Statistics 592C, Problem Set 1

Wellner; 1/6/99

Reading:	VdV&W, Sections 2.1 - 2.2, pages 80 - 106.
Due:	Wednesday, January 13, 1999.

1. Correct line -2, page 95, Van der Vaart and Wellner (1996): show that if $p \le q$, then
$$\|X\|_{\psi_p} \le (\log 2)^{1/q-1/p} \|X\|_{\psi_q}\,.$$

Solution: This is essentially Problem 2.2.5, page 105. For me, the difficult part of this is showing that the function ϕ defined by
$$\phi(y) = 2^{(\log(1+y)/\log 2)^{p/q}} - 1$$
is concave. To show this, first note that
$$\phi(y) = e^{a(\log(1+y))^b} - 1$$
where $b = p/q < 1$ and $a = (\log 2)^{1-p/q} < 1$. Then direct calculation yields
$$\phi'(y) = e^{a(\log(1+y))^b}\left\{ ab(\log(1+y))^{b-1}\frac{1}{1+y}\right\}$$
and
$$\begin{aligned}\phi''(y) &= e^{a(\log(1+y))^b}\frac{ab}{(1+y)^2}(\log(1+y))^{b-2} \\ &\qquad \left\{(\log(1+y))^b ab + (b-1) - \log(1+y)\right\}\,.\end{aligned}$$
To show that ϕ is concave it now suffices to show that $\phi''(y) \le 0$, or equivalently, letting $v \equiv \log(1+y)$, that
$$g(v) \equiv abv^b - v + (b-1) \le 0$$
for all $0 \le v < \infty$. For $v \ge 1$, $v^b \le v$, and hence
$$\begin{aligned} g(v) &\le abv - v + (b-1) \\ &= v(ab-1) + (b-1) < 0\,. \end{aligned}$$

For $v \in [0,1]$ we compute

$$g'(v) = ab^2 v^{b-1} - 1; \qquad g''(v) = ab^2(b-1)v^{b-2} < 0.$$

Hence g is maximized at $v_0 = (ab^2)^{1/(1-b)}$ and the maximum value is

$$\begin{aligned} g(v_0) &= ab(ab^2)^{b/(1-b)} - (ab^2)^{1/(1-b)} + b - 1 \\ &= (b-1)\{1 - a^{1/(1-b)} b^{(1+b)/b}\} < 0\,. \end{aligned}$$

Thus $g(v) < 0$ for all $v \in [0, \infty)$ and ϕ is concave.

2. Show that the constant in line -1, page 95 of VdV and W can be taken to be $\Gamma(p+1)^{1/p}$.

   Solution: Let $p! \equiv \Gamma(p+1)$ for $p > 0$. Since $\psi_1(x) = e^x - 1 \geq x^p/p!$ for any $p \geq 1$ and $x \geq 0$, it follows that $x^p \leq p!\psi_1(x)$. Hence

$$\begin{aligned} E\left\{\frac{|X|}{(p!)^{1/p}||X||_{\psi_1}}\right\}^p &= E\left\{\frac{|X|^p}{(p!)||X||_{\psi_1}^p}\right\} \\ &= \frac{1}{p!} E\left\{\frac{|X|}{||X||_{\psi_1}}\right\}^p \\ &\leq \frac{1}{p!} \cdot p! \cdot E\psi_1\left(\frac{|X|}{||X||_{\psi_1}}\right) = 1\,, \end{aligned}$$

   and this yields

$$E|X|^p \leq (p!)||X||_{\psi_1}^p = \Gamma(p+1)||X||_{\psi_1}^p$$

   or $||X||_p \leq \Gamma(p+1)^{1/p}||X||_{\psi_1}$.

3. Problem 2.3.3 page 120, VdV and W.

   Solution: (i) Let $\epsilon > 0$; then

$$\begin{aligned} E\{n^{-1/2} \max_{1 \leq i \leq n} |X_{ni}|\} &= \int_0^\infty P(\max_{i \leq n} |X_{ni}| > t\sqrt{n})dt \\ &= \int_0^\epsilon P(\quad)dt + \int_\epsilon^\infty P(\quad)dt \end{aligned}$$

$$
\begin{aligned}
&\le \epsilon + \int_\epsilon^\infty n^{-1} \sum_1^n n t^2 P(|X_{ni}| > t\sqrt{n}) t^{-2} dt \\
&\le \epsilon + \sup_{x \ge \epsilon\sqrt{n}} \frac{1}{n} \sum_1^n x^2 P(|X_{ni}| > x) \int_\epsilon^\infty t^{-2} dt \\
&\le \epsilon + \frac{1}{\epsilon} \sup_{x \ge \epsilon\sqrt{n}} \frac{1}{n} \sum_1^n x^2 P(|X_{ni}| > x) \\
&\to \epsilon
\end{aligned}
$$

