*STATISTICS AND RELATED TOPICS*
*M. Csörgö, D.A. Dawson, J.N.K. Rao, A.K.Md.E. Saleh (eds.)*


# MEAN RESIDUAL LIFE

W. J. Hall and Jon A. Wellner

Statistics Department
University of Rochester
Rochester, New York
U.S.A.

The mean residual life function $e$ at age $x$ is defined to be the expected remaining life given survival to age $x$; it is a function of interest in actuarial studies, survivorship analysis, and reliability. Here we characterize those functions which can arise as mean residual life functions, present and study an "inversion formula" which expresses the survival function in terms of $e$, and collect a variety of facts about $e$ and other residual moments: inequalities for $e$, new characterizations of the exponential distribution, inequalities for coefficients of variation, and limiting behavior of $e$ at 'great age'. We also discuss applications to parametric modelling.

## 1. INTRODUCTION

Let $X$ be a non-negative random variable with right continuous distribution function (df) $F$, and survival function $\overline{F} = 1 - F$, on $\mathbb{R}^+$ and suppose that $F(0) = 0$ and $\mu \equiv E(X) = \int_0^\infty x dF(x) = \int_0^\infty \overline{F}(x)dx < \infty$; write $T \equiv T_F \equiv \inf\{x: F(x) = 1\} \leq \infty$. The <u>mean residual life</u> (MRL) function or remaining life expectancy function at age $x$ is defined as

$$e(x) = e_F(x) \equiv E(X-x|X>x) = \int_x^\infty \overline{F}dI/\overline{F}(x), \qquad \text{for } x \geq 0, \tag{1.1}$$

and $e(x) \equiv 0$ whenever $\overline{F}(x) = 0$. We use I to denote the identity function and Lebesgue measure on $\mathbb{R}^+$.

The discretized version of the MRL function $e$ has had considerable use in life table analysis (see e.g. Chiang, 1968, pages 189 and 213-214; Gross and Clark, 1975, page 25ff), and estimation of $e = e_F$ on the basis of samples from $F$ has

<u>AMS 1970 Subject classifications</u>: Primary 62P05, 62N05, 62E10; Secondary 62G99
<u>Key words and phrases</u>: life expectancy, characterizations, residual variance, coefficient of variation, new better than used in expectation, renewal theory, regular variation, inequalities.
Research of the second author supported by National Science Foundation Grant MCS 77-02255.

recently been considered by Yang (1978) and Hall and Wellner (1980). Here we are concerned with the behavior of $e$ as a function (of $x$ and $F$). For example, what functions $e$ can arise as MRL functions? (What is the image or range $E$ of the set $F$ of all df's with finite mean under the function $F \to e_F$?) Does the mean residual life function $e_F$ determine the df $F$ completely? (Is the function $F \to e_F$ one-to-one?) These questions are answered in Section 3, along with a brief review of the related literature.

The remainder of the paper presents a number of related properties of $e$, and of other residual moments, and applications thereof. Among them are elementary inequalities for $e$ (Section 2); some characterizations of the exponential distribution, some inequalities for coefficients of variation, and Pyke's variance formula (Section 4); characterization of linear segments in $e$, decomposition of $e$, and relations with renewal theory (Section 5); and limiting properties of $e$ 'at great age' (Section 6). Use of some of the results of Sections 5 and 6 in parametric modelling appears in Section 7.

## 2. BOUNDS FOR MRL

The following elementary inequalities yield bounds for the MRL function $e_F$: Since $e_F(x) + x = E(X \mid X > x)$, we have $\{e_F(x) + x\}\overline{F}(x) = E(X \cdot 1_{(X > x)}) = \mu - E(X \cdot 1_{(X \leq x)})$. But $E(X \cdot 1_{(X > x)}) \leq T_F\overline{F}(x)$, $\leq \mu$, and $\leq (EX^r)^{1/r}\overline{F}(x)^{1-(1/r)}$ for $r > 1$, the last by Hölder's inequality. Similarly, $E(X \cdot 1_{(X \leq x)}) \leq xF(x)$ and also $\leq (EX^r)^{1/r}F(x)^{1-(1/r)}$ for $r > 1$. These five inequalities, together with conditions for equality, yield (a) - (e) below; (f) - intuitively trivial as is (a) - follows from (d).

<u>Proposition 1.</u> If $F$ is non-degenerate with mean $\mu$ and $\nu_r \equiv EX^r \leq \infty$,

(a) $e_F(x) \leq (T - x)^+$ for all $x$, with equality if and only if $F(x) = F(T-)$ or 1;

(b) $e_F(x) \leq (\mu/\overline{F}(x)) - x$ for all $x$ with equality if and only if $F(x) = 0$;

(c) $e_F(x) < (\nu_r/\overline{F}(x))^{1/r} - x$ for all $x$ and any $r > 1$;

(d) $e_F(x) \geq (\mu - x)^+/\overline{F}(x)$ for $x < T$ with equality if and only if $F(x) = 0$;

(e) $e_F(x) > \{\mu - F(x)(\nu_r/F(x))^{1/r}\}/\overline{F}(x) - x$ for $x < T$ and any $r > 1$;

(f) $e_F(x) \geq (\mu - x)^+$ for all $x$, with equality if and only if $F(x) = 0$ or 1.

If $F$ is degenerate at $\mu$, $e_F(x) = (\mu - x)^+$ for all $x$.

## 3. CHARACTERIZATIONS OF MRL; THE INVERSION FORMULA

We first present some properties of the MRL function $e_F$; some are apparently trivial but are included to provide the basis for a characterization theorem.

