Statistics 581, Problem Set 8 Solutions

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1. (a) Show that if $\theta_n = cn^{-1/2}$ and T_n is the Hodges super-efficient estimator discussed in class, then the sequence $\{\sqrt{n}(T_n - \theta_n)\}$ is uniformly square-integrable.
(b) Let $R_n(\theta) \equiv nE_\theta(T_n - \theta)^2$ where T_n is the Hodges superefficient estimator as in Example 3.3.1 (so $T_n = \delta_n$ of Example 2.5, Lehmann and Casella pages 440 - 443). Show that $R_n(n^{-1/4}) \to \infty$ as $n \to \infty$.

Solution: (a) First recall that (with $\delta_n = T_n$) since $\sqrt{n}(\overline{X} - \theta) \stackrel{d}{=} Z \sim N(0,1)$ we can write

$$\begin{aligned}
\sqrt{n}(T_n - \theta) &= \sqrt{n}(\overline{X}_n 1_{[|\overline{X}_n| > n^{-1/4}]} + a\overline{X}_n 1_{[|\overline{X}_n| \le n^{-1/4}]} - \theta) \\
&\stackrel{d}{=} Z 1_{[|Z + \theta\sqrt{n}| > n^{1/4}]} + [aZ + \sqrt{n}\theta(a-1)] 1_{[|Z+\theta\sqrt{n}| \le n^{1/4}]} \\
&= Z + [(a-1)Z + (a-1)\sqrt{n}\theta] 1_{[|Z+\theta\sqrt{n}| \le n^{1/4}]} \\
&= Z - (1-a)[Z + \sqrt{n}\theta] 1_{[|Z+\theta\sqrt{n}| \le n^{1/4}]}.
\end{aligned}$$

Thus (as we showed in class) when $\theta_n = cn^{-1/2}$ we have

$$\begin{aligned}
\sqrt{n}(T_n - \theta_n) &\stackrel{d}{=} Z 1_{[|Z+c| > n^{1/4}]} + [aZ + c(a-1)] 1_{[|Z+c| \le n^{1/4}]} \\
&= Z + [(a-1)Z + (a-1)c] 1_{[|Z+c| \le n^{1/4}]} \\
&= Z - (1-a)[Z + c] 1_{[|Z+c| \le n^{1/4}]}. \qquad (1)
\end{aligned}$$

Thus

$$\begin{aligned}
Y_n &\equiv \left\{\sqrt{n}(T_n - \theta_n)\right\}^2 \\
&\stackrel{d}{=} \left\{Z - (1-a)[Z+c] 1_{[|Z+c| \le n^{1/4}]}\right\}^2 \\
&\le 2\left(Z^2 + (1-a)^2 (Z+c)^2\right) \equiv Y
\end{aligned}$$

where

$$E(Y) = 2\left(E(Z^2) + (1-a)^2 E(Z+c)^2\right) < \infty.$$

Thus

$$\limsup_{n\to\infty} E\{Y_n 1_{[Y_n \ge \lambda]}\} \le E\{Y 1_{[Y \ge \lambda]}\} \to 0$$

as $\lambda \to \infty$. Hence $\{Y_n\}$ is uniformly integrable; that is, $\{\sqrt{n}(T_n - \theta_n)\}$ is uniformly square-integrable.

(b) (a') Note that the identity (1) in (a) above holds. Thus

$$\begin{aligned}
b_n(\theta) &= E_\theta(T_n) - \theta \\
&= n^{-1/2}\left\{EZ - (1-a)E[Z + \sqrt{n}\theta] 1_{[|Z+\theta\sqrt{n}| \le n^{1/4}]}\right\} \\
&= -\frac{1-a}{\sqrt{n}} E[Z + \sqrt{n}\theta] 1_{[|Z+\theta\sqrt{n}| \le n^{1/4}]} \\
&= -\frac{1-a}{\sqrt{n}} \int_{-n^{1/4}}^{n^{1/4}} x\phi(x - \sqrt{n}\theta)dx
\end{aligned}$$

since $Z + \theta\sqrt{n} \sim N(\theta\sqrt{n}, 1)$.
(b') Differentiating the result in (a') gives

$$\begin{aligned} b_n'(\theta) &= -\frac{1-a}{\sqrt{n}} \int_{-n^{1/4}}^{n^{1/4}} x\phi'(x - \sqrt{n}\theta)(-\sqrt{n})\ dx \\ &= -(1-a) \int_{-n^{1/4}}^{n^{1/4}} x(x-\sqrt{n}\theta)\phi(x-\sqrt{n}\theta)\ dx \qquad \text{since} \quad \phi'(x) = -x\phi(x) \\ &\to 0 \qquad \text{if} \quad \theta \neq 0 \end{aligned}$$

