3 Examples.

Example 3.1 (One-sample $t-$test) Suppose that $X_1, \ldots, X_n$ are i.i.d. with $E(X_1) = \mu$ and $Var(X_1) = \sigma^2$. Consider testing $H : \mu \le \mu_0$ versus $K : \mu > \mu_0$. The normal theory test is "reject H if $T_n \ge t_{n-1,\alpha}$" where

$$T_n \equiv \frac{\sqrt{n}(\overline{X}_n - \mu_0)}{S_n}$$

with $S_n^2 \equiv (n-1)^{-1}\sum_1^n (X_i - \overline{X}_n)^2$ and where $P(t_{n-1} \ge t_{n-1,\alpha}) = \alpha$. We are interested in the behavior of this test when the X_i's are *not normally distributed*.
(a) What if $\mu = \mu_0$ is true? Note that by the Lindeberg central limit theorem $\sqrt{n}(\overline{X}_n - \mu_0) \to_d N(0, \sigma^2)$ when $\mu = \mu_0$ is true, and by the WLLN and Slutsky's theorem

$$S_n^2 = \frac{n}{n-1}\left\{\frac{1}{n}\sum_{i=1}^n (X_i - \mu_0)^2 - (\overline{X} - \mu_0)^2\right\} \to_p 1\left\{\sigma^2 - 0\right\} = \sigma^2.$$

Thus by Slutsky's theorem again, $T_n \to_d Z \sim N(0,1)$ when $\mu = \mu_0$ is true, and we have $P_{\mu_0}(T_n \ge t_{n-1,\alpha}) \to P(Z \ge z_\alpha) = \alpha$.
(b) What if $\mu > \mu_0$ is true, with μ fixed? In this case

$$T_n = \frac{\sqrt{n}(\overline{X} - \mu)}{S_n} + \frac{\sqrt{n}(\mu - \mu_0)}{S_n} \to_p Z + \infty/\sigma = \infty,$$

so $P_\mu(T_n > t_{n-1,\alpha}) \to P(Z + \infty > z_\alpha) = 1$.
(c) What if $\mu = \mu_n > \mu_0$ with $\sqrt{n}(\mu_n - \mu_0) \to c > 0$? Then it will usually hold that

$$T_n = \frac{\sqrt{n}(\overline{X} - \mu_n)}{S_n} + \frac{\sqrt{n}(\mu_n - \mu_0)}{S_n} \to_d Z + c/\sigma$$

where we may need to apply a Lindeberg-Feller or Liapunov CLT to justify the convergence to normality in the first term. If this holds, then

$$P_{\mu_n}(T_n > t_{n-1,\alpha}) \to P(Z + c/\sigma > z_\alpha) = P(Z > z_\alpha - c/\sigma) = 1 - \Phi(z_\alpha - c/\sigma)$$

gives the limiting power of the test under the local alternatives μ_n. Note that $1 - \Phi(z_\alpha - c/\sigma) > \alpha$ for $c > 0$.

Example 3.2 (One sample normal - theory test of variance) Now suppose that $X_1, \ldots, X_n$ are i.i.d. with $E(X_1) = \mu$, $Var(X_1) = \sigma^2$, and $\mu_4 \equiv E(X_1 - \mu)^4 < \infty$.
(a) Now with $Y_i \equiv (X_i - \mu)^2 \sim (\sigma^2, \mu_4 - \sigma^4)$,

$$\begin{aligned}
\frac{\sqrt{n}(n^{-1}\sum_1^n (X_i - \mu)^2 - \sigma^2)}{\sqrt{2}\sigma^2} &= \frac{\sqrt{n}(\overline{Y}_n - \sigma^2)}{\sqrt{2}\sigma^2} \\
&\to_d \frac{N(0, \mu_4 - \sigma^4)}{\sqrt{2}\sigma^2} \\
&= N\left(0, \frac{\mu_4 - \sigma^4}{2\sigma^4}\right) \\
&= N\left(0, \frac{2\sigma^4 + \mu_4 - 3\sigma^4}{2\sigma^4}\right) \\
&= N(0, 1 + 2^{-1}\gamma_2) \quad \text{with} \quad \gamma_2 \equiv \frac{\mu_4}{\sigma^4} - 3.
\end{aligned}$$

