Statistics 581, Handout
Gamma, Digamma, and Polygamma
Wellner; 11/15/00, revised 11/03/06, 11/05/08, 10/30/14

We begin with the usual Gamma function Γ defined for $\alpha > 0$ by

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$$

Now differentiation across this identity with respect to α and interchanging the derivative and integral on the right side yields, since

$$\frac{\partial}{\partial\alpha} t^\alpha = \frac{\partial}{\partial\alpha} \exp(\alpha \log t) = t^\alpha \log t,$$

$$\Gamma'(\alpha) = \int_0^\infty t^{\alpha-1} (\log t) e^{-t}\, dt.$$

Hence it follows that the *digamma function* ψ, defined as the derivative of $\log \Gamma$, is the expected value of $\log W$ where $W \sim \text{Gamma}(\alpha, 1)$:

$$\begin{aligned}
\psi(\alpha) &\equiv \frac{d}{d\alpha} \log \Gamma(\alpha) = \frac{\Gamma'(\alpha)}{\Gamma(\alpha)} \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (\log t) e^{-t}\, dt = E(\log W)
\end{aligned}$$

where $W \sim \text{Gamma}(\alpha, 1)$. Alternatively, if $Y \sim \text{Exp}(1)$, then

$$\Gamma'(\alpha) = \psi(\alpha)\Gamma(\alpha) = E\{Y^{\alpha-1}(\log Y)\}\,.$$

Differentiation of Γ a second time yields

$$\Gamma''(\alpha) = \int_0^\infty t^{\alpha-1} (\log t)^2 e^{-t}\, dt,$$

and hence

$$\frac{\Gamma''(\alpha)}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} (\log t)^2 e^{-t}\, dt = E[(\log W)^2] = \frac{E[Y^{\alpha-1}(\log Y)^2]}{\Gamma(\alpha)}\,.$$

Thus, computing the derivative of the digamma function ψ, sometimes called the *trigamma function*, we find that $\psi'(\alpha)$ equals $Var(\log W)$ where $W \sim \text{Gamma}(\alpha, 1)$:

$$\begin{aligned}
\psi'(\alpha) &= \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \left(\frac{\Gamma'(\alpha)}{\Gamma(\alpha)}\right)^2 \\
&= \frac{\Gamma''(\alpha)}{\Gamma(\alpha)} - \psi^2(\alpha) \\
&= E[(\log W)^2] - \{E(\log W)\}^2 = \text{Var}(\log W).
\end{aligned}$$

Note that this implies that $\log \Gamma(\alpha)$ is a convex function of α, a result which also follows from Artin's theorem; see e.g. Marshall, Olkin, and Arnold (2011), page 649.

Various special values of these functions are available from e.g. Abramowitz and Stegun (1964) or Mathematica. For example, $\psi(1) = -\gamma$, $\psi(2) = 1 - \gamma$ where γ is Euler's constant, $\gamma = .577216...$; while $\psi'(1) = \pi^2/6$, $\psi'(2) = \pi^2/6 - 1$.

For our Weibull example we need to compute

$$\begin{aligned} E[(Y-1)^2 \log(Y)] &= E[Y^2 \log Y] - 2E[Y \log Y] + E[\log Y] \\ &= \psi(3)\Gamma(3) - 2\psi(2)\Gamma(2) + \psi(1)\Gamma(1) \\ &= 2\psi(3) - 2\psi(2) + \psi(1) \\ &= 2(3/2 - \gamma) - 2(1-\gamma) + (-\gamma) \\ &= 1 - \gamma\,, \end{aligned}$$

and hence $a \equiv -E[(Y-1)^2 \log Y] = -(1-\gamma)$. We can also express

$$b^2 \equiv E\{[1 - (Y-1)\log Y]^2\}$$

in terms of $\psi(\alpha)$, $\psi(\alpha)$, and $\Gamma(\alpha)$ for $\alpha = 1, 2, 3$, but it is perhaps easier to compute this latter integral directly in Mathematica. Here is a bit of Mathematica input / output :

```
In[19]:=
PolyGamma[1]
PolyGamma[2]
PolyGamma[3]
PolyGamma[4]
(* PolyGamma[z]  is the Mathematica name for digamma[z] .   *)

PolyGamma[1, 1]
PolyGamma[1, 2]
PolyGamma[1, 3]
PolyGamma[1, 4]
(* PolyGamma[n, z]  is the Mathematica name for the n -
    th derivative  of digamma[z] .   *)
Integrate[(1 - (x - 1)*Log[x])^2* Exp[-x], {x, 0, Infinity}]
Integrate[(x - 1)^2*Log[x]*Exp[-x], {x, 0, Infinity}]
Out[19]= -EulerGamma
Out[20]= 1 - EulerGamma
Out[21]= (3/2 - EulerGamma)
Out[22]= (11/6 - EulerGamma)
Out[23]= ([Pi]^2/6)
Out[24]= ((-1) + \[Pi]^2/6 )
Out[25]= ((-(5/4)) + [Pi]^2/6)
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Out[26]= ((-(49/36)) + [Pi]^2/6)
Out[27]= (1 - 2\ EulerGamma + EulerGamma^2 + [Pi]^2/6)
Out[28]= 1-EulerGamma
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References:

Abramowitz, M. and Stegun, I. A. (1964). *Handbook of Mathematical Functions.* National Bureau of Standards 55, Washington, D.C.

Marshall, A. W., Olkin, I., and Arnold, B. (2011). *Inequalities: Theory of Majorization and Applications.* Springer, New York.

Wolfram, S. (1996). *The Mathematica Book.* Wolfram Research, Chicago.