by the dominated convergence theorem since $x(x-\sqrt{n}\theta)\phi(x-\sqrt{n}\theta)1_{[-n^{1/4},n^{1/4}]}(x) \to 0$ for each fixed x and is dominated by the integrable function $4e^{-1}\phi(x)/(|\theta| \wedge 1)$ (for $n \geq (3/|\theta|)^4$).
Details of this domination: For $|x| \leq n^{1/4}$ it follows that

$$|x||x - \sqrt{n}\theta| \leq n^{1/4}| - n^{1/4} - \sqrt{n}\theta| \leq n^{1/2} + n^{3/4}|\theta| \leq 2n^{3/4}(|\theta| \vee 1)$$

while

$$\begin{aligned} \phi(x - \sqrt{n}\theta) &= \phi(x)\exp(\sqrt{n}\theta x - n\theta^2/2) \\ &\leq \phi(x)\exp(|\theta|n^{3/4} - n\theta^2/2) \\ &= \phi(x)\exp(|\theta|n^{3/4}(1 - n^{1/4}|\theta|/2)) \\ &\leq \phi(x)\exp(-\frac{1}{2}|\theta|n^{3/4}) \qquad \text{if} \quad 1 - n^{1/4}|\theta|/2 < -1/2 \end{aligned}$$

or, equivalently, when $n > (3/|\theta|)^4$. Combining these two bounds yields

$$\begin{aligned} |x||x-\sqrt{n}\theta|\phi(x-\sqrt{n}\theta) &\leq \phi(x)n^{3/4}2(|\theta| \vee 1)\exp(-|\theta|n^{3/4}/2) \\ &= \phi(x)\begin{cases} 2n^{3/4}\exp(-|\theta|n^{3/4}/2) & \text{if} \quad |\theta| < 1 \\ 2n^{3/4}|\theta|\exp(-|\theta|n^{3/4}/2) & \text{if} \quad |\theta| \geq 1 \end{cases} \\ &= \phi(x)\begin{cases} (4/|\theta|)(n^{3/4}|\theta|/2)\exp(-|\theta|n^{3/4}/2) & \text{if} \quad |\theta| < 1 \\ 4(n^{3/4}|\theta|/2)\exp(-|\theta|n^{3/4}/2) & \text{if} \quad |\theta| \geq 1 \end{cases} \\ &\leq \frac{4e^{-1}}{|\theta| \wedge 1}\phi(x). \end{aligned}$$

(b), Second (more elegant) solution: from the lecture notes, 3.3 (3), it follows that

$$R_n(\theta) = E[n(T_n - \theta)^2] = nVar[T_n] + nb_n(\theta)^2 \geq a^2 + nb_n(\theta)^2.$$

Using the formula for $b_n(\theta)$ from part (a) above, it follows that it is enough to show that

$$\left| \int_{-n^{1/4}}^{n^{1/4}} x\phi(x - n^{1/4})dx \right| \to \infty.$$

But we have, with $Z \sim N(0,1)$ (and hence $E|Z| < \infty$),

$$\left| \int_{-n^{1/4}}^{n^{1/4}} x\phi(x - n^{1/4})dx \right| = \left| \int_{-2n^{1/4}}^{0} (y + n^{1/4})\phi(y)dy \right|$$

$$
\begin{aligned}
&\geq \left| n^{1/4} \int_{-2n^{1/4}}^{0} \phi(y)dy \right| - \left| \int_{-2n^{1/4}}^{0} y\phi(y)dy \right| \\
&\geq n^{1/4}(\Phi(0) - \Phi(-2n^{1/4})) - E|Z| \\
&\rightarrow \infty.
\end{aligned}
$$

2. (Super-efficiency at two parameter values) Suppose that $X_1, \ldots, X_n$ are i.i.d. $N(\theta, 1)$ where $\theta \in \mathbb{R}$) Let $a, b \in [0, 1)$ and define the estimator T_n as follows:

$$
T_n = \begin{cases} \overline{X}_n & \text{if } |\overline{X}_n - 1| > n^{-1/4} \text{ and } |\overline{X}_n + 1| > n^{-1/4}, \\ a\overline{X}_n + (1-a) & \text{if } |\overline{X}_n - 1| \leq n^{-1/4}, \\ b\overline{X}_n + (1-b)(-1) & \text{if } |\overline{X}_n + 1| \leq n^{-1/4}. \end{cases}
$$

(a) Find the limiting distribution of $\sqrt{n}(T_n - \theta)$ when:
 (i) $\theta \neq 1$ and $\theta \neq -1$; (ii) $\theta = 1$; (iii) $\theta = -1$.
(b) Find the limiting distribution of $\sqrt{n}(T_n - \theta_n)$ when:
 (i) $\theta_n = 1 + cn^{-1/2}$; (ii) $\theta_n = -1 + cn^{-1/2}$.
(c) Could we have super-efficiency at a countable collection of parameter values?