as $n \to \infty$ by the hypothesis. Since $\epsilon > 0$ is arbitrary, this implies that

$$E\{n^{-1/2} \max_{1 \le i \le n} |X_{ni}|\} \to 0 \,.$$

(ii) Let $\epsilon > 0$. Then

$$
\begin{aligned}
P(\max_{1 \le i \le n} |X_{ni}| \le \epsilon\sqrt{n}) &= \prod_{i=1}^n P(|X_{ni}| \le \epsilon\sqrt{n}) \\
&= \prod_{i=1}^n (1 - P(|X_{ni}| > \epsilon\sqrt{n})) \\
&= (1 - P(|X_{n1}| > \epsilon\sqrt{n}))^n
\end{aligned}
$$

since the X_{ni}'s are row-wise i.i.d. By the condition that $nP(|X_{ni}| > \epsilon\sqrt{n}) \to 0$ we find that the last line is

$$\left(1 - \frac{nP(|X_{n1}| > \epsilon\sqrt{n}}{n}\right)^n \to e^0 = 1.$$

Hence we have $n^{-1/2} \max_{i \le n} |X_{ni}| \to_p 0$.
Conversely, if $n^{-1/2} \max_{i \le n} |X_{ni}| \to_p 0$, it follows from (1) that

$$
\begin{aligned}
P(|X_{n1}| > \epsilon\sqrt{n}) &= (1 - (1 - P(\max_{i \le n} |X_{ni}| > \epsilon\sqrt{n}))^{1/n} \\
&= 1 - (1 - o(1))^{1/n} = o(1/n) \,.
\end{aligned}
$$

4. Problem 2.3.4, page 120, VdV and W.

Solution: (i) First I'll show that (a) implies (b). Let $\epsilon > 0$. Then

$$\begin{aligned}
P(n^{-1}\max_{1\le i\le n}|\xi_{ni}| > \epsilon) &= 1 - P(\max_{1\le i\le n}|\xi_{ni}| \le n\epsilon) \\
&= 1 - P(|\xi_1| \le n\epsilon)^n \\
&= 1 - (1 - P(|\xi_1| > n\epsilon))^n \\
&= 1 - \left(1 - \frac{n\epsilon P(|\xi_1| > n\epsilon)/\epsilon}{n}\right)^n \\
&\to 1 - e^{-0} = 1 - 1 = 0
\end{aligned}$$

by (a). Thus (b) holds.
Now (b) implies (a): if (b) holds, then by (*), we have

$$0 = 1 - \exp(-\lim_{n\to\infty} nP(|\xi_1| > n\epsilon)),$$

and hence $nP(|\xi_1| > n\epsilon)) \to 0$ for every $\epsilon > 0$, and in particular for $\epsilon = 1$. Since

$$xP(|\xi_1| > x) \le [[x]]P(|\xi_1| > [x]) = \frac{[[x]]}{[x]} \cdot [x]P(|\xi_1| > [x])$$

where $[[x]]$ ≡smallest integer $> x$ and $[x]$ ≡largest integer $\le x$, this implies (a).
To show that (a) implies (c), write

$$\begin{aligned}
E\frac{1}{n^r}\max_{1\le i\le n}|\xi_i|^r &= \int_0^\infty P(\max_{1\le i\le n}|\xi_i|^r > tn^r)dt \\
&\le \int_0^\epsilon nP(|\xi_1| > t^{1/r}n)dt + \int_\epsilon^\infty nP(|\xi_1| > t^{1/r}n)dt \\
&\equiv A_n + B_n\,.
\end{aligned}$$

Now by changing variables, and using (a)

$$\begin{aligned}
B_n &= \int_{n\epsilon}^\infty x^{1/r}n^{1-1/r}P(|\xi_1| > x^{1/r}n^{1-1/r})x^{-1/r}dx \cdot n^{-(1-1/r)} \\
&\le C\int_{n\epsilon}^\infty x^{-1/r}dx \cdot n^{-(1-1/r)} \\
&= C\epsilon^{1-1/r}
\end{aligned}$$

