Statistics 581, Problem Set 2
Wellner; 10/3/2018

Reading: Course Notes: Chapter 2, pages 1 - 22.
Ferguson, pages 8-35.
van der Vaart, Sections 2.2 - 2.9 (pages 12 - 24).

Due: Wednesday, October 10, 2018.

1. Suppose that $X_1, X_2, \ldots$ is a sequence of random variables such that $X_1 \sim \text{Uniform}(0,1)$, and for $n = 1, 2, \ldots$ the conditional distribution of X_{n+1} given $X_1, \ldots, X_n$ is uniform on $[0, cX_n]$ for a number $c \in (\sqrt{3}, 2)$.
(a) Compute $E(X_n^r)$ for $r > 0$.
(b) Show that X_n converges to 0 in mean, but X_n does not converge to 0 in quadratic mean.
(c) Does $X_n \rightarrow_{a.s.} 0$?

2. Wellner 581 Course Notes, Chapter 1, Exercise 4.1, page 19. (Show just the first equality in each case; we will do the second equalities later.)

3. Ferguson, ACILST, #4, page 6:
(a) Give an example of random variables X_n such that $E|X_n| \rightarrow 0$ and $E|X_n|^2 \rightarrow 1$.
(b) Give an example of a sequence of random variables X_n such that $X_n \rightarrow_p 0$ and $EX_n \rightarrow 0$, but $X_n \rightarrow_{a.s.} 0$ fails.
(c) Suppose that Y has a standard Cauchy distribution with density $f(y) = (\pi(1+y^2))^{-1}$. Find a sequence of random variables Y_n such that $Y_n \rightarrow_2 Y$, but Y_n does not converge to Y almost surely.

4. vdV, *Asymp. Statist.*, problem 5, page 24: Find an example of a sequence (X_n, Y_n) such that $X_n \rightarrow_d X$, $Y_n \rightarrow_d Y$, but (X_n, Y_n) does not converge in distribution.

5. (See vdV, *Asymp. Stat.*, section 11.1, pages 153 - 156.)
Suppose that Y is a random variable with $E(Y^2) < \infty$, let X be another random variable on the same probability space as Y, and consider finding a (measurable) function g of X with $Eg^2(X) < \infty$ so that $E(Y - g(X))^2$ is "small".
(a) Show that
$$\inf_{g:\mathbb{R}\rightarrow\mathbb{R}, Eg^2(X)<\infty} E(Y - g(X))^2 = E(Y - E(Y|X))^2$$
so that the minimizer is exactly $g_0(X) \equiv E(Y|X)$.
(b) Show that $E\{(Y - E(Y|X))g(X)\} = 0$ for all $g(X) \in L_2(P)$.
(c) Interpret the results in (a) and (b) geometrically (i.e. in the Hilbert

space $L_2(P)$ of square integrable random variables with the inner product $\langle X, Y \rangle \equiv E(XY)$.

6. **Optional Bonus Problem 1:**
Suppose that Y is a random variable with $E(Y^2) < \infty$.
(a) Show that

$$Var(Y) = E\{Var(Y|X)\} + Var\{E(Y|X)\};$$

i.e.

$$E(Y - EY)^2 = E\{E[(Y - E(Y|X))^2|X]\} + E\{[E(Y|X) - E(Y)]^2\}.$$

(b) Interpret (a) geometrically.
(c) Suppose that $Y \sim \chi_n^2(\delta)$. Compute $E(Y)$ and $Var(Y)$.
Hint: Use $E(Y) = E\{E(Y|X)\}$ and (a).
(d) Show that

$$\frac{\chi_n^2(\delta) - (n + \delta)}{\sqrt{2n + 4\delta}} \to_d N(0, 1)$$

as either $n \to \infty$ or $\delta \to \infty$.

7. **Optional Bonus Problem 2:** Suppose that $\xi_1, \ldots \xi_n$ are i.i.d. Uniform$(0, 1)$, and let $U_{(0)} \equiv 0 \leq U_{(1)} \leq \cdots \leq U_{(n)} \leq 1 \equiv U_{(n+1)}$ denote the *order statistics* of the X_i's. Let $D_j \equiv X_{(j)} - X_{(j-1)}$ for $j = 1, \ldots, n+1$ denote the *spacings*.
(a) Show that $(D_1, \ldots, D_{n+1}) \stackrel{d}{=} (Y_1, \ldots, Y_{n+1})/S_{n+1}$ where $S_{n+1} \equiv \sum_{j=1}^{n+1} Y_i$ and $Y_1, \ldots, Y_{n+1}$ are i.i.d. exponential(1) random variables.
(b) Show that for any fixed $k \geq 1$ we have

$$(nD_1, \ldots, nD_k) \to_d (Y_1, \ldots, Y_k).$$

(c) Find the joint density of $(U_{(1)}, U_{(2)})$.
(d) Using the density you found in (c), show that the joint density of $n(U_{(1)}, U_{(2)})$ converges pointwise to the joint density of $(Y_1, Y_1 + Y_2)$ for two independent exponential random variables Y_1 and Y_2.

8. **Optional Bonus Problem 3:** Wellner 581 Course Notes, Chapter 1, Exercise 3.2, page 16.