Solution: (a) Note that $\sqrt{n}(\overline{X}_n - \theta) \stackrel{d}{=} Z \sim N(0, 1)$ for all $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. Thus we find that

$$
\begin{aligned}
\sqrt{n}(T_n - \theta) &= \sqrt{n}(\overline{X}_n - \theta) 1_{[|\overline{X}_n - 1| > n^{1/4}]} \cdot 1_{[|\overline{X}_n + 1| > n^{1/4}]} \\
&\quad + \sqrt{n}(a\overline{X}_n + (1-a) - \theta) 1_{[|\overline{X}_n - 1| \leq n^{1/4}]} \\
&\quad + \sqrt{n}(a\overline{X}_n - (1-b) - \theta) 1_{[|\overline{X}_n + 1| \leq n^{1/4}]} \\
&\stackrel{d}{=} Z \cdot 1_{[\sqrt{n}|\overline{X}_n - \theta + \theta - 1| > n^{1/4}]} 1_{[\sqrt{n}|\overline{X}_n - \theta + \theta + 1| > n^{1/4}]} \\
&\quad + \left\{ a\sqrt{n}(\overline{X}_n - \theta) + \sqrt{n}(a\theta - \theta + (1-a)) \right\} 1_{[|\sqrt{n}(\overline{X}_n - \theta) + \sqrt{n}(\theta - 1)| \leq n^{1/4}]} \\
&\quad + \left\{ b\sqrt{n}(\overline{X}_n - \theta) + \sqrt{n}(b\theta - \theta - (1-b)) \right\} 1_{[|\sqrt{n}(\overline{X}_n - \theta) + \sqrt{n}(\theta + 1)| \leq n^{1/4}]} \\
&\rightarrow_d \left\{ \begin{array}{ll} Z & \text{if } \theta \neq 1, \ \theta \neq -1, \\ aZ & \text{if } \theta = 1, \\ bZ & \text{if } \theta = -1, \end{array} \right\} \\
&\sim N(0, V^2(\theta))
\end{aligned}
$$

where

$$
V^2(\theta) = 1_{\{-1,1\}}(\theta) + a^2 1_{\{1\}}(\theta) + b^2 1_{\{-1\}}(\theta).
$$

(b) If $\theta = \theta_n = 1 + cn^{1/2}$,

$$
\begin{aligned}
\sqrt{n}(T_n - \theta_n) &\stackrel{d}{=} Z 1_{[|Z+c| > n^{1/4}]} + (aZ + c(a-1)) 1_{[|Z+c| \leq n^{1/4}]} + o_p(1) \\
&\rightarrow_d aZ + c(a-1) \sim N(c(a-1), a^2).
\end{aligned}
$$

In the same way, if $\theta = \theta_n = -1 + cn^{1/2}$, we find that

$$
\sqrt{n}(T_n - \theta_n) \rightarrow_d bZ + c(b-1) \sim N(c(b-1), b^2).
$$

(c) A similar construction works to yield superefficiency at all $\theta \in \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$.

3. Suppose that $X_1, \ldots, X_n$ are i.i.d. with distribution function F having a continuous density function f. Let $\mathbb{F}_n$ be the empirical distribution function of the X_i's, suppose that b_n is a sequence of positive numbers, and let

$$\hat{f}_n(x) = \frac{\mathbb{F}_n(x+b_n) - \mathbb{F}_n(x-b_n)}{2b_n}.$$

(a) Compute $E\{\hat{f}_n(x)\}$ and $Var(\hat{f}_n(x))$.
(b) Show that $E\hat{f}_n(x) \to f(x)$ if $b_n \to 0$.
(c) Show that $Var(\hat{f}_n(x)) \to 0$ if $b_n \to 0$ and $nb_n \to \infty$.
(d) Use some appropriate central limit theorem to show that (perhaps under some suitable further conditions that you might need to specify)

$$\sqrt{2nb_n}(\hat{f}_n(x) - E\hat{f}_n(x)) \to_d N(0, f(x)).$$

Hint: Write $\hat{f}_n(x)$ in terms of some Bernoulli random variables and identify $p = p_n$.

