Statistics 491, Problem Set 8

Wellner; 11/13/13

Reading: Ross; Chapter 4, pages 243 - 290
Durrett; Chapter 1, pages 58 - 91 .

Due: Wednesday, November 20, 2013.

1. Durrett, chapter 1, problem 1.75, page 91.
The opposite of the aging chain is the renewal chain with state space $\{0, 1, 2, \ldots\}$ in which $P(i, i-1) = 1$ when $i > 0$. The only nontrivial part of the transition probability matrix is $P(0, i) = p_i$ (with $\sum_{i=0}^{\infty} p_i = 1$). Show that this chain is always recurrent, but is positive recurrent if and only if $\sum_{n=0}^{\infty} np_n < \infty$.

2. Durrett, chapter 1, problem 1.77, page 91.
Consider a branching process as defined in Example 1.8 in which each family has a number of children that follows a shifted geometric distribution $p_k = p(1-p)^k$ for $k \geq 0$ (which counts the number of failures before the first success when success has probability p). Compute the probability that starting from one individual the chain will be absorbed at 0. What does this yield when $p = 1/3$? When $p = 2/5$? Plot the probability generating function ϕ of $\{p_k\}$ for these two values of p and compare to the identity function.

3. Durrett, chapter 1, problem 1.56, page 87.
A bank classifies loans as paid in full (F), in good standing (G), in arrears (A), or as a bad debt (B). Loans move between the categories according the following transition probability matrix:

$$\mathbf{P} = \begin{array}{c|cccc} & \mathbf{F} & \mathbf{G} & \mathbf{A} & \mathbf{B} \\ \mathbf{F} & 1 & 0 & 0 & 0 \\ \mathbf{G} & 0.1 & 0.8 & 0.1 & 0 \\ \mathbf{A} & 0.1 & 0.4 & 0.4 & 0.1 \\ \mathbf{B} & 0 & 0 & 0 & 1 \end{array}$$

What fraction of loans in good standing are eventually paid in full? What is the answer for those in arrears?

4. Durrett, chapter 1, problem 1.53, page 86.
Consider the Ehrenfest chain, example 1.2, with transition probabilities $P(i, i+1) = (N-i)/N$ and $P(i, i-1) = i/N$ for $0 \leq i \leq N$. Let $\mu_n = E_x X_n$.

(a) Show that $\mu_{n+1} = 1 + (1 - 2/N)\mu_n$. (b) Use this and induction to conclude that

$$\mu_n = \frac{N}{2} + \left(1 - \frac{2}{N}\right)^n (x - N/2).$$

From this we see that the mean μ_n converges exponentially rapidly to the equilibrium value of $N/2$ with the error at time n being $(1 - 2/N)^n(x - N/2)$.

5. Durrett, chapter 1, problem 1.72, page 90.
   Consider the Markov chain with state space $\{0, 1, 2, \ldots\}$ and transition probability matrix $\mathbf{P}$ given by

$$P(m, m+1) = \frac{1}{2}\left(1 - \frac{1}{m+2}\right) \quad \text{for} \quad m \geq 0,$$
$$P(m, m-1) = \frac{1}{2}\left(1 + \frac{1}{m+2}\right) \quad \text{for} \quad m \geq 1.$$

   and $P(0, 0) = 1 - P(0, 1) = 3/4$. Find the stationary distribution π.

6. Durrett, chapter 1, problem 1.73, page 90.
   Consider the Markov chain with state space $\{1, 2, \ldots\}$ and transition probability matrix $\mathbf{P}$ given by

$$P(m, m+1) = \frac{m}{2m+2} \quad \text{for} \quad m \geq 1,$$
$$P(m, m-1) = \frac{1}{2} \quad \text{for} \quad m \geq 2,$$
$$P(m, m) = \frac{1}{2m+2} \quad \text{for} \quad m \geq 2.$$

   and $P(1, 1) = 1 - P(1, 2) = 3/4$. Show that there is no stationary distribution π.

7. **Optional bonus problem 1:** Durrett, chapter 1, problem 1.70, page 90.

8. **Optional bonus problem 2:** Durrett, chapter 1, problem 1.71, page 90.