Proposition 2. (Properties of MRL). (a) $e$ is non-negative and right-continuous, and $e(0) = \mu > 0$; (b) $v \equiv e + I$ is non-decreasing; (c) $e$ has left limits everywhere in $(0,\infty)$ and has positive increments at discontinuities, if any; (d) $e(x-) > 0$ for $x \in (0,T)$; if $T < \infty$, $e(T-) = 0$ and $e$ is continuous at $T$; (e) $\overline{F}(x) = \{e(0)/e(x)\}\exp\{-\int_0^x(1/e)dI\}$ for all $x < T$; (f) $\int_0^x(1/e)dI \to \infty$ as $x \to T$.

Proof. The continuity of the numerator in (1.1) and the right continuity and positivity of the denominator establishes much of (a). To prove (b), consider $t > 0$ and $x + t < T$; then $v(x+t) - v(x) \geq \{(\int_{x+t}^{\infty}\overline{F}dI - \int_x^{\infty}\overline{F}dI/\overline{F}(x)\} + t =$ $\{-\int_x^{x+t}\overline{F}dI/\overline{F}(x)\} + t \geq 0$. If $x < T \leq x + t$, this remains true, and if $T \leq x$ (b) is trivial. Now (c) follows from (b).

For $x < T$, $e(x-) \geq \int_{x-}^{T}\overline{F}dI > 0$. If $x < T < \infty$, $v(x) \leq v(T) = T$ so $e(x) \leq T - x$ and $e(T-) = 0$. Hence $e$ is continuous at $T$ (when $T < \infty$), and (d) is proved.

To prove (e), write $k(x) \equiv \int_x^{\infty}\overline{F}dI = \overline{F}(x)e(x)$; then

$$\int_0^x(1/e)dI = -\int_0^x d(\log k(t)) = -\log\{k(x)/k(0)\} = -\log\{\overline{F}(x)e(x)/e(0)\}, \tag{3.1}$$

from which (e) follows after negative exponentiation. Now $k(x) \to 0$ as $x \to T$, so the right side in (3.1) $\to \infty$, proving (f). □

The function $v(x)$ in (b) is the <u>mean age at death conditional on survival to age $x$</u>; it is intuitive that it should be monotone. It increases from $\mu$ at $x = 0$ to $T$ at $x = T$ $(\leq \infty)$, and is flat only on intervals having zero probability.

The formula (e) shows that $\overline{F}$ may be recovered from $e_F$, and hence a one-to-one correspondence exists between survival functions (with $\mu \to \infty$) and MRL functions. This formula has been discovered, rediscovered, and generalized repeatedly: it was alluded to by Cox (1962, Exercise 1, page 128); given explicitly and proved very simply by Meilijson (1972); also presented in Swartz (1973), Laurent (1974), and Galambos and Kotz (1978); "extended" to expectations of the form $E\{h(X)|X>x\}$, $h$ a fixed function, $X$ with df $F$ absolutely continuous, by a series of authors including Hamdan (1972), Gupta (1975), and Shanbag and Rao (1975) (see the discussion on pages 21 and 32 of Galambos and Kotz in this regard); and extended to df's on $(-\infty,\infty)$ in Kotz and Shanbag (1980). We demonstrate a variety of uses of the inversion formula (e) in the next two sections.

The property (f) describes a limitation on how fast $e$ can grow; thus, for each $k \geq 1$, $e(x) \sim c\, x(\log x) \cdots (\log_{k-1}x)(\log_k x)^{1+\varepsilon}$ is seen to be possible for $\varepsilon = 0$, impossible for $\varepsilon > 0$. Behavior of $e_F$ 'at great age' is pursued in Section 6.

We now proceed to find a list of characteristic properties, in that any function $e$ satisfying them will be a MRL function for some survival function $\overline{F}$, namely

$\overline{F}$ defined by the inversion formula (e). Since property (c) above follows directly from (b), it need not be explicitly required; likewise, if $e(T) = 0$ for some $T < \infty$, then (b) implies $\int_0^T (1/e)dI = \infty$ (since $e(x) \le T - x$) and so (f) need only be required when e is strictly positive everywhere. We thus are led to:

Characterization Theorem. Suppose $e: \mathbb{R}^+ \to \mathbb{R}^+$ satisfies (a) e is right-continuous and $e(0) > 0$; (b) $v \equiv e + I$ is non-decreasing; (c) if $e(x-) = 0$ for some $x = x_o$, then $e(x) = 0$ on $[x_o, \infty)$; (d) if $e(x-) > 0$ for all x, then $\int_0^\infty (1/e)dI = \infty$. Let $T \equiv \inf\{x: e(x-) = 0\} \le \infty$, and define $\overline{F}$ by (e) for $x < T$ and $\overline{F}(x) \equiv 0$ for $x \ge T$. Then $F \equiv 1 - \overline{F}$ is a df on $\mathbb{R}^+$ with $F(0) = 0$, $T_F = T$, finite mean $\mu_F = e(0)$ and MRL function $e_F = e$.

For related results, see Theorem 2.1 of Bhattacharjee (1980) (who has a more complex list of characteristic properties) and Proposition 2 of Kotz and Shanbag (1980) (more general and more complex).

Counterexamples can be constructed to demonstrate that none of (a) - (d) can be omitted. In particular, if $e(x) = 1 + x^2$, $x \ge 0$, then $\int_0^\infty (1/e)dI = \arctan(x) \to \pi/2 < \infty$ as $x \to \infty$. According to Proposition 2(f), e cannot be a MRL function; nevertheless, the inversion formula (e) may be used to define a df F whose MRL function $e_F$ turns out to be $(1 + x^2)\{1 - \exp(\arctan(x) - \frac{\pi}{2})\} \ne e(x)$.

Proof. We need to prove: (a') $\overline{F}$ is non-negative, right-continuous, with $\overline{F}(0) = 1$; (b') $\overline{F}$ is non-increasing; (c') $\overline{F} > 0$ for $x < T$, and if $T = \infty$, $\overline{F}(x) \to 0$ as $x \to \infty$; and (d') $\mu_F < \infty$ and $e_F = e$.