(b) Since

$$n^{-1}\sum_1^n (X_i-\overline{X})^2 = n^{-1}\sum_1^n (X_i-\mu)^2 - (\overline{X}-\mu)^2$$

we have

$$\begin{aligned}\frac{\sqrt{n}(n^{-1}\sum_1^n (X_i-\overline{X}_n)^2-\sigma^2)}{\sqrt{2}\sigma^2} &= \frac{\sqrt{n}(\overline{Y}_n-\sigma^2)}{\sqrt{2}\sigma^2} - \frac{\sqrt{n}(\overline{X}_n-\mu)(\overline{X}_n-\mu)}{\sqrt{2}\sigma^2}\\ &\to_d N(0,1+2^{-1}\gamma_2) - N(0,1)\cdot 0 = N(0,1+2^{-1}\gamma_2)\end{aligned}$$

by Slutsky's theorem. Thus with $S_n^2 \equiv (n-1)^{-1}\sum_1^n (X_i-\overline{X}_n)^2$,

$$\frac{\sqrt{n}(S_n^2-\sigma^2)}{\sqrt{2}\sigma^2} \to_d N(0,1+2^{-1}\gamma_2).$$

Now consider testing $H:\sigma=\sigma_0$ versus $K:\sigma>\sigma_0$. If $X_i \sim N(\mu,\sigma_0^2)$, then $(n-1)S_n^2/\sigma_0^2 \sim \chi^2_{n-1}$ under H, so the usual normal theory test is "reject H if $(n-1)S_n^2/\sigma_0^2 > \chi^2_{n-1,\alpha}$". Then since $\gamma_2(N(\mu,\sigma^2))=0$ we have

$$\begin{aligned}\alpha &= P_{\sigma_0,Norm}\left(\frac{(n-1)S_n^2}{\sigma_0^2} > \chi^2_{n-1,\alpha}\right)\\ &= P_{\sigma_0,Norm}\left(\sqrt{\frac{n}{2}}\left(\frac{S_n^2}{\sigma_0^2}-1\right) > \sqrt{\frac{n}{2}}\left(\frac{\chi^2_{n-1,\alpha}}{n-1}-1\right)\right)\\ &\to P(Z\ge z_\alpha)=\alpha,\end{aligned}$$

which forces

$$\sqrt{\frac{n}{2}}\left(\frac{\chi^2_{n-1,\alpha}}{n-1}-1\right)\to z_\alpha.$$

Now suppose we carried out the normal theory test, but the X_i's are *not* normal. Then, under H,

$$\begin{aligned}P_{\sigma_0}\left(\frac{(n-1)S_n^2}{\sigma_0^2} > \chi^2_{n-1,\alpha}\right) &= P_{\sigma_0}\left(\sqrt{\frac{n}{2}}\left(\frac{S_n^2}{\sigma_0^2}-1\right) > \sqrt{\frac{n}{2}}\left(\frac{\chi^2_{n-1,\alpha}}{n-1}-1\right)\right)\\ &\to P(N(0,1+2^{-1}\gamma_2)\ge z_\alpha)\ne\alpha\end{aligned}$$

when $\gamma_2\ne 0$. In general the asymptotic size is smaller than α if $\gamma_2<0$, but the asymptotic size is greater than α if $\gamma_2>0$.

Example 3.3 (Two-sample tests for means) Suppose that $X_1,\ldots,X_m$ are i.i.d. with mean μ and variance σ^2, and that $Y_1,\ldots,Y_n$ are i.i.d. with mean ν and variance τ^2, independent of the X_j's. If we suppose that $\lambda_N \equiv m/N \equiv m/(m+n)\to\lambda\in[0,1]$, then

$$\begin{aligned}\sqrt{\frac{mn}{N}}\left(\overline{X}_m-\overline{Y}_n-(\mu-\nu)\right) &= \sqrt{\frac{n}{N}}\sqrt{m}(\overline{X}_m-\mu) - \sqrt{\frac{m}{N}}\sqrt{n}(\overline{Y}_n-\nu)\\ &\to_d \sqrt{1-\lambda}Z_1-\sqrt{\lambda}Z_2 \quad (Z_1,Z_2)\sim N_2(0,\mathrm{diag}(\sigma^2,\tau^2))\\ &\sim N(0,(1-\lambda)\sigma^2+\lambda\tau^2).\end{aligned}$$

