**Statistics 581, Problem Set 6**

Wellner; 10/31/2018

**Reminder:** Midterm exam: Friday, November 2.
**Reading:** Lecture Notes Chapter 3, sections 1-2;
Ferguson, ACILST, chapters 19-20, pages 126 - 139;
vdVaart, Asym. Stat., sections 8.1-8.3, pages 108-112.
**Due:** Wednesday, November 7, 2018.

1. A. Compute and plot the score for location $-f'(x)/f(x)$ when:
   (a) $f = \phi$, the standard normal density;
   (b) $f(x) = \exp(-x)/(1+\exp(-x))^2$ (logistic);
   (c) $f(x) = (1/2)\exp(-|x|)$ (double exponential);
   (d) $f(x) = t_k$, the $t-$density with $k-$degrees of freedom;
   (e) $f(x) = \exp(-x)\exp(-\exp(-x))$;
   (f) $f(x) = 2\phi(x)\Phi(ax)$ where $\Phi(x)$ is the standard normal distribution function and $a > 0$;
   (g) $f(x) = 1/(\pi(1+x^2))$, the standard Cauchy density.
   B. A density $f$ is called *log-concave* if $\log f$ is a concave function. Let $s < 0$. A density $f$ is called *$s-$concave* if $f^s$ is convex. Which of the densities in (a) - (f) are log-concave? Which of the densities in (a) - (f) are $s-$concave for some $s < 0$? Which of the densities in (a) - (f) are symmetric about 0?

2. Suppose that $Z \sim N(0,1)$ and, for $\mu \in R$ and $\sigma > 0$, that $X = \mu + \sigma Z \sim P_{\mu,\sigma} = N(\mu, \sigma^2)$.
   (a) Compute the likelihood ratio
$$\frac{dP_{\mu,\sigma}}{dP_{0,\sigma}}(x) = \frac{\sigma^{-1}\phi((x-\mu)/\sigma)}{\sigma^{-1}\phi(x/\sigma)} \qquad \text{and} \qquad Y \equiv \log \frac{dP_{\mu,\sigma}}{dP_{0,\sigma}}(X)\,.$$
   What is the distribution of $Y$ under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$?
   (b) Plot the function
$$l(\mu; X) \equiv \log \frac{dP_{\mu,\sigma}}{dP_{0,\sigma}}(X)$$
   as a function of $\mu$.
   (c) Find the maximum value of the function $l(\mu; X)$ in (b) (as a function of $\mu$) and the value of $\mu \equiv \hat{\mu}$ which achieves the maximum.
   (d) What is the distribution of $\hat{\mu}$ under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$? What is the distribution of $l(\hat{\mu}; X)$ under $P_{0,\sigma}$ and under $P_{\mu,\sigma}$?

3. Suppose that $X, X_1, X_2, \ldots, X_n$ are independent Exponential($\lambda$) random variables:
$$P(X \ge x) = \exp(-\lambda x), \qquad x > 0.$$
   (a) Show that the $r$-th moment of $X$, $\mu_r \equiv \mu_r(\lambda)$ is given by
$$\mu_r(\lambda) = EX^r = \frac{\Gamma(r+1)}{\lambda^r}.$$

(b) Use the moment calculation in (a) to show that

$$\frac{\mu_r(\lambda)}{\mu_{r+1}(\lambda)} = \frac{\lambda}{r+1}$$

and hence that the family of estimators $\{\hat{\lambda}_n^{(k)}\}_{k\ge 0}$ given by

$$\hat{\lambda}_n^{(k)} \equiv (k+1)\frac{\overline{X^k}_n}{\overline{X^{k+1}}_n} \equiv (k+1)\frac{n^{-1}\sum_1^n X_i^k}{n^{-1}\sum_1^n X_i^{k+1}}$$

are all consistent estimators of $\lambda$: $\hat{\lambda}_n^{(k)} \to_p \lambda$ for each $k = 0, 1, 2, \ldots$.
(c) Show that

$$\sqrt{n}(\hat{\lambda}_n^{(k)} - \lambda) \to_d N(0, \sigma_k^2(\lambda)) \quad \text{as} \quad n \to \infty$$

and compute $\sigma_k^2(\lambda)$ explicitly as a function of $k$ and $\lambda$.
(d) What is the asymptotic relative efficiency of $\hat{\lambda}_n^{(k)}$ to $\widehat{\lambda}_n \equiv \hat{\lambda}_n^{(0)} = 1/\overline{X}_n$ for $k > 1$?
(e) Now suppose that $X, X_1, \ldots, X_n$ are i.i.d. with distribution function $F$ on $(0, \infty)$ where $F$ is not an exponential distribution function. Specify hypotheses on $F$ (or $X$) which guarantee that $\hat{\lambda}_n^{(k)} \to_p$ some natural parameter, say $\lambda_k(F)$ defined in terms of $F$. What hypothesis will be needed to guarantee that $\sqrt{n}(\hat{\lambda}_n^{(k)} - \lambda_k(F)) \to_d N(0, V^2)$ for some $V^2$?

4. **Optional bonus problem 1:** Ferguson, ACILST, problem 6, page 93, plus the following:
(d) Construct a family of estimators $\tilde{\theta}_n$ of $\theta$ based on the sample quantile function $\mathbb{F}_n^{-1}(t)$. Show that your estimators are consistent and asymptotically normal. Give a formula for the asymptotic variance of your estimators.

5. **Optional bonus problem 2:** Consider a function $T : \mathcal{F} \to \mathbb{R}$ where $\mathcal{F}$ is some (sub) class of distribution functions $F$ (examples include the mean, $T(F) = \mu(F) = \int x dF(x)$, the variance $T(F) = \sigma^2(F) = \int (x - \int y dF(y))^2 dF(x)$, the median $T(F) = F^{-1}(1/2)$, linear combinations of order statistics $T(F) = \int_0^1 F^{-1}(u)w(u)du$, the *mean residual life function* at $x > 0$ $T(F) \equiv e(x, F) \equiv \int_{(x,\infty)}(1 - F(u))du/(1 - F(x)) = E(X - x|X > x)$, and so forth). [The mean residual life function gives the mean life conditional on surviving beyond $x$.] The "principle of substitution" says that $T(F)$ can be estimated by $T(\widehat{F}_n)$ for some estimator $\widehat{F}_n$ of $F$. If $T$ is sufficiently "smooth", then frequently the empirical distribution function $\mathbb{F}_n$ can be taken as the estimator $\hat{F}_n$ of $F$.

Give a treatment of consistency and asymptotic normality of the estimator $e(x, \mathbb{F}_n)$ of $e(x, F)$ based on our results from Sections 2.4 and 2.6. You may assume that with $X \sim F$ on $(0, \infty)$ we have $E_F X < \infty$, $E_F X^2 < \infty$, and $1 - F(x) > 0$ (as well as any other additional assumptions you need). What is the joint limiting distribution of $\sqrt{n}(e(x, \mathbb{F}_n) - e(x, F), e(y, \mathbb{F}_n) - e(y, F))$ for $0 < x < y < \infty$?