Now (a') follows from (a) and (e). To prove (b'), consider $0 < t \le t + x < T$; then $\overline{F}(x+t)/\overline{F}(x) = \{e(x)/e(x+t)\}\exp\{-\int_x^{x+t}(1/e)dI\}$. But

$$\begin{aligned}\int_x^{x+t}(1/e)dI &= \int_x^{x+t}(v(u) - u)^{-1}du \\ &\ge \int_x^{x+t}(v(x+t) - u)^{-1}du \quad \text{by (b)} \\ &\ge \log\{[e(x+t) + t]/e(x+t)\}\end{aligned} \tag{3.2}$$

and hence

$$\int_x^{x+t}(1/e)dI \ge \log\{e(x)/e(x+t)\}, \tag{3.3}$$

again by (b). Therefore $\overline{F}(x+t)/\overline{F}(x) \le \{e(x)/e(x+t)\}\exp\{-\log[e(x)/e(x+t)]\} = 1$, proving (b') for $x < T$. The case of $x \ge T$ is trivial.

For $x < T$, there exists $\varepsilon > 0$ for which $\inf\{e(t): 0<t<x\} \ge \varepsilon$; hence $\int_0^x (1/e)dI < \infty$ and $\overline{F}(x) > 0$. Now consider (3.2) with $x = 0$ and $t < T = \infty$: $\int_0^t (1/e)dI > \log\{[e(t) + t]/e(t)\}$ so that $\overline{F}(t) \le \{e(0)/e(t)\} \cdot \{e(t)/[e(t) + t]\}$ which decreases to 0 as $t \to \infty$ by (b). Note that this last inequality

$\bar{F}(x) \le e(0)/v(x)$ is equivalent to $e(x) \le (e(0)/\bar{F}(x)) - x$, stated in Proposition 1(b) for a MRL function $e = e_F$.

To prove (d'), we first show that $e$ has property (f) (already assumed in (d) if $T = \infty$): simply apply (3.3) with $x = 0$ and let $t \to T$. Now note from (e) that $-\log\{\bar{F}(x)e(x)/e(0)\} = \int_0^x (1/e)dI$ for $x < T$ (and hence $\bar{F}e \downarrow 0$). The right side has derivative $1/e(x)$. Hence the left side is differentiable, and equating derivatives yields $\bar{F}(x) = -(d/dx)\{\bar{F}(x)e(x)\}$, and therefore

$$\int_x^T \bar{F}dI = -\bar{F}e\Big|_x^T = \bar{F}(x)e(x) \qquad \text{for all } x < T. \tag{3.4}$$

In particular, $\mu_F \equiv \int_0^\infty \bar{F}dI = \int_0^T FdI < \infty$, and dividing (3.4) by $\bar{F}(x)$ yields $e_F = e$. □

How irregular or ill-behaved can $e_F$ be? It inherits continuity and differentiability properties from $F$ at all points except $T = T_F$; and although $e + I$ is monotone, $e$ may oscillate with $0 \le \liminf_{x\to\infty} e(x) < \limsup_{x\to\infty} e(x) \le \infty$, with one or both equalities holding. For further discussion see Section 5.

## 4. RESIDUAL MOMENT FORMULAS AND SOME CHARACTERIZATIONS

Introduce the notation

$$\bar{F}^{(r)}(x) \equiv \int_x^\infty \bar{F}^{(r-1)}dI \;(\le \infty), \qquad \text{for } r = 1,2,\ldots \tag{4.0}$$

where $\bar{F}^{(0)} \equiv \bar{F}$, and $\nu_r \equiv E_F X^r$ $(\le \infty)$. When normalized, these are the survival functions corresponding to the df's $F_{(r)}$ of Smith (1959, page 6). We find by successive integration by parts that

$$\nu_r = r!\,\bar{F}^{(r)}(0) \qquad \text{for } r = 1,2,\ldots \tag{4.1}$$

with $\bar{F}^{(r)}$ finite if and only if $\nu_r$ is finite. Hence $(d/dx)\bar{F}^{(r)} = -\bar{F}^{(r-1)}$.

Now introduce the <u>residual life distribution at age a</u>:

$$\bar{F}_a(x) = P(X > a + x \mid X > a) = \bar{F}(a + x)/\bar{F}(a)$$

for $a < T$, and let $X_a$ represent a random variable with df $F_a$. Appending the subscript 'a' on previous symbols we have $\bar{F}_a^{(r)}(x) = \int_x^\infty \bar{F}_a^{(r-1)}dI$ which equals $\bar{F}^{(r)}(a + x)/\bar{F}(a)$ by induction on $r$. Hence (4.1) yields:

<u>Proposition 3</u>. For $r = 0,1,\ldots$ and $a < T$,

$$\nu_{r,a} = r!\,\bar{F}^{(r)}(a)/\bar{F}(a). \tag{4.2}$$

In particular,

$$\mu_a \equiv \nu_{1,a} = e(a) = \bar{F}^{(1)}(a)/\bar{F}(a), \tag{4.3}$$

$$\nu_{2,a} = 2\bar{F}^{(2)}(a)/\bar{F}(a), \tag{4.4}$$

$$\sigma_F^2(a) \equiv \sigma_a^2 = \nu_{2,a} - \mu_a^2 = \mu_a\{2\frac{\overline{F}^{(2)}(a)}{\overline{F}^{(1)}(a)} - \mu_a\}. \tag{4.5}$$

In the following pages we have used the notation $k \equiv \overline{F}^{(1)}$ , $K \equiv \overline{F}^{(2)}$, so $K' = -k$, $k' = -\overline{F}$, etc.