Thus we see that

$$\begin{aligned}\frac{\overline{X}_m - \overline{Y}_n - (\mu - \nu)}{\sqrt{\frac{S_X^2}{m} + \frac{S_Y^2}{n}}} &= \frac{\sqrt{\frac{mn}{N}}\left(\overline{X}_m - \overline{Y}_n - (\mu - \nu)\right)}{\sqrt{\frac{n}{N}S_X^2 + \frac{m}{N}S_Y^2}} \\ &\to_d N(0,1)\end{aligned}$$

by Slutsky's theorem. On the other hand, again by Slutsky's theorem,

$$\begin{aligned}T_{m,n}(\mu,\nu) &\equiv \frac{\sqrt{\frac{mn}{N}}\left(\overline{X}_m - \overline{Y}_n - (\mu - \nu)\right)}{\sqrt{\frac{(m-1)S_X^2 + (n-1)S_Y^2}{N-2}}} \\ &\to_d \frac{N(0,(1-\lambda)\sigma^2 + \lambda\tau^2)}{\sqrt{\lambda\sigma^2 + (1-\lambda)\tau^2}} \\ &= N\left(0, \frac{(1-\lambda)\sigma^2 + \lambda\tau^2}{\lambda\sigma^2 + (1-\lambda)\tau^2}\right) \\ &\neq N(0,1)\end{aligned}$$

unless $\lambda = 1/2$ or $\sigma^2 = \tau^2$.

Since the two-sample $t-$test of $H : \mu \le \nu$ versus $K : \mu > \nu$ rejects H if $T_{m,n}(0,0) > t_{N-2,\alpha}$, it follows that the test is *not size (or level) robust* against violations of the assumption $\sigma^2 = \tau^2$ when $\lambda \neq 1/2$.

Example 3.4 (Simple linear regression with non-normal errors.) Suppose that $Y_i = \alpha + \beta(x_i - \overline{x}) + \epsilon_i$ for $i = 1, \ldots, n$ where $\overline{x} \equiv n^{-1}\sum_1^n x_i$ and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d. with mean zero and finite variance, $Var(\epsilon_1) = \sigma^2$; the ϵ_i's are *not* assumed to be normally distributed. In matrix form

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \equiv \begin{pmatrix} 1 & x_1 - \overline{x} \\ \vdots & \vdots \\ 1 & x_n - \overline{x}\end{pmatrix}\begin{pmatrix}\alpha \\ \beta\end{pmatrix} + \boldsymbol{\epsilon}.$$

The least squares estimators $\hat{\alpha}$ and $\hat{\beta}$ of α and β are given by

$$\hat{\alpha} = \overline{Y}, \qquad \hat{\beta} = \frac{\sum_1^n (x_i - \overline{x})Y_i}{\sum_1^n (x_i - \overline{x})^2}.$$

Claim: if $\max_{1\le i\le n}(x_i - \overline{x})^2 / \sum_1^n (x_i - \overline{x})^2 \to 0$, then

$$(1) \qquad (\mathbf{X}\mathbf{X}^T)^{1/2}(\hat{\boldsymbol{\beta}} - \boldsymbol{\beta}) = \begin{pmatrix}\sqrt{n}(\hat{\alpha} - \alpha) \\ \sqrt{\sum_1^n (x_i - \overline{x})^2}(\hat{\beta} - \beta)\end{pmatrix} \to_d N_2(0, \sigma^2 I_2).$$

