Statistics 581, Problem Set 3

Wellner; 10/10/2018

Reading: Course Notes: Chapter 2, pages 15 - 30;
Ferguson ACILST pages 26-65;
Van der Vaart Asymp. Statistics, Chapter 3, pages 12 - 34.
Due: Wednesday, October 17, 2018.

1. Ferguson, ACILST, page 34, problem 1(a) (modified slightly):
   Suppose that $X_1, X_2, \ldots$ are i.i.d. in R^2 with distribution giving probability θ_1 to $(1,0)'$, probability θ_2 to $(0,1)'$, θ_3 to $(-1,0)'$ and θ_4 to $(0,-1)'$ where $\theta_j \ge 0$ for $j = 1,2,3,4$ and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 1$.
   (a) Find $\mu = E(X_1)$.
   (b) Compute $E(X_1 X_1^T)$ and $\Sigma = E(X_1 - \mu)(X_1 - \mu)^T$.
   (c) Find the limiting distribution of $\sqrt{n}(\overline{X}_n - \mu)$ and describe the resulting approximation to the distribution of $\overline{X}_n$.
   (d) Find values of $(\theta_1, \ldots, \theta_4)$ such that Σ has rank 1 and $\det(\Sigma) = 0$.

2. (Van der Vaart, page 24)
   (a) Suppose that X_n and Y_n are independent random vectors with $X_n \to_d X$ and $Y_n \to_d Y$. Show that $(X_n, Y_n) \to_d (X, Y)$ where X and Y are independent.
   (b) Suppose that $P(X_n = i/n) = 1/n$ for $i = 1, 2, \ldots, n$. show that $X_n \to_d X \sim$ Uniform$(0,1)$.
   (c) Consider the X_n's as in (b). Show that there exists a Borel set B such that $P(X_n \in B) = 1$ but $P(X \in B) = 0$. In particular, with $P_n = \mathcal{L}(X_n)$ and $P = \mathcal{L}(X)$, $d_{TV}(P_n, P) = 1$ for each n.

3. ACILST, page 34, problem 5: Suppose that $X_1, X_2, \ldots$ are i.i.d. random variables with mean μ and variance $\sigma^2 < \infty$. Let $T_n = \sum_{j=1}^n z_{nj} X_j$ where the $\{z_{nj}\}_{j=1}^n$ are given numbers. Let $\mu_n = E(T_n)$ and $\sigma_n^2 = Var(T_n)$. Use the Lindeberg-Feller central limit theorem to show that $(T_n - \mu_n)/\sigma_n \to_d Z \sim N(0,1)$ if $\max_{1 \le j \le n} z_{nj}^2 / \sum_{j=1}^n z_{nj}^2 \to 0$ as $n \to \infty$.

4. (a) ACILST, problem 4, page 49: Let $X_1, \ldots, X_n$ be a sample of size n from the beta distribution Beta $(\theta, 1)$ with $\theta > 0$. Show that the method of moments estimate of θ is $\hat{\theta}_n = \overline{X}_n/(1 - \overline{X}_n)$.
   (b) Find the asymptotic distribution of $\hat{\theta}_n$.
   (c) Is $\hat{\theta}_n$ asymptotically linear? If so, find the influence function of $\hat{\theta}_n$.
   (d) Find the Cramér-Rao lower bound for estimation of θ and compare it to the asymptotic variance you found in (b).

5. Suppose that $X_1, \ldots, X_n$ are i.i.d. with $E(X_1) = \mu$, $Var(X_1) = \sigma^2 < \infty$, and $E|X_1|^6 < \infty$. Let $M_{j,n} \equiv n^{-1} \sum_{i=1}^n (X_i - \overline{X}_n)^j$ be the $j-$th sample central moment for $j \in \{2, 3, \ldots\}$.
   (a) ACILST, page 49, problem 3.
   (b) Find the joint asymptotic distribution of $\sqrt{n}((\overline{X}_n, M_{2,n}, M_{3,n})^T - (\mu, \sigma^2, m_3)^T)$ where $m_3 \equiv E(X_1 - \mu)^3$ is the population 3rd central moment.

(c) Find the asymptotic distribution of $\sqrt{n}(\kappa_{3,n} - \kappa_3)$ where $\kappa_{3,n} \equiv M_{3,n}/M_{2,n}^{3/2}$ is the sample skewness and $\kappa_3 \equiv m_3/\sigma^3$ is the population skewness. (See vdV Example 3.5, page 29.)

6. **Optional bonus problem 1.**
   (a) Statistics 581 Course Notes, Exercise 1.5, page 12.
   (b) Statistics 581 Course Notes, Exercise 1.6, page 12.
   (c) Statistics 581 Course Notes, Exercise 1.7, page 12.

7. **Optional bonus problem 2.**
   vdV, *Asymp. Statist.*, problem 4, page 34: Find the limit distribution of the sample kurtosis $k_n = n^{-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)^4/S^4 - 3$ and design an asymptotic level α test of normality based on k_n. (Note Aad's warning about the sample size required to make the normal approximation work.)