Now for some characterizations of the exponential distribution. It is elementary (e.g. from the inversion formula (e)) that a constant $e_F$ characterizes the exponential distribution. It likewise follows from the differential equation (4.2) with $r = 1$: $\mu = -k(x)/k'(x)$ for all $x$. A constant residual life moment of any order does likewise; see Theorem 2.3.2, page 33 of Galambos and Kotz (1978) and the accompanying discussion:

Proposition 4. Suppose $r$ is a positive integer. Then $\nu_{r,a} = \nu$ ($> 0$) for all $a \in \mathbb{R}^+$ if and only if $F$ is exponential.

Proof. This follows directly by expressing (4.2) as a differential equation, recalling that $\overline{F}(x) = (-1)^r(d^r/dx^r)\overline{F}^{(r)}(x)$, namely $\nu_r = r!\ (-1)^r g(x)/g^{(r)}(x)$ where $g = \overline{F}^{(r)}$ and $g^{(r)}$ is the $r^{th}$ derivative of $g$. □

If we only ask that $e_F$ be constant a.e.(F), then other distributions are possible. Specifically, the geometric distribution on a lattice in $(0,\infty)$ has $e_F$ constant on the lattice but has slope -1 off the lattice. (This is a characterization, among distributions with positive probability everywhere on the lattice.) But other e's (and hence F's) may be constructed: take $e$ constant except on an interval $[a,b)$ where it is continuous with slope -1; then $F$ is exponential, except flat on the interval and with a mass point at the right end of the interval; additional mass points may be inserted inside the interval. Other discrete distributions may also be constructed, non-lattice or lattice with 'holes'.

Also, the exponential distribution has a constant residual variance. Does this uniquely characterize the exponential law? Not quite, since it may be verified that a geometric distribution on a lattice has a constant residual variance. (The residual distribution remains geometric, but on a 'shifted lattice' which of course does not affect the variance.) We prove the following characterization and return to this question at the end of this section:

Proposition 5. If $F$ is strictly increasing on $\mathbb{R}^+$ and $\sigma_F^2(x) = \sigma^2$ for all $x$, then $F$ is exponential with mean $\mu = \sigma$.

Proof. Equation (4.5) yields

$$r(x) \equiv (\sigma^2(x)/e(x)) + e(x) = 2\, K(x)/k(x). \tag{4.6}$$

Since $k$ has derivative $-\overline{F}$ and $K$ has derivative $-k$, the right side has derivative $-2 + 2\overline{F}(x)K(x)/k^2(x) = -2 + r(x)/e(x)$ (using (4.6)) $= -s(x)$ where $s(x) \equiv 1 - \sigma^2/e^2(x)$. Hence from (4.6) $r$ also has derivative $-s$.

Since $e$ is right-continuous, we find, for $\delta > 0$,

$$\frac{r(x+\delta) - r(x)}{\delta} = \frac{e(x+\delta) - e(x)}{\delta}\{1 - \frac{\sigma^2}{e(x+\delta)e(x)}\}$$

$$= \frac{e(x+\delta) - e(x)}{\delta}\{s(x) + o(1)\} \quad \text{as} \quad \delta \to 0.$$

Since the left side has limit $-s(x)$, $e$ has a right-derivative $e'(x)$ and $\{e'(x) + 1\}s(x) = 0$.

Now $F$ strictly increasing may be shown to imply that $v \equiv e + I$ is strictly increasing, from which it follows that $e'(x) + 1$ is positive. Hence $s(x) = 0$ or $e(x) = \sigma$. □

We now go to the residual coefficient of variation $\gamma_a = \sigma_a/\mu_a$. It is identically unity for an exponential distribution, and this again is a characterization; this and some other related characterizations are given in

Proposition 6. $\sigma_F(x) = \gamma e_F(x)$ for all $x$ in $\mathbb{R}^+$ and some $\gamma > 0$ if and only if $e_F(x) = (\mu + cx)^+$ with $c = (\gamma^2 - 1)/(\gamma^2 + 1)$, and hence $F$ is Pareto (if $\gamma > 1$), exponential ($\gamma = 1$), rescaled beta$(\alpha,\beta)$ with $\alpha = 1$ ($0 < \gamma < 1$), or degenerate ($\gamma = 0$), respectively.

Hence a constant residual coefficient of variation characterizes the distributions listed, up to a scale factor.

Proof. Verification that $\gamma_F(x)$ is constant for each listed distribution is elementary, using (4.5).

That $e(x) = (\mu + cx)^+$ occurs if and only if $F$ is of one of the given types follows from the inversion formula (e). For such an $e_F$, the corresponding $\sigma_F^2(x)$ may be found from (4.5) after substituting the inversion formula as noted in Section 5, and hence the constancy of $\gamma_F(x)$ established.

For the converse, consider $x < T$, replace the left side of (4.5) with $\gamma^2 e^2(x)$ to obtain $\beta e(x) = K(x)/k(x)$ where $\beta = (1/2)(\gamma^2 + 1)$. Differentiating both sides leads to $\beta e'(x) = -1 + \beta$, and hence to $e(x) = \mu + cx$ for $x < T$, and by continuity $e(x) = (\mu + cx)^+$ for all $x$. □

Several classes of distributions are defined in terms of the MRL: NBUE ($e_F(x) \le e_F(0) = \mu$ for all $x$), NWUE ($e_F(x) \ge \mu$ for all $x$), IMRL ($e_F \uparrow$), and DMRL ($e_F \downarrow$). These are larger than the classes IFRA and DFRA respectively (if $\mu < \infty$).

Watson and Wells (1961) show that $\gamma_F$ is at most, or at least, unity in the IFR and DFR classes. Barlow and Proschan (1975), page 117, extended these inequalities to the IFRA and DFRA classes. We show that the same inequalities hold for the even larger NBUE and NWUE classes. (See Bryson and Siddiqui (1969) for the relationships of these classes; and Haines and Singpurwalla (1974) and Klefsjo (1979) for other classes and related material.)