Here is a partial proof. Now

$$\sqrt{n}(\hat{\alpha} - \alpha) = \sqrt{n}(\overline{Y} - \alpha) = \sqrt{n}\overline{\epsilon} \to_d N(0, \sigma^2)$$

by the Lindeberg CLT. Thus the first coordinate in (1) converges to the claimed limit marginally. We now use the Lindeberg-Feller CLT to show that the same is true for the second coordinate, and hence the two claimed marginal convergences hold. Note that

$$\sqrt{\sum_1^n (x_i - \overline{x})^2}(\hat{\beta} - \beta) = \sigma\frac{\sum_1^n (x_i - \overline{x})\epsilon_i}{\sigma\sqrt{\sum_1^n (x_i - \overline{x})^2}} \equiv \sigma\frac{S_n}{\sigma_n}$$

in the context of the Lindeberg-Feller CLT where $X_{n,i} \equiv (x_i - \overline{x})\epsilon_i$ for $i = 1, \ldots$,n, and hence $\mu_{n,i} = EX_{n,i} = 0$, $\sigma^2_{n,i} = Var(X_{n,i}) = (x_i - \overline{x})^2\sigma^2$, and $\sigma^2_n = \sigma^2 \sum_1^n (x_i - \overline{x})^2$. Thus we need to verify the condition $LF_n(\delta) \to 0$ for every $\delta > 0$ where

$$LF_n(\delta) \equiv \frac{1}{\sigma_n^2} \sum_1^n E\{|X_{n,i}|^2 1_{[|X_{n,i}| \geq \delta\sigma_n]}\}.$$

But in the present case,

$$\begin{aligned}
LF_n(\delta) &= \frac{1}{\sigma^2 \sum_1^n (x_i - \overline{x})^2} \sum_{i=1}^n (x_i - \overline{x})^2 E\left\{\epsilon_i^2 1\{|x_i - \overline{x}||\epsilon_i| \geq \delta\sigma \sqrt{\sum_1^n (x_i - \overline{x})^2\}}\right\} \\
&= \frac{1}{\sigma^2 \sum_1^n (x_i - \overline{x})^2} \sum_{i=1}^n (x_i - \overline{x})^2 E\left\{\epsilon_1^2 1\{|\epsilon_1| \geq \frac{\delta\sigma}{\sqrt{\frac{|x_i - \overline{x}|^2}{\sum_1^n (x_i - \overline{x})^2}}}\}\right\} \\
&\leq \frac{1}{\sigma^2 \sum_1^n (x_i - \overline{x})^2} \sum_{i=1}^n (x_i - \overline{x})^2 E\left\{\epsilon_1^2 1\left\{|\epsilon_1| \geq \frac{\delta\sigma}{\sqrt{\max_{1 \leq i \leq n} \frac{|x_i - \overline{x}|^2}{\sum_1^n (x_i - \overline{x})^2}}}\right\}\right\} \\
&= \frac{1}{\sigma^2} E\left\{\epsilon_1^2 1\{|\epsilon_1| \geq \delta\sigma / \sqrt{\max_{1 \leq i \leq n} |x_i - \overline{x}|^2 / \sum_1^n (x_i - \overline{x})^2\}}\right\} \\
&\to 0
\end{aligned}$$

for every $\delta > 0$ by the DCT since the integrand converges a.s. to zero by the hypothesis and since $E(\epsilon_1^2) < \infty$, so ϵ_1^2 gives an integrable dominating function. Thus the second coordinate satisfies the claimed marginal convergence. All that remains to be shown is the claimed joint convergence.

Conclusion: the normal theory tests and confidence intervals for α and β have the right asymptotic size and coverage probabilities as long as $\sigma^2 < \infty$ and $\max_{1 \leq i \leq n} |x_i - \overline{x}|^2 / \sum_1^n (x_i - \overline{x})^2 \to 0$.

Example 3.5 (Multiple linear regression with non-normal errors). Insert?

