X_i = 1\{\text{ith ball drawn is White}\}$ for $i = 1, \dots, n$, the random variables X_i each have a (marginal) Bernoulli(p) distribution with $p = W/N$, but are *dependent* random variables. What is the joint distribution of (X_1, \dots, X_n) ? The distribution of the total number of white balls in n draws from the urn, $S_n = \sum_{i=1}^n X_i$, is Hypergeometric (n, W, N) . Find $E(S_n)$ and $Var(S_n)$ and express both in terms of n , W , and N .

6. Durrett, chapter 1, problem 1.46, page 85.

Bernoulli-Laplace model of diffusion: Consider two urns each of which contains m balls; b of these $2m$ balls are black, and the remaining $2m - b$ are white. We say that the system is in state i if the first urn contains i black balls and $m - i$ white balls, while the second contains $b - i$ black balls and $m - b + i$ white balls. Each trial consists of choosing a ball at random from each urn and exchanging the two. Let X_n be the state of the system after n exchanges have been made. X_n is a Markov chain.

(a) Compute the transition matrix of X_n .

(b) Verify that the stationary distribution is given by

$$\pi(i) = \frac{\binom{b}{i} \binom{2m-b}{m-i}}{\binom{2m}{m}}.$$

What is the name of this distribution? If Y is a discrete random variable with the mass function $\pi(i)$, what is $E(Y)$? What is $Var(Y)$? (c) Can you give a simple intuitive explanation why the formula in (b) gives the right answer?