Proposition 7. Suppose that $F$ is in the NWUE [NBUE, resp.] class. Then $\gamma_F \geq 1$ [ $\leq 1$ resp.], and the exponential distribution is the unique member of these classes with $\gamma = 1$. (If $\mu = \infty$ or $\mu < \infty = \sigma$, $\gamma \equiv \infty$.)

Proof. Assume $F$ is NWUE. Then $e(x) \geq \mu$ for all $x \in \mathbb{R}^+$ and hence $k(x) \geq \mu\overline{F}(x)$. Integrating this inequality ($0$ to $\infty$) and using (4.1), we find $EX^2 \geq 2\mu^2$ or $\sigma^2 \geq \mu^2$. Equality holds only if $e(x) = \mu$ for all $x$. Similarly for $F$ NBUE. □

An application in renewal theory is that $M(t) - (t/\mu)$ ($M$ the renewal function) has a positive limit $(\gamma_F - 1)$ as $t \to \infty$ whenever $F$ is NBUE ($F$ is also required to be non-arithmetic); see Karlin and Taylor (1975), page 195.

Proposition 7 can be extended to residual coefficients of variation: Define the class NWUE(a) as $\{F: e_F(a+x) \geq e_F(a)$ for all $x \in \mathbb{R}^+\}$ and similarly NBUE(a); hence $F$ is in NWUE(a) if $F_a$ is in NWUE, and $F$ is in NWUE(a) for every $a \geq 0$ if and only if $F$ is in IMRL. From Proposition 7, $F \in$ NWUE(a) implies $\sigma_a \geq \mu_a$, with equality if and only if $F_a$ is exponential:

Proposition 8. If $F$ is IMRL [DMRL, resp.], then $\gamma_a \equiv \sigma_a/\mu_a \geq 1$ [$\leq 1$, resp.] for every $a \in \mathbb{R}^+$.

It is easily seen that $\gamma_a \geq 1$ [ $\leq 1$, resp.] does not imply that $e$ is non-decreasing [non-increasing, resp.]; in fact, if $e$ is such that $\sigma_0^2 = \infty$, then $\sigma_a^2 = \infty$ and $\gamma_a = \infty$ for all $a \geq 0$, but $e$ need not be monotone. Or, take $e$ arbitrary continuous with $e + I$ increasing on $[0,x_o]$, $e(x) = e(x_o) + 2(x - x_o)$ on $[x_o,\infty)$. Yet another counterexample is the geometric distribution: $\gamma(x) \geq 1$ for all $x$, but $e$ decreases off the integers.

We close this section with an extension of 'Pyke's formula for the variance'--a curious formula relating $\sigma_F^2$ to $e_F$--and comment again on distributions with constant residual variance. The continuous case version of this formula appears in Pyke (1965, page 422).

Proposition 9. $\quad \sigma_F^2 = E_F e(X)e(X-)$ , $\sigma_F^2(x) = E_F[e(X)e(X-)|X > x]$ .

Proof. It may be verified (by integrating over the continuity set for $F$ and each of the discontinuity points) that $d(1/\overline{F}(x)) = [\overline{F}(x)\overline{F}(x-)]^{-1}dF(x)$ a.e.(F). Writing $Ee(X)e(X-)$ as a triple integral, applying Fubini and the formula just given, yields $\iint[\overline{F}(s\vee t) - \overline{F}(s)\overline{F}(t)]dsdt$, which equals $\sigma^2$ since $\iint\overline{F}(s\vee t)dsdt = \iiint_{s\vee t}^{\infty}dF(u)dsdt = \int u^2dF(u)$ by Fubini. This proves the formula for $\sigma^2$; the residual variance formula holds similarly. □

Whether $E_F e(X)^2$, or even its finiteness, is related to $\sigma_F^2$ when $F$ is not continuous is not known.

The formulas in Proposition 9 enable characterization of distributions with constant residual variance, alternative to Proposition 5. Specifically, it follows readily that $\sigma_F^2(x) = \sigma_F^2$ for all x, or a.e.(F), if e(x)e(x-) is constant a.e.(F). Among continuous F's, the exponentials are the only such distributions; this is consistent with Proposition 5. But discontinuous distributions are possible--e.g. the geometric distribution has this property as already noted; for it, e(x)e(x-) is easily seen to be constant on the lattice.

Other discrete distributions with constant residual variance can be constructed by modifying a geometric distribution--removing one mass point, or translating a mass point, and adjusting subsequent masses appropriately to preserve the "$e(x)e(x-) = \sigma^2$ a.e." property.

To construct a mixed distribution with constant residual variance, start with an exponential with $e(x) \equiv \mu$, say. Choose an interval I = (a,b) with length < $\mu$ and set $e(x) = \mu - (x-a)$ on [a,c) and $= \mu + b - c - (x-c)$ on [c,b], with c in I so chosen that $e(c)e(c-) = \mu^2$ -- i.e., by solving the quadratic equation $(\mu+b-c)\cdot(\mu-c+a) = \mu^2$ for c. The resulting e is a MRL function, of a distribution F with $e(x)e(x-) = \mu^2$a.e. This F is exponential on [0,a) and on [b,∞), has a mass point at c, and (a,c) and (c,b) are null intervals. To have constant residual variance, F <u>must</u> be exponential on any interval on which it is strictly increasing.