Example 3.6 (The correlation coefficient). Suppose that $(X_1, Y_1)^T, \ldots, (X_n, Y_n)^T$ are i.i.d. with means $E(X_1, Y_1) = (\mu_X, \mu_Y)$, covariance matrix

$$\begin{pmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & \sigma_Y^2 \end{pmatrix},$$

and $E|X_1|^4 < \infty$, $E|Y_1|^4 < \infty$. Let

$$S_{XY} \equiv n^{-1} \sum_1^n (X_i - \overline{X})(Y_i - \overline{Y}), \quad S_{XX} \equiv n^{-1} \sum_1^n (X_i - \overline{X})^2, \quad S_{YY} \equiv n^{-1} \sum_1^n (Y_i - \overline{Y})^2,$$

and consider the sample correlation $r \equiv r_n$ defined by

$$r_n \equiv \frac{S_{XY}}{\sqrt{S_{XX} S_{YY}}}.$$

since r_n is invariant with respect to linear transformations of each axis, we may assume without loss of generality that $\mu_X = \mu_Y = 0$ and $\sigma_X^2 = \sigma_Y^2 = 1$; if not replace (X_i, Y_i) by $((X_i - \mu_X)/\sigma_X, (Y_i - \mu_Y)/\sigma_Y)$, $i = 1, \ldots, n$. Note that

$$\begin{aligned}\sqrt{n}\begin{pmatrix} S_{XY} - \rho \\ S_{XX} - 1 \\ S_{YY} - 1 \end{pmatrix} &= \sqrt{n}\begin{pmatrix} \overline{XY} - \rho \\ \overline{X^2} - 1 \\ \overline{Y^2} - 1 \end{pmatrix} + \begin{pmatrix} o_p(1) \\ o_p(1) \\ o_p(1) \end{pmatrix} \\ &\to_d \underline{Z} \sim N_3(0, \Sigma)\end{aligned}$$

by the multivariate CLT where

$$\Sigma = \begin{pmatrix} E(X^2Y^2) - \rho^2 & E(X^3Y) - \rho & E(XY^3) - \rho \\ E(X^3Y) - \rho & E(X^4) - 1 & E(X^2Y^2) - 1 \\ E(XY^3) - \rho & E(X^2Y^2) - 1 & E(Y^4) - 1 \end{pmatrix}.$$

Since $g(u, v, w) \equiv u/\sqrt{vw}$ has $\nabla g(\rho, 1, 1) = (1, -\rho/2, -\rho/2)$, it follows by the g' theorem (or delta method) that

$$\sqrt{n}(r_n - \rho) \to_d \nabla g(\rho, 1, 1)\underline{Z} = Z_1 - \frac{\rho}{2}(Z_2 + Z_3).$$

Note that if X_1 and Y_1 are independent, then $\rho = 0$ and the covariance matrix Σ becomes

$$\Sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu_{4,X} & 0 \\ 0 & 0 & \mu_{4,Y} \end{pmatrix},$$

and hence $\sqrt{n}(r_n - 0) = \sqrt{n} r_n \to_d N(0, 1)$. Thus under independence the test of $H : \rho = 0$ versus $K : \rho > 0$ that rejects if $\sqrt{n} r_n > z_\alpha$ has asymptotic level α even if the true distribution is not Gaussian (but we have $E|X_1|^4 + E|Y_1|^4 < \infty$). If the true distribution of the (X_i, Y_i) pairs is Normal, then Σ becomes

$$\Sigma = \begin{pmatrix} 1 + \rho^2 & 2\rho & 2\rho \\ 2\rho & 2 & 2\rho^2 \\ 2\rho & 2\rho^2 & 2 \end{pmatrix},$$

and hence

$$\sqrt{n}(r_n - \rho) \to_d N(0, (1, -\rho/2, -\rho/2)\Sigma(1, -\rho/2 - \rho/2)^T) = N(0, (1 - \rho^2)^2).$$

Finally, it is often useful to transform the distribution of r_n (which always has support in $[-1, 1]$) to the whole line $\mathbb{R}$: if $g(x) \equiv 2^{-1} \log((1 + x)/(1 - x))$, then $g'(x) = 1/(1 - x^2)$, and hence, under normality of the (X_i, Y_i)'s,

$$\sqrt{n}(g(r_n) - g(\rho)) \to_d N(0, 1).$$

If we let $Z_n \equiv g(r_n)$ and $\xi \equiv g(\rho)$ (this is sometimes known as *Fisher's Z-transform*), then

$$\sqrt{n - 3}\left(Z_n - \xi - \frac{\rho}{2(n - 1)}\right) \approx N(0, 1)$$

is an excellent approximation.