Solution: (a) First note that $2nb_n = n(\mathbb{F}_n(x+b_n) - \mathbb{F}_n(x-b_n))$ is a Binomial(n, p_n) random variable with $p_n = F(x+b_n) - F(x-b_n)$. Hence if $b_n \to 0$

$$\begin{aligned} E\widehat{f}_n(x) &= \frac{F(x+b_n) - F(x-b_n)}{2b_n} = \frac{p_n}{2nb_n} \\ &= \frac{1}{2}\left\{\frac{F(x+b_n) - F(x)}{b_n} + \frac{F(x) - F(x-b_n)}{b_n}\right\} \\ &\to \frac{1}{2}\{f(x) + f(x)\} = f(x). \end{aligned}$$

(b) Furthermore

$$\begin{aligned} Var(\widehat{f}_n(x)) &= \frac{np_n(1-p_n)}{(2nb_n)^2} \\ &= \frac{1}{2nb_n}\frac{p_n}{2b_n}(1-p_n) \\ &\to 0 \cdot f(x) \cdot 1 = 0 \end{aligned}$$

if $nb_n \to \infty$ and $b_n \to 0$.
(c) Since $2nb_n\widehat{f}_n(x) = \sum_{i=1}^n X_{ni}$ where $X_{ni} \sim$ Bernoulli(p_n), it follows that $\sigma_{ni}^2 = p_n(1-p_n)$ so that $\sigma_n^2 = \text{Var}(\sum_{i=1}^n X_{ni}) = np_n(1-p_n)$, and

$$\begin{aligned} \gamma_n \equiv \sum_{i=1}^n \gamma_{ni} &= \sum_{i=1}^n E|X_{ni} - \mu_{ni}|^3 \\ &= np_n(1-p_n)\{(1-p_n)^2 + p_n^2\} \\ &\le 2np_n(1-p_n) \end{aligned}$$

so that

$$\gamma_n/\sigma^3 \le \frac{2}{\sqrt{np_n(1-p_n)}} = \frac{2}{\sqrt{nb_n(p_n/b_n)(1-p_n)}} \to 0$$

if $b_n \to 0$ and $nb_n \to \infty$. Thus, by the Liapunov CLT,

$$\frac{2nb_n(\widehat{f}_n(x) - E\widehat{f}(x))}{\sqrt{np_n(1-p_n)}} \to N(0,1)$$

if $b_n \to 0$ and $nb_n \to \infty$. Thus

$$\begin{aligned} \sqrt{2nb_n}(\widehat{f}_n(x) - E\widehat{f}_n(x)) &= \frac{2nb_n(\widehat{f}_n(x) - E\widehat{f}_n(x))}{\sqrt{np_n(1-p_n)}}\sqrt{\frac{np_n(1-p_n)}{2nb_n}} \\ &\to N(0,1)\sqrt{f(x)} = N(0, f(x)). \end{aligned}$$

4. Suppose that $(T|Z) \sim$ Weibull$(\lambda^{-1}e^{-\gamma Z}, \beta)$, and $Z \sim G_\eta$ on R with density g_η with respect to some dominating measure μ. Thus the conditional cumulative hazard function $\Lambda(t|z)$ is given by

$$\Lambda_{\gamma,\lambda,\beta}(t|z) = (\lambda e^{\gamma Z} t)^\beta = \lambda^\beta e^{\beta\gamma Z} t^\beta$$

and hence

$$\lambda_{\gamma,\lambda,\beta}(t|z) = \lambda^\beta e^{\beta\gamma Z} \beta t^{\beta-1} .$$

(Recall that $\lambda(t) = f(t)/(1-F(t))$ and

$$\Lambda(t) \equiv \int_0^t \lambda(s)ds = \int_0^t (1-F(s))^{-1} dF(s) = -\log(1-F(t))$$

if F is continuous.) Thus it makes sense to re-parametrize by defining $\theta_1 \equiv \beta\gamma$ (this is the parameter of interest since it reflects the effect of the covariate Z), $\theta_2 \equiv \lambda^\beta$, and $\theta_3 \equiv \beta$. This yields

$$\lambda_\theta(t|z) = \theta_3\theta_2 \exp(\theta_1 z) t^{\theta_3 - 1}$$

You may assume that

$$a(z) \equiv (\partial/\partial\eta) \log g_\eta(z)$$

exists and $E\{a^2(Z)\} < \infty$. Thus Z is a "covariate" or "predictor variable", θ_1 is a "regression parameter" which affects the intensity of the (conditionally) Weibull variable T, and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ where $\theta_4 \equiv \eta$.
(a) Derive the joint density $p_\theta(t, z)$ of (T, Z) for the re-parametrized model.
(b) Find the information matrix for θ. What does the structure of this matrix say about the effect of $\eta = \theta_4$ being known or unknown about the estimation of $\theta_1, \theta_2, \theta_3$?
(c) Find the information and information bound for θ_1 if the parameters θ_2 and θ_3 are known.
(d) What is the information bound for θ_1 if just θ_3 is known to be equal to 1?
(e) Find the efficient score function and the efficient influence function for estimation of θ_1 when θ_3 is known.
(f) Find the information $I_{11\cdot(2,3)}$ and information bound for θ_1 if the parameters θ_2 and θ_3 are unknown. (Here both θ_2 and θ_3 are in "the second block".)
(g) Find the efficient score function and the efficient influence function for estimation

of θ_1 when θ_2 and θ_3 are unknown.
(h) Specialize the calculations in (d) - (g) to the case when $Z \sim$ Bernoulli(θ_4) and compare the information bounds.