## 5. APPLICATIONS OF THE INVERSION FORMULA

Recall the inversion formula from (e):

$$\overline{F}(x) = \{e(0)/e(x)\}\exp\{-\textstyle\int_0^x(1/e)dI\} \qquad \text{for } x < T \qquad (5.1)$$

or $k(x) = \mu\cdot\exp\{-\int_0^x(1/e)dI\}$. Applying (5.1) to the residual survival function $\overline{F}_a$ yields

$$\overline{F}_a(x) = \{e(a)/e(a+x)\}\exp\{-\textstyle\int_a^{a+x}(1/e)dI\} \qquad \text{for } x < T - a. \qquad (5.2)$$

Use of these formulas (4.1) - (4.5) is sometimes convenient. Thus from (4.1), $\nu_2 = 2\mu\int_0^\infty \exp\{-\int_0^x(1/e)dI\}dx$ and similarly for $\nu_2(a)$; the first of these yields an alternative easy proof of Proposition 7, and both are useful when e has a convenient form, as noted already in the proof of Proposition 6.

We now apply the inversion formula in the form (5.2) to infer properties of $\overline{F}_a$ on an interval [0,b] from properties of $e_F$ on J = [a,a+b], and conversely. Since $\overline{F}(a+x) = \overline{F}(a)\overline{F}_a(x)$, we equivalently relate properties of $\overline{F}$ and $e_F$ on J. Specifically, the next proposition characterizes MRL's containing linear segments.

<u>Proposition 10</u>. Let $J \equiv [a,a+b]$ (or $[a,\infty)$), $a \geq 0$, $a + b \leq T_F$. If

(1) $$e_F(x) = \lambda - cx \quad \text{on } J \qquad (-\infty < c \leq 1),$$

then

$$(2) \qquad \overline{F}(x) = \begin{cases} \overline{F}(a)\{(\lambda - cx)/(\lambda - ca)\}^{(1/c)-1} & \text{on } J \text{ if } c \neq 0 \\ \overline{F}(a)\exp(-(x-a)/\lambda) & \text{on } J \text{ if } c \neq 0 \end{cases}$$

and, trivially,

$$(3) \qquad \lambda = e_F(a+b) + ca + cb \qquad (\text{if } b < \infty).$$

Conversely, if (2) and (3) hold for some $c$, or if (2) holds and $\overline{F}(a+b) = 0$, then $e_F$ is linear (if $c \neq 0$) or constant (if $c = 0$) on $J$; specifically, (1) holds.

Remarks. The case $c = 1$ corresponds to $P_F(J) = 0$; the case $0 < c < 1$ corresponds to $F$ beta $(1,(1/c)-1)$ on $J$; $c = 0$ to $F$ exponential on $J$; and $c < 0$ to $F$ Pareto on $J$. The converse without (3) is not true (if $\overline{F}(a+b) > 0$), as will become apparent in the proof; this is not surprising: for (1) to hold at $x = a+b$ imposes a condition on the (mean of the) residual distribution beyond $a+b$, and (2) makes no imposition on this residual distribution.

Proof. We prove the case with $c = 0$; the other case is similar when $a+b < T_F$, and easier when $a+b = T_F$.

From the inversion formula, we have for $x \in J$, $\overline{F}(x)/\overline{F}(a) = \exp(-(x-a)/\lambda)$ which is (2). Conversely, assume (2) (and $b \leq \infty$); then for $x \in J$

$$\begin{aligned} \int_x^\infty \overline{F}dI &= \int_x^{a+b} \overline{F}dI + \int_{a+b}^\infty \overline{F}dI \\ &= \overline{F}(a)\int_x^{a+b}\exp\{-(t-a)/\lambda\}dt + \overline{F}(a+b)e_F(a+b) \\ &= \overline{F}(a)\ \lambda\ \exp\{-(x-a)/\lambda\} - \overline{F}(a)\ \lambda\ \exp\{-b/\lambda\} \\ &\qquad + \overline{F}(a)\exp\{-b/\lambda\}e_F(a+b). \end{aligned}$$

Therefore on $J$ $e_F(x) = \lambda - \exp\{x-a)/\lambda\}\exp\{-b/\lambda\}\{\lambda - e_F(a+b)\}$ and (1) follows if and only if (3) is assumed or $b = \infty$. □

Example: If $e_F(x) = \mu - c(x \wedge a)$ for some $a > 0$, then (from (2), or directly from the inversion formula)

$$\overline{F}(x) = \{1 - (c/\mu)(x \wedge a)\}^{(1/c)-1}\exp\{-(x-a)^+/(\mu-ca)\} \quad \text{on } \mathbb{R}^+. \qquad (5.3)$$

Conversely, if $\overline{F}(x) = \{1 - (c/\mu)(x \wedge a)\}^{(1/c)-1}\exp\{-(x-a)^+/(\mu-ca)\}$ on $\mathbb{R}^+$, then $e_F(x)$ is constant for $x \geq a$; also, $e_F(x)$ is linear on $[0,a]$ if and only if $\mu = e_F(a) + ca$, i.e., $\mu = \lambda + ca$. Also see Section 7.

For related results, see Proposition 9 of Kotz and Shanbag (1980) and the references therein. The relationship of $e_F$ and $\overline{F}$ on $[a,\infty)$, for large $a$, is pursued similarly in Section 6.

We now discuss decomposition of $e$ and $F$. From the definition of $e$, or from the inversion formula, and with $v = e + I$, we find

$$dv = (e/\overline{F})dF \quad \text{or} \quad dF = (\overline{F}/e)dv$$

and $dv = de + dI$ (treating e as v-I, the difference between two increasing functions). Hence, except for a possible mass point at $T_F$ (if $< \infty$), where e and $\overline{F}$ vanish, absolutely continuous, singular and discrete components occur together or in neither of F and v. However, an absolutely continuous component in v need not correspond to one in e, and conversely; for if e (or F) is discrete or singular, v and hence F (or e) has an absolutely continuous component.

We thus conclude: If F has a discrete [singular, resp.] component then so does e, except a discontinuity in F at $T_F$ does not lead to a discontinuity in e, and conversely. Either e or F alone can have an absolutely continuous component.