Example 3.7 (Chi-square test of a simple null hypothesis). Suppose that $\underline{\Delta}_1, \ldots, \underline{\Delta}_n, \ldots$ are i.i.d. Multinomial$_k(1, \underline{p})$ random vectors so that $\underline{\Delta}_i$ vector contains only zeros and 1's, but only one 1. Thus

$$\underline{N}_n \equiv \sum_{i=1}^{n} \underline{\Delta}_i \sim \text{Mult}_k(n, \underline{p}),$$

and $\hat{\underline{p}}_n \equiv n^{-1} N_n \to_{p,a.s.} \underline{p}$.

Consider testing $H : \underline{p} = \underline{p}_0$ versus $K : \underline{p} \neq \underline{p}_0$. One simple test statistic is the (Pearson) chi-square statistic Q_n defined by

$$Q_n \equiv \sum_{j=1}^{k} \frac{(N_j - np_{0,j})^2}{np_{0,j}}.$$

To carry out the test based on Q_n we need to know the distribution of Q_n under the null hypothesis either exactly (which is complicated: its depends on k, n, and $\underline{p}_0$), or at least asymptotically which is relative easy. We claim that: when $\underline{p} = \underline{p}_0$,

$$Q_n \to_d \chi^2_{k-1}.$$

Here is the proof. Set

$$\underline{Z}_n = \left(\frac{N_1 - np_{0,1}}{\sqrt{np_{0,1}}}, \ldots, \frac{N_1 - np_{0,k}}{\sqrt{np_{0,k}}} \right)^T \equiv \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \underline{Y}_i$$

where The $\underline{Y}_i$'s are i.i.d. with $E(\underline{Y}_i) = \underline{0}$, and covariance matrix $\Sigma = I - \sqrt{\underline{p}_0}\sqrt{\underline{p}_0}^T$ where $\sqrt{\underline{p}_0} = (\sqrt{p_{0,1}}, \ldots, \sqrt{p_{0,k}})^T$. Thus

$$\underline{Z}_n \to_d \underline{Z} \sim N_k(0, \Sigma)$$

by the multivariate CLT. Now $Q_n = \underline{Z}_n^T \underline{Z}_n$ is a continuous function of $\underline{Z}_n$, so $Q_n = \underline{Z}_n^T \underline{Z}_n \to_d = \underline{Z}^T \underline{Z} \equiv Q$ by the Mann-Wald theorem. It remains to show that the distribution of Q is χ^2_{k-1}.

To see this, note that for any orthogonal matrix Γ we have

$$Q = \underline{Z}^T \underline{Z} = (\Gamma \underline{Z})^T (\Gamma \underline{Z}) \equiv \underline{V}^T \underline{V}$$

where $\underline{V} \sim N(0, \Gamma \Sigma \Gamma^T)$. Here is a convenient choice of Γ: choose Γ to be the orthogonal matrix with first row $\sqrt{\underline{p}_0}^T$, and filled out with orthogonal rows. Then

$$\begin{aligned} \Gamma \Sigma \Gamma^T &= \Gamma I \Gamma^T - \Gamma \sqrt{\underline{p}_0}\sqrt{\underline{p}_0}^T \Gamma^T = I - (1, 0, \ldots, 0)^T (1, 0, \ldots, 0) \\ &= \begin{pmatrix} 0 & \underline{0}^T \\ \underline{0} & I_{(k-1)\times(k-1)} \end{pmatrix}. \end{aligned}$$

Thus $V_1 = 0$ with probability 1 and $V_2, \ldots, V_k$ are i.i.d. $N(0, 1)$. It follows that $Q = \sum_{j=2}^{k} V_j^2 \sim \chi^2_{k-1}$. Thus we have

$$P_{\underline{p}_0}(Q_n \geq \chi^2_{k-1,\alpha}) \to P(\chi^2_{k-1} \geq \chi^2_{k-1,\alpha}) = \alpha.$$

What if $\underline{p} \neq \underline{p}_0$? In this case, since $\hat{\underline{p}}_n \to_p \underline{p}$, we can write

$$n^{-1} Q_n = \sum_{j=1}^{k} \frac{(\hat{p}_j - p_{0,j})^2}{p_{0,j}} \to_p \sum_{j=1}^{k} \frac{(p_j - p_{0,j})^2}{p_{0,j}} \equiv q > 0.$$

Thus $Q_n \to_p \infty$, and it follows that

$$P_{\underline{p}}(Q_n \ge \chi^2_{k-1,\alpha}) \to 1$$

as $n \to \infty$; i.e. the test is (power) consistent.