Solution: (a) Integrating $\lambda_\theta(t|z)$ with respect to t gives

$$\Lambda_\theta(t|z) = \theta_2 \exp(\theta_1 z) t^{\theta_3} ,$$

and hence the conditional survival function $1 - F_\theta(t|z)$ is given by

$$1 - F_\theta(t|z) = \exp(-\Lambda_\theta(t|z)) = \exp(-\theta_2 \exp(\theta_1 z) t^{\theta_3}) . \tag{2}$$

It follows that

$$f_\theta(t|z) = \theta_2\theta_3 e^{\theta_1 z} t^{\theta_3 - 1} \exp(-\theta_2 e^{\theta_1 z} t^{\theta_3}) ,$$

and hence that

$$\begin{aligned} p_\theta(y,z) &= f_\theta(y|z) g_\eta(z) = \theta_2\theta_3 e^{\theta_1 z} t^{\theta_3 - 1} \exp(-\theta_2 e^{\theta_1 z} t^{\theta_3}) g_\eta(z) \\ &= = \theta_2\theta_3 e^{\theta_1 z} t^{\theta_3 - 1} \exp(-\theta_2 e^{\theta_1 z} t^{\theta_3}) g_{\theta_4}(z) . \end{aligned}$$

(b) We first calculate the scores for θ. Note that the random variable $W \equiv \theta_2 \exp(\theta_1 Z) Y^{\theta_3}$ has, conditionally on Z, a standard Exponential(1) distribution:

$$P_\theta(W > w|Z) = P_\theta(\theta_2 \exp(\theta_1 Z) Y^{\theta_3} > w|Z) = e^{-w}$$

by (2). We calculate

$$\begin{aligned} l(\theta|Y,Z) &= \log p_\theta(Y,Z) \\ &= \log\theta_2 + \log\theta_3 + \theta_1 Z + (\theta_3 - 1)\log Y - \theta_2 e^{\theta_1 Z} Y^{\theta_3} + \log g_{\theta_4}(Z) , \\ \dot{\mathrm{l}}_1(Y,Z) &= Z - Z\theta_2 e^{\theta_1 Z} Y^{\theta_3} = Z(1 - W) , \\ \dot{\mathrm{l}}_2(Y,Z) &= \frac{1}{\theta_2} - \frac{\theta_2 e^{\theta_1 Z} Y^{\theta_3}}{\theta_2} = \frac{1}{\theta_2}(1 - W) , \\ \dot{\mathrm{l}}_3(Y,Z) &= \frac{1}{\theta_3} + \log Y - \theta_2 e^{\theta_1 Z} Y^{\theta_3} \log Y \\ &= \frac{1}{\theta_3} + \log Y \{1 - \theta_2 e^{\theta_1 Z} Y^{\theta_3}\} \\ &= \frac{1}{\theta_3}\left\{1 + \log\frac{\theta_2 e^{\theta_1 Z} Y^{\theta_3}}{\theta_2 e^{\theta_1 Z}}\{1 - W\}\right\} \\ &= \frac{1}{\theta_3}\{1 + \{\log W - \log(\theta_2 e^{\theta_1 Z})\}\{1 - W\}\} \\ &= \frac{1}{\theta_3}\{[1 - (W-1)\log W] + (W-1)\log(\theta_2 e^{\theta_1 Z})\} \\ \dot{\mathrm{l}}_4(Y,Z) &= a(Z) = a(Z,\eta) . \end{aligned}$$

Moreover,

$$\begin{aligned} \ddot{\mathrm{l}}_{13}(Y,Z) &= -Z\theta_2 e^{\theta_1 Z} Y^{\theta_3} \log Y = -Z\frac{1}{\theta_3}\theta_2 e^{\theta_1 Z} Y^{\theta_3} \log\left(\frac{\theta_2 e^{\theta_1 Z} Y^{\theta_3}}{\theta_2 e^{\theta_1 Z}}\right) \\ &= -\frac{Z}{\theta_3} W\{\log W - \log(\theta_2 e^{\theta_1 Z})\} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{z}{\theta_3} W\{\log W - \log(\theta_2) - \theta_1 Z\} \\
\ddot{\mathbf{l}}_{23}(Y,Z) &= -e^{\theta_1 Z} Y^{\theta_3} \log Y = -\frac{1}{\theta_2\theta_3}\theta_2 e^{\theta_1 Z} Y^{\theta_3} \log\left(\frac{\theta_2 e^{\theta_1 Z} Y^{\theta_3}}{\theta_2 e^{\theta_1 Z}}\right) \\
&= -\frac{1}{\theta_2\theta_3} W\{\log W - \log(\theta_2 e^{\theta_1 Z})\} \\
&= -\frac{1}{\theta_2\theta_3} W\{\log W - \log(\theta_2) - \theta_1 Z\}\,, \\
\ddot{\mathbf{l}}_{33}(Y,Z) &= -\frac{1}{\theta_3^2}\{1 + W[\log W - \log(\theta_2 e^{\theta_1 Z}]^2\}\,.
\end{aligned}
$$

