Statistics 591B, Problem Set 1

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Due: Thursday, October 19, 2017

1. BLM, page 47, problem 2.7: Prove that if $Z \sim N(0,1)$ is a centered normal random variable with variance 1, then

$$\frac{z}{1+z^2}\phi(z) \le P(Z \ge z) \le \left\{ \begin{array}{l} \frac{1}{2}\exp\left(-\frac{z^2}{2}\right), \\ z^{-1}\phi(z) \end{array} \right.$$

for all $z \ge 0$.

2. BLM, page 47, problem 2.7: Elementary inequalities:

$$\begin{aligned} -\log(1-u) - u &\le \frac{u^2}{2(1-u)} \quad \text{for} \quad u \in (0,1); \\ \overline{h}(u) = (1+u)\log(1+u) - u &\ge \frac{u^2}{2(1+u/3)} \quad \text{for} \quad u > 0; \\ h_1(u) = 1 + u - \sqrt{1+2u} &\ge \frac{u^2}{2(1+u)}, \quad \text{for} \quad u > 0. \end{aligned}$$

3. Verify the elementary calculation required to show that

$$\psi^*(t) \equiv \sup_{0<\lambda<1/c}\left(t\lambda - \frac{\nu\lambda^2}{2(1-c\lambda)}\right) = \frac{\nu}{c^2}h_1\left(\frac{ct}{\nu}\right)$$

where $h_1(u) = 1 + u - \sqrt{1+2u}$, $u > 0$.

4. Let (T, d) be a pseudo-metric space. Show that

$$N(\epsilon, T, d) \le D(\epsilon, T, d) \le N(\epsilon/2, T, d).$$

5. Let $Z_1, Z_2, \ldots$ be i.i.d. $N(0,1)$ random variables, and define $X_n \equiv Z_n/\sqrt{1+\log n}$ for $n \ge 1$. Show that $\{X_n : \ n \ge 1\}$ is almost surely bounded and $E\|X\| \equiv E\sup_{n\ge 1}|X_n| < \infty$, but (with $T = \mathbb{N}^+ \equiv \{1,2,3,\ldots\}$ and $d^2(n,m) \equiv E(X_n - X_m)^2$)

$$\int_0^{\text{diam}(T)} \sqrt{\log N(\epsilon, T, d)}\, d\epsilon = \infty.$$

Hint: Let $\sigma_n^2 \equiv 1/(1+\log n)$. Show that $P(|X_n| \ge \lambda) \le \exp(-\lambda^2/2\sigma_n^2) = n^{-\lambda^2/2}\exp(-\lambda^2/2)$, and hence find a bound for $P(\sup_{n\ge 1}|X_n| > \lambda)$ for $\lambda \ge 2$.