But for any fixed ϵ the condition (a) implies that $A_n \to 0$ as $n \to \infty$ by the dominated convergence theorem. Hence by letting $n \to \infty$ and then $\epsilon \to \infty$, it follows that (c) holds. .sp .2 Finally, (c) implies (b) by Markov's inequality. This completes the proof for (i).
(ii) I'll first show that (a) is equivalent to (b). Let $\epsilon > 0$. Now $E|\xi_1| < \infty$ if and only if $E|\xi_1|/\epsilon < \infty$ if and only if $\sum_1^\infty P(|\xi_1| > \epsilon n) < \infty$, and since the ξ_i's are i.i.d., this holds if and only if $\sum_1^\infty P(|\xi_n| > \epsilon n) < \infty$, which, by Borel-Cantelli is equivalent to $|\xi_n|/n \to_{a.s.} 0$. But

$$\begin{aligned} \frac{1}{n}\max_{1\le i\le n}|\xi_i| &\le \frac{1}{n}\max_{1\le i\le n_0}|xi_i| + \frac{1}{n}\max_{n_0\le i\le n}|\xi_i| \\ &\le \frac{1}{n}\max_{1\le i\le n_0}|xi_i| + \sup_{n_0\le i<\infty}\frac{|\xi_i|}{i} \\ &\to_{a.s.} 0 \end{aligned}$$

by first letting n_0 be large to get the second term small (using $|\xi_n|/n \to_{a.s.} 0$), and then letting n be large to make the first term small. Hence (a) implies (b). Conversely, if (b) holds, then $|\xi_n|/n \to_{a.s.} 0$, and this implies that (a) holds by the equivalence argued above. Thus (a) and (b) are equivalent.

To see that (b) implies (c), note that since (b) implies (a) we know that $E|\xi_1| < \infty$; then

$$M_n \equiv \frac{1}{n}\max_{1\le i\le n}|\xi_i| \le \frac{1}{n}\sum_1^n |\xi_i| \equiv Y_n$$

where $Y_n \to_{a.s.} E|\xi_1| < \infty$ and $EY_n = E|\xi_1|$. Since $Y_n \ge 0$, these together imply that Y_n is uniformly integrable; see e.g. Billingsley (1968), Theorem 5.4, page 32. But since $M_n \le Y_n$, this implies that M_n is also uniformly integrable. Since $M_n \to_{a.s.} 0$ by (b), this implies that $E(M_n) \to E(0) = 0$; i.e. (c) holds.
Finally, (c) implies (a) almost trivially:

$$\frac{1}{n}E|\xi_1| \le \frac{1}{n}E\max_{1\le i\le n}|\xi_{ni}| \to 0\,,$$

so in particular the right side is finite, and hence so is $E|\xi_1|$.
Here is another proof that (a) implies (c):

$$EM_n \;=\; \int_0^\infty \frac{1}{n}P(\max_{1\le i\le n}|\xi_i| > t)dt$$

where the integrand converges to 0 for each fixed t and is bounded by

$$\frac{1}{n}\sum_{i=1}^{n} P(|\xi_i| > t) = P(|\xi_1| > t)$$

with $\int_0^\infty P(|\xi_1| > t)dt = E|\xi_1| < \infty$. Hence we conclude that $EM_n \to 0$ by the dominated convergence theorem.
(iii) (a) Note that $P(|\xi_1| > x) = o(x^{-r})$ if and only if $tP(|\xi_1|^r > t) = o(1)$. Hence by part (i) (with $|\xi_1|$ replaced by $|\xi_1|^r$), this condition is equivalent to

$$n^{-1} \max_{1\le i\le n} |\xi_i|^r \to_p 0\,,$$

and to

$$E\left\{ n^{-1} \max_{1\le i\le n} |\xi_i|^r \right\}^{1/r} \to 0\,.$$

(b) This follows immediately from part (ii) with $|\xi_1|$ replaced by $|\xi_1|^r$.

5. Show that if X is a random variable which is weak - L_2 (i.e. $x^2P(|X| > x) \to 0$ as $x \to \infty$), then $E|X|^r < \infty$ for all $r < 2$.

   Solution: For any $0 < r < 2$ we have

$$\begin{aligned} E|X|^r &= \int_0^\infty rx^{r-1}P(|X| > x)dx \\ &\le \tau^r + \int_\tau^\infty rx^{r-3}x^2P(|X| > x)dx \\ &\le \tau^r + \sup_{x\ge\tau} x^2P(|X| > x)\int_\tau^\infty rx^{r-3}dx \\ &\le \tau^r + \frac{r}{2-r}\tau^{r-2} < \infty \end{aligned}$$

   by choosing τ so large that $\sup_{x\ge\tau} x^2P(|X| > x) \le 1$.