Thus we calculate easily:

$$
\begin{aligned}
I_{11}(\theta) &= E_\theta(\dot{\mathbf{l}}_1(Y,Z)^2) = E_\theta\{E[Z^2(1-W)^2|Z]\} \\
&= E\{Z^2 E[(1-W)^2|Z]\} = E(Z^2)\,, \\
I_{22}(\theta) &= E_\theta(\dot{\mathbf{l}}_2(Y,Z)^2) = E_\theta\{E[\theta_2^{-2}(1-W)^2|Z]\} = \theta_2^{-2}\,, \\
I_{33}(\theta) &= \theta_3^{-2}\left\{1 + E[W(\log W)^2] - 2E(W\log W)\{\log\theta_2 + \theta_1 E(Z)\}\right. \\
&\qquad \left. + E\{(\log\theta_2 + \theta_1 Z)^2\}\right\} \\
&= \theta_3^{-2}\left\{1 + B^2 - 2A\{\log\theta_2 + \theta_1 E(Z)\} + E\{(\log\theta_2 + \theta_1 Z)^2\}\right\} \\
I_{12}(\theta) &= E_\theta(\dot{\mathbf{l}}_1(Y,Z)\dot{\mathbf{l}}_2(Y,Z)) = E_\theta\{E[Z\theta_2^{-1}(1-W)^2|Z]\} = \theta_2^{-1}E(Z)\,, \\
I_{13}(\theta) &= -E_\theta\{\ddot{\mathbf{l}}_{13}(Y,Z)\} \\
&= \theta_3^{-1}\{E(Z)[A - \log\theta_2] - \theta_1 E(Z^2)\}\,, \\
I_{23}(\theta) &= -E_\theta\{\ddot{\mathbf{l}}_{23}(Y,Z)\} \\
&= (\theta_2\theta_3)^{-1}\{A - \log\theta_2 - \theta_1 E(Z)\}
\end{aligned}
$$

where

$$
\begin{aligned}
A &\equiv E\{W\log W\} = \int_0^\infty (w\log w)\exp(-w)dw = 1-\gamma\,, \\
B^2 &\equiv E\{W(\log W)^2\} = \pi^2/6 + (1-\gamma)^2 - 1\,.
\end{aligned}
$$

Note that since $\dot{\mathbf{l}}_4(y,z) = a(z)$ is just a function of Z, it follows easily that for $j = 1,2,3$ we also have

$$
\begin{aligned}
I_{j4}(\theta) &= E_\theta\{\dot{\mathbf{l}}_j(Y,Z)\dot{\mathbf{l}}_4(Y,Z)\} \\
&= E\{g_j(W,Z)a(Z)\} = E\{E[g_j(W,Z)a(Z)|Z]\} \\
&= E\{a(Z)E[g_j(W,Z)|Z]\} = E\{a(Z)\cdot 0\} = 0\,,
\end{aligned}
$$

Because of this orthogonality, the information bounds for $(\theta_1, \theta_2, \theta_3)$ are the same when $\theta_4 = \eta$ is unknown as when it is known.
(c) If θ_2 and θ_3 are known, then the information bound for estimation of θ_1 is just $I_{11}^{-1}(\theta) = 1/E(Z^2)$. It follows that the information matrix for θ is of the following form:

$$
I(\theta) = \begin{pmatrix}
E(Z^2) & \theta_2^{-1}E(Z) & \theta_3^{-1}C & 0 \\
\theta_2^{-1}E(Z) & \theta_2^{-2} & (\theta_2\theta_3)^{-1}D & 0 \\
\theta_3^{-1}C & (\theta_2\theta_3)^{-1}D & \theta_3^{-2}E & 0 \\
0 & 0 & 0 & Ea^2(Z)
\end{pmatrix}
$$

where

$$\begin{aligned} C &= E(Z)(A - \log\theta_2) - \theta_1 E(Z^2) \\ D &= A - \log\theta_2 - \theta_1 E(Z) \\ E &= 1 + B^2 - 2A(\log\theta_2 + \theta_1 E(Z)) + E(\log\theta_2 + \theta_1 Z)^2 \,. \end{aligned}$$

(d) If $\theta_3 = 1$ is known, then the information bound for θ_1 is $I_{11\cdot 2}^{-1}$ where

$$\begin{aligned} I_{11\cdot 2}(\theta) &= I_{11} - I_{12} I_{22}^{-1} I_{21} \\ &= E(Z^2) - (E(Z)/\theta_2)^2 \theta_2^2 = E(Z^2) - (EZ)^2 = Var(Z) \,. \end{aligned}$$