What if $\underline{p} = \underline{p}_n$ satisfies $\underline{p}_n = \underline{p}_0 + \underline{c}n^{-1/2}$ where $\underline{1}^T\underline{c} = 0$ and hence $\underline{1}^T\underline{p}_n = 1$ for all n? In this case, by using the Cramér-Wold device together with either the Liapunov CLT or the Lindeberg-Feller CLT,

$$\begin{aligned}
\underline{Z}_n &= \left(\frac{N_1 - np_{0,1}}{\sqrt{np_{0,1}}}, \ldots, \frac{N_1 - np_{0,k}}{\sqrt{np_{0,k}}}\right)^T \\
&= \left(\frac{N_1 - np_{n,1}}{\sqrt{np_{0,1}}}, \ldots, \frac{N_1 - np_{n,k}}{\sqrt{np_{0,k}}}\right)^T + \mathrm{diag}(1/\sqrt{\underline{p}_0})\sqrt{n}(\underline{p}_n - \underline{p}_0) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \underline{Y}_{n,i} + \mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c} \\
&\to_d \underline{Z} + \mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c}
\end{aligned}$$

where $\underline{Z} \sim N_k(0, \Sigma)$ with $\Sigma = I - \sqrt{\underline{p}_0}\sqrt{\underline{p}_0}^T$. Thus it follows by the Mann-Wald theorem that under the local alternatives $\underline{p}_n$ we have

$$\begin{aligned}
Q_n &= \underline{Z}_n^T\underline{Z}_n \to_d (\underline{Z} + \mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c})^T(\underline{Z} + \mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c}) \\
&= \left(\Gamma(\underline{Z} + \mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c})\right)^T \left(\Gamma(\underline{Z} + \mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c})\right) \\
&\equiv \underline{V}^T\underline{V}
\end{aligned}$$

where

$$\underline{V} \sim N_k\left(\Gamma\mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c}, \begin{pmatrix} 0 & \underline{0}^T \\ \underline{0} & I_{(k-1)\times(k-1)} \end{pmatrix}\right) \equiv N_k\left(\underline{\mu}, \begin{pmatrix} 0 & \underline{0}^T \\ \underline{0} & I_{(k-1)\times(k-1)} \end{pmatrix}\right)$$

Noting that $\underline{\mu}^T\underline{\mu} = \underline{c}^T\mathrm{diag}(1/\underline{p}_0)\underline{c} = \sum_{j=1}^k c_j^2/p_{0,j}$ and

$$\mu_1 = \sqrt{\underline{p}_0}^T\mathrm{diag}(1/\sqrt{\underline{p}_0})\underline{c} = \underline{1}^T\underline{c} = 0,$$

it follows that $\underline{V}^T\underline{V} \sim \chi^2_{k-1}(\delta)$ with $\delta = \sum_1^k c_j^2/p_{0,j}$. We conclude that

$$P_{\underline{p}_n}(Q_n \ge \chi^2_{k-1,\alpha}) \to P(\chi^2_{k-1}(\delta) > \chi^2_{k-1,\alpha}).$$

This leads to approximating the power of the chi-square test based on Q_n by

$$P_{\underline{p}}(Q_n \ge \chi^2_{k-1,\alpha}) \approx P(\chi^2_{k-1}(\delta_n) > \chi^2_{k-1,\alpha})$$

with $\delta_n \equiv \sum_{j=1}^k c_{n,j}^2/p_{0,j}$ where $\underline{c}_n \equiv \sqrt{n}(\underline{p} - \underline{p}_0)$.