Brown (1980, page 238) noted that if F is in the IMRL class, then it must have an absolutely continuous component and raised the question whether a singular component is possible. The formula $dF = (\overline{F}/e)(de + dI)$, when e is increasing, provides an alternative proof of Brown's claim. Moreover, both discrete and singular components are seen to be possible: let $e_1$ be 1+G for a discrete df G, and $e_2 = (2-H)^{-1}$ for a singular df H, and let $e = e_1 + e_2$. Then F corresponding to e is in the IMRL class and has discrete, singular, and absolutely continuous components.

A final application of the inversion formula, in renewal theory, is as follows: When watching a cumulative sum of iid rv's until 'just before', and 'just after', it crosses a level t, $\delta_t$ (= t - 'the sum before crossing') is defined as the <u>current life at time t</u>, and $\gamma_t$ (= 'the sum after crossing' - t) is defined as the <u>excess life at time t</u>. It is well-known (e.g. Karlin and Taylor (1975), page 193) that

$$\lim_{t\to\infty} P(\delta_t \geq y,\ \gamma_t \geq x) = \int_{x+y}^{\infty} \overline{F}dI/\mu = \overline{G}(x+y)$$

for all $x,y \geq 0$, where $G(x) \equiv \mu^{-1}\int_0^x \overline{F}dI$; this is the df $F_{(1)}$ of Smith (1959)- see (4.0) in Section 4. Alternative expressions for $\overline{G}(x)$ are $k_F(x)/\mu \equiv \overline{F}(x)e_F(x)/\mu$ and $\exp\{-\int_0^x (1/e)dI\}$, the latter from the inversion formula. It follows immediately that $e_F$ is related to the hazard function of G as noted by Meilijson (1972) and by Bhattacharjee (1980). The df G has a monotone density $g = \overline{F}/\mu$ and its hazard function $\lambda_G \equiv g/\overline{G}$ equals $1/e_F$. Conversely, if g is an arbitrary non-increasing density on $(0,\infty)$ (right-continuous without loss of generality) with hazard function $\lambda_G$, then $\overline{F}(x) \equiv g(x)/g(0)$ defines a df F with $e_F = 1/\lambda_G$. Meilijson used these facts to give a very simple proof of the inversion formula (e).

## 6. MRL 'AT GREAT AGE'

Now suppose that $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ as well as $\mu = EX < \infty$. Recall that

$\overline{F}$ is said to be regularly-varying (at infinity) with exponent $-\gamma$, and we write $\overline{F} \in R_{-\gamma}$, if $\overline{F}(tx)/\overline{F}(t) \to x^{-\gamma}$ for all $x \geq 0$ as $t \to \infty$; $\overline{F}$ is regularly-varying with exponent $-\infty$, written $\overline{F} \in R_{-\infty}$, if $\overline{F}(tx)/\overline{F}(t) \to \infty, 1, 0$ according as $x <,=,> 1$ respectively; and that $\overline{F}$ satisfies the weak law of large numbers, written $\overline{F} \in$ WLLN, if $\overline{F}(t+x)/\overline{F}(t) \to 0$ for all $x > 0$. The following proposition is simply a restatement of various theorems concerning functions of regular variation -- which are conveniently stated in de Haan (1975).

Proposition 11. If $\overline{F}(x) > 0$ for all $x \in \mathbb{R}^+$ and $\mu = EX < \infty$ then, as $x \to \infty$,

(a) $e(x)/x \to \infty$ if $\overline{F} \in R_{-1}$,

(b) $e(x)/x \to c \in (0,\infty)$ if and only if $\overline{F} \in R_{-1-(1/c)}$,

(c) $e(x)/x \to 0$ if and only if $\overline{F} \in R_{-\infty}$,

(d) $e(x) \to 0$ if and only if $\overline{F} \in$ WLLN.

Now, in addition, suppose that $f = F'$ exists for large $x$ and define $\lambda \equiv f/\overline{F}$, the hazard function. Then, as $x \to \infty$,

(e) $e(x)/x \to 0$ if $x\lambda(x) \to \infty$, and

(f) $e(x) \to 0$ if $\lambda(x) \to \infty$.

Finally, make the further assumption that $f$ is non-increasing; then, as $x \to \infty$,

(g) $e(x)/x \to 0$ if and only if $x\lambda(x) \to \infty$, and

(h) $e(x) \to 0$ if and only if $\lambda(x) \to \infty$.

Proof. All of (a) - (h) are simply restatements of results in de Haan (1975): (a) and (b) are in Theorem 1.2.1, page 15; (c) is in Theorems 1.3.1, page 26, and 2.9.3, page 116; (d) is Theorem 2.9.3, page 119; (e) and (g) are given in Theorem 2.9.2, page 118; and (f) and (h) are given in Theorem 2.9.4, page 120. □

It would be interesting to know other sufficient conditions for limsup $e(x)/x$ to be finite; this would imply for example that $E_F e(X)^2 \leq E_F X^2$ (see Proposition 9 above and the remarks after Condition 1a in Hall and Wellner (1980)).

For best results concerning the residual distribution at great age, see Balkema and de Haan (1974).

The principal shortcoming of Proposition 11 is that many quite different MRL functions satisfy (c), (e), and (g): $e(x)/x \to 0$ is a relatively crude description of the behavior of $e$ for large $x$. The next proposition presents an attempt to 'separate out' or distinguish a more complete range of possible behavior of $e$, and correspondingly of $\overline{F}$, at infinity.

Proposition 12. Suppose that (i) $e_F \sim e_G$ and (ii) $-\log(e_G\overline{G}) = O(-\log\overline{G})$. Then $-\log(\overline{F}) \sim -\log(\overline{G})$, or equivalently, $\overline{F}(x) = \overline{G}(x)^{1+o(1)}$.

