Statistics 591B, Problem Set 2

Wellner; 10/19/2017

Due: Thursday, November 2, 2017

1. One remaining basic inequality comparison: verify that $\overline{h}(x) \equiv h(1+x)$ and $h_1(x) \equiv 1 + x - \sqrt{1+2x}$ satisfy
$$\overline{h}(x) \ge 9 h_1(x/3) \text{ for all } \quad x \ge 0.$$

2. (a) The moment condition for (the extended form of) Bernstein's inequality is: $E|X_i|^k \le k! c^{k-2} v_i/2$ for every $k \ge 2$ and all $i \le n$ and constants $c > 0$ and v_i. Show that this holds if we have
$$E\left(e^{|X_i|/c} - 1 - \frac{|X_i|}{c}\right) c^2 \le \frac{1}{2} v_i .$$
On the other hand, show that if the moment condition holds, then the previous display holds with c replaced by $2c$ and v_i replaced by $2v_i$.
(b) Show that the moment condition of Bernstein's inequality holds with $c = M/3$ if $|X_i| \le M$ with probability 1.

3. (a) We showed in class on 10/17/2017 that if $Y \sim \text{Poisson}(\nu)$, then
$$P(Y - \nu \ge t) \le \exp(-\nu h(1 + t/\nu)) = \exp\left(-\frac{t^2}{2\nu}\psi(t/\nu)\right) \quad \text{for all } t > 0$$
where $h(x) = x(\log x - 1) + 1$ and $\psi(x) = 2x^{-2} h(1+x)$. Show that
$$\begin{aligned} P(-(Y-\nu) \ge t) &\le \exp(-\nu h(1 - t/\nu)) = \exp\left(-\frac{t^2}{2\nu}\psi(-t/\nu)\right) \\ &\le \exp\left(-\frac{t^2}{2\nu}\right) \quad \text{for all } 0 < t \le \nu. \end{aligned}$$
(b) Now suppose that $X_1, \ldots, X_n$ are independent non-negative random variables. Let $S_n \equiv \sum_{i=1}^n (X_i - E(X_i))$, and let $\nu \equiv \sum_{i=1}^n E(X_i^2)$. Show that
$$P(-S \ge t) \le \exp\left(-\frac{t^2}{2\nu}\right) \quad \text{for all } \quad t > 0.$$
Can this be improved?

4. Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables. In the notation used in class on 10/17/2017, let $\psi_{X_1}(\lambda) \equiv \log E e^{\lambda X_1}$ and let $\psi^*_{X_1}(t) = \sup_\lambda \{\lambda t - \psi_{X_1}(\lambda)$. Let $Z \equiv \sum_1^n X_i$. Show that $\psi_Z(\lambda) = n\psi_{X_1}(\lambda)$ and $\psi^*_Z(t) = n\psi^*_{X_1}(t/n)$.

5. Suppose that $Z = \sum_{i=1}^n (X_i - p)$ where $X_i \sim \text{Bern}(p)$ for $1 \le i \le n$.
(a) Use the previous problem to show that $P(Z \ge t) \le \exp(-nh_p(p + t/n))$ for all $t > 0$ where

$$h_p(a) = a \log\left(\frac{a}{p}\right) + (1-a)\log\left(\frac{1-a}{1-p}\right).$$

(b) The log-MGF of $Z_1 = X_1 - p$ is $\psi_{Z_1}(\lambda) = \log(pe^\lambda + (1-p)) - \lambda p$. Use the inequality $\log(1+w) \le w$ for all $w > -1$ to find an alternative inequality to the one you found in (a). The alternative inequality should involve the function $\overline{h}$ or h as in Bennett's inequality.