Statistics 593A, Problem Set 4

Wellner; 5/15/2014

Due: Thursday, May 29, 2014

1. BLM, page 336, problem 11.1: Show that it is not necessarily true that the variance of the supremum of an empirical process is upper bounded by the wimpy variance. *Hint:* consider $n = 1$, $\mathcal{T} = \{1, 2\}$, and binary-valued random variables.

2. BLM, page 337, problem 11.8: (Nemirovski's inequality in the non-symmetric case) Using the notation of Theorem 11.2, prove that even if the X_i's are not assumed to be symmetric but centered,

$$E\|S_n\|_\infty^2 \le 2(1 + 4\log(2d))\Sigma^2.$$

3. (a) Suppose that $X_1, \ldots, X_n$ are i.i.d. P on $\mathbb{R}$ with $E|X_1| < \infty$. Consider the measure of dispersion $D_n \equiv n^{-1}\sum_{i=1}^n |X_i - \overline{X}_n|$ Use a Glivenko-Cantelli theorem to show that $D_n \rightarrow_{a.s.} E|X_1 - \mu| \equiv d$ where $\mu = E(X_1)$.
(b) Now suppose that the X_i's and D_n are as defined in (a), but now assume that $E(X_1^2) < \infty$. Use a Donsker theorem to show that $\sqrt{n}(D_n - d) \rightarrow_d$ something and identify "something". *Hint:* This is from Pollard (1989), *Statistical Science* **4**, 341-366.

4. BLM, page 212, problem 6.11: (A log-Sobolev inequality for the exponential distribution) Assume that X is exponentially distributed; i.e. X has density $\exp(-x)$ for $x > 0$. Prove that if $f : [0, \infty) \rightarrow \mathbb{R}$ is differentiable, then

$$\text{Ent}(f(X)^2) \le 4E\{x(f'(X))^2\}.$$

Hint: Use the fact that if X_1 and X_2 are independent $N(0, 1)$ random variables, $(X_1^2 + X_2^2)2$ is exponentially distributed, and use the Gaussian log-Sobolev inequality.

5. **Bonus problem:** BLM, page 213, problem 6.12 (Square root of a Poisson random variable). Let X be a Poisson random variable (with parameter ν). Prove that for $0 \le \lambda < 1/2$

$$\log E\exp(\lambda(\sqrt{X} - E\sqrt{X})) \le \frac{\lambda^2}{1 - 2\lambda}.$$