Thus $I_{11\cdot 2}^{-1} = 1/Var(Z)$.
(e) When θ_3 is known, the efficient score function and the efficient influence function for estimation of θ_1 are given by

$$\begin{aligned} \dot{\mathrm{l}}_1^*(Y, Z) &= \dot{\mathrm{l}}_1 - I_{12} I_{22}^{-1} \dot{\mathrm{l}}_2 \\ &= Z(1 - W) - \theta_2^{-1} E(Z) \theta_2^2 \frac{1}{\theta_2}(1 - W) \\ &= Z(1 - W) - E(Z)(1 - W) = (Z - E(Z))(1 - W) \,, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mathrm{l}}_1(Y, Z) &= I_{11\cdot 2}^{-1} \dot{\mathrm{l}}_1^*(Y, Z) \\ &= \frac{1}{Var(Z)}(Z - E(Z))(1 - W) \,. \end{aligned}$$

(f) When both the parameters θ_2 and θ_3 are unknown, the information $I_{11\cdot(2,3)}$ is given by

$$\begin{aligned} I_{1\cdot(2,3)} &\equiv I_{11\cdot 2} \qquad \text{where the "second block" contains both } \theta_2, \theta_3 \\ &= I_{11} - I_{12} I_{22}^{-1} I_{21} \end{aligned} \tag{3}$$

where

$$\begin{aligned} I_{12} &= (\theta_2^{-1} E(Z), \theta_3^{-1} C) \,, \\ I_{22}^{-1} &= \begin{pmatrix} \theta_2^2 E & -\theta_2\theta_3 D \\ -\theta_2\theta_3 D & \theta_3^2 \end{pmatrix} \frac{1}{E - D^2} \,. \end{aligned}$$

Thus the second term in (3) is

$$\left\{[E(Z)]^2 E - 2E(Z)CD + C^2\right\} / (E - D^2) \,. \tag{4}$$

Now the denominator is

$$\begin{aligned} E - D^2 &= 1 + B^2 - 2A(\log\theta_2 + \theta_1 E(Z)) + E(\log\theta_2 + \theta_1 Z)^2 \\ &\qquad - (A - \log\theta_2 - \theta_1 E(Z))^2 \\ &= 1 + B^2 - 2A(\log\theta_2 + \theta_1 E(Z)) + E(\log\theta_2 + \theta_1 Z)^2 \\ &\qquad - [A^2 - 2A(\log\theta_2 + \theta_1 E(Z)) + (\log\theta_2 + \theta_1 E(Z))^2 \\ &= 1 + B^2 - A^2 + Var[\log\theta_2 + \theta_1 Z] \\ &= \pi^2/6 + \theta_1^2 Var(Z) \,, \end{aligned}$$

and, upon noting that

$$\begin{aligned} C - E(Z)D &= E(Z)(A - \log\theta_2) - \theta_1 E(Z^2) - \{E(Z)(A - \log\theta_2) - \theta_1 [E(Z)]^2\} \\ &= -\theta_1 Var(Z)\,, \end{aligned}$$

it follows that the numerator of (4) is

$$\begin{aligned} C^2 - 2E(Z)CD + [E(Z)]^2 E &= C^2 - 2E(Z)CD + [E(Z)]^2 D^2 + [E(Z)]^2 (E - D^2) \\ &= (C - E(Z)D)^2 + [E(Z)]^2 \{\pi^2/6 + \theta_1^2 Var(Z)\} \\ &= \theta_1^2 [Var(Z)]^2 + [E(Z)]^2 \{\pi^2/6 + \theta_1^2 Var(Z)\}\,. \end{aligned}$$

It follows that the information for θ_1 when θ_2 and θ_3 are unknown is equal to

$$\begin{aligned} I_{11\cdot(2,3)} &= E(Z^2) - \frac{[E(Z)]^2 \{\pi^2/6 + \theta_1^2 Var(Z)\}}{\pi^2/6 + \theta_1^2 Var(Z)} \\ &= \frac{\pi^2/6}{\pi^2/6 + \theta_1^2 Var(Z)} Var(Z) \le Var(Z) \le E(Z^2) \end{aligned}$$

with equality in the first inequality if and only if $\theta_1 = 0$. Note that the information decreases as θ_1 increases, and it converges to $\pi^2/(6\theta_1^2)$ as $Var(Z) \to \infty$.
(g) When θ_2 and θ_3 are unknown the efficient score function for θ_1 is, with the "second block" containing both θ_2 and θ_3,

$$\begin{aligned} \mathbf{l}_1^* &= \dot{\mathbf{l}}_1 - I_{12} I_{22}^{-1} \dot{\mathbf{l}}_2 \\ &= \dot{\mathbf{l}}_1 - (\theta_2(E(Z)E - CD), \theta_3(C - DE(Z)) \dot{\mathbf{l}}_2 / (E - D^2) \\ &= Z(1 - W) - \frac{E(Z)E - CD}{E - D^2}(1 - W) \\ &\quad + \frac{\theta_1 Var(Z)}{\pi^2/6 + \theta_1^2 Var(Z)} \left\{[1 - (W - 1)\log W] + (W - 1)\log(\theta_2 e^{\theta_1 Z})\right\} \\ &= \left\{ Z - \frac{E(Z)E - CD + \log(\theta_2 e^{\theta_1 Z})}{\pi^2/6 + \theta_1^2 Var(Z)} \right\} (1 - W) \\ &\quad + \frac{\theta_1^2 Var(Z)}{\pi^2/6 + \theta_1^2 Var(Z)} \{1 - (W - 1)\log W\}\,. \end{aligned}$$

(h) When $Z \sim$ Bernoulli(η), then

$$\begin{aligned} &I_{11} = E(Z^2) = \eta = \theta_4\,, \\ &I_{11\cdot 2} = Var(Z) = \eta(1 - \eta) = \theta_4(1 - \theta_4)\,, \\ &I_{11\cdot(2,3)} = \frac{\pi^2/6}{\pi^2/6 + \theta_1^2 Var(Z)} Var(Z) \\ &\qquad\quad = \frac{\pi^2/6}{\pi^2/6 + \theta_1^2 \eta(1 - \eta)} \eta(1 - \eta)\,. \end{aligned}$$

The corresponding information bounds are given by the reciprocals of these quantities. See the following figures for comparisons of the information and information bounds.

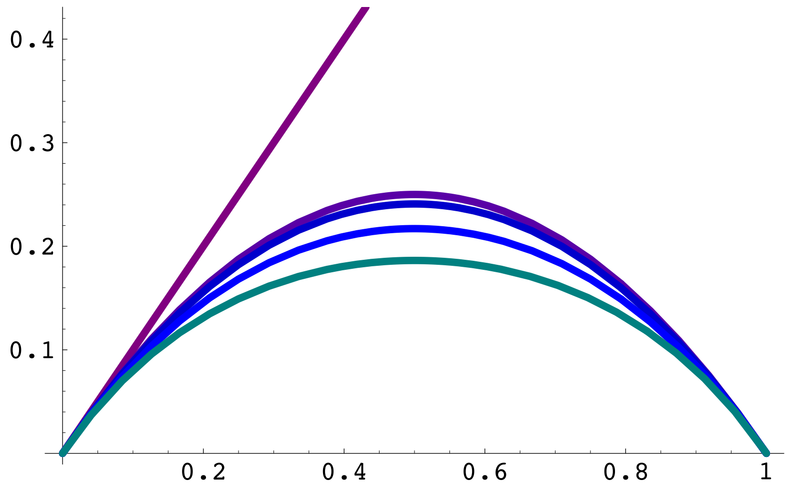

Figure 1: Plots of I_{11}, $I_{11\cdot 2}$, and $I_{11\cdot(2,3)}$ as a function of $\eta = \theta_4$, and for $\theta_1 = .5,\ 1.0, 1.5$

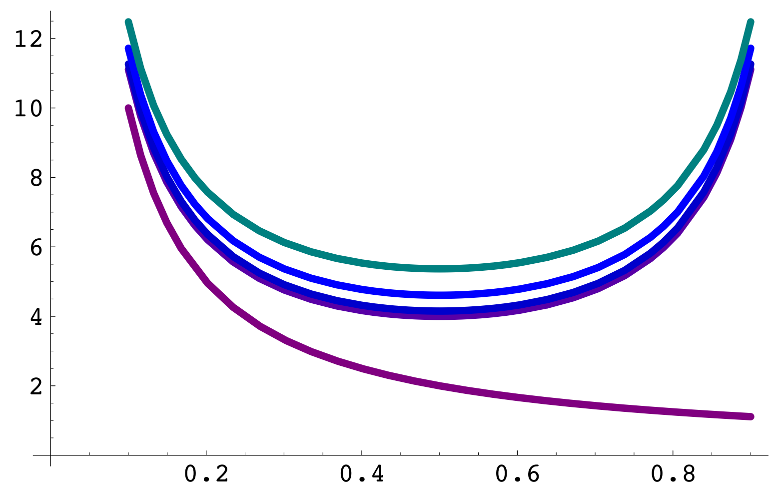

Figure 2: Plots of I_{11}^{-1}, $I_{11\cdot 2}^{-1}$, and $I_{11\cdot(2,3)}^{-1}$ as a function of $\eta = \theta_4$, and for $\theta_1 = .5,\ 1.0, 1.5$