A simple sufficient condition for (ii) is $\liminf_{x\to\infty} e_G(x) > 0$; another (see the Lemma below) is: $e_G$ has a derivative $e_G'$ with a limit at $\infty$.

Proof. Let $\eta(x) = \{e_G(x)/e_F(x)\} - 1 = o(1)$ by (i). Then the inversion formula yields

$$r(x) \equiv \frac{\overline{F}(x)}{\overline{G}(x)} = \frac{\mu_F}{\mu_G}\{1 + \eta(x)\}\exp\{-\int_0^x (\eta/e_G)dI\} .$$

But $|\int_0^x (\eta/e_G)dI| \le |\int_0^t (\eta/e_G)dI| + \delta\int_t^x (1/e_G)dI$ if $|\eta(x)| < \delta$ for all $x > t = t_\delta$ (for large x) and the right side $= A(t) + \delta\int_0^x (1/e_G)dI$. Now, by the inversion formula and (ii), $\int_0^x (1/e_G)dI = -\log\{e_G(x)\overline{G}(x)/\mu_G\} = O(-\log\overline{G})$ and hence $\int_0^x (\eta/e_G)dI = o(-\log\overline{G}(x))$. Then $\log(r(x)) = O(1) - \int_0^x (\eta/e_G)dI = o(-\log\overline{G}(x))$ or $-\log\overline{F} = -\log\overline{G} + o(-\log\overline{G})$. □

Lemma. Suppose F (or $e_F$) is absolutely continuous and $e_F'$ has a limit ($\le \infty$) at infinity. Then $\lim e_F\lambda_F \ge 1$ and $\lim\{-\log(e_F\overline{F})/-\log\overline{F}\} \le 1$ where $\lambda_F \equiv f/\overline{F}$.

Proof. $e'(x) = -1 + \lambda(x)e(x)$ and $\lim e' \ge 0$ (since $\lim e' < 0$ implies $\lim e < 0$). Therefore $\lim(e\lambda) \ge 1$ and $\lim(1/e) \le 1$. But, by L'Hôpital's rule,

$$\lim[\log(e\overline{F})]/[\log(\overline{F})] = \lim[\log\int_x^\infty \overline{F}dI]/[\log(\overline{F}(x))] = \lim(1/e\lambda). \;\square$$

The following Corollary is an immediate consequence of Proposition 12. Note that $e_F(x)/x \to 0$ in cases (a), (b), and (e), but the related $-\log\overline{F}$'s are asymptotic to quite different $-\log\overline{G}$'s.

Corollary.

(a) If $e_F(x) \to c$ as $x \to \infty$, then $\overline{F}(x) = \overline{G}(x)^{1+o(1)}$ where $\overline{G}(x) - e^{-x/c}$.

(b) If $e_F(x) \sim cx^{1-\theta}$ for some $c \in (0,\infty)$, $\theta > 0$, then $\overline{F}(x) = \overline{G}(x)^{1+o(1)}$ where $\overline{G}(x) = \exp\{-\alpha x^\theta\}$, $a > 0$, a Weibull survival function.

(c) If $e_F(x) \sim cx$ for some $c \in (0,\infty)$, then $\overline{F}(x) = \overline{G}(x)^{1+o(1)}$ where $\overline{G}(x) = (1 + bx)^{-1/b}$, $b = c(c+1)^{-1}$, a Pareto survival function.

(d) If $e_F(x) \sim c\,x\log(x)$ for some $c \in (0,\infty)$, then $\overline{F}(x) = \overline{G}(x)^{1+o(1)}$ where $\overline{G}(x) = x^{-1}\{\log(ex)\}^{-\tau}$, $\tau = 1+(1/c)$.

(e) If $e_F(x) \sim \sigma^2 x/\log(x)$, $\sigma^2 > 0$, then $\overline{F} = \overline{G}^{1+o(1)}$ where $\overline{G}(x) = P(\exp(\sigma Z + \mu) > x)$, $Z \sim N(0,1)$, a lognormal survival function.

(See Watson and Wells (1961) page 289 in regard to (e).) Note that in (a) we could also have taken G to be a Gamma$(\beta,1/c)$ df for some $\beta > 0$ (i.e. $\overline{G}(x) = \int_x^\infty c^{-\beta}\Gamma(\beta)^{-1}t^{\beta-1}e^{-t/c}dt$); all of these gamma df's have $e_G(x) \to c$ as $x \to \infty$.

## 7. USE OF MRL IN MODELLING

Various authors have (at least implicitly) suggested that knowledge of the characteristic forms of MRL functions may be useful in modelling (e.g. Bhattacharjee (1980), Bryson and Siddiqui (1969), Chhikara and Folks (1977), Laurent (1974), Muth (1977), and Watson and Wells (1961)). We add a few comments supporting this view.

Specifically, if the empirical MRL function has some apparently linear segments, Proposition 10 characterizes the survival function on these segments (see example below). Also, the behavior of the empirical MRL 'at great age' may suggest a corresponding tail behavior of the underlying survival function, asserted in the Corollary of the previous section.

Of course, histograms, density estimates, empirical survival functions, total time on test plots, and empirical (cumulative) hazard functions may likewise be used to advantage in parametric modelling. We only suggest that the empirical MRL function is a useful addition to this arsenal -- one which identifies certain kinds of behavior more readily than others. For example, a flat (or linear) tail on the MRL suggests a gamma (or Pareto or beta) tail on $F$, features not so readily determined from other empirical plots.

As an example, consider the survival times of 72 guinea pigs injected with tubercle bacilli (Bjerkedal, 1960, regimen 4.3), illustrated in Figure 1 of Hall and Wellner (1980). The empirical plot of the MRL is not too different from that in the example of Section 5 above, namely, two line segments, the latter one horizontal: $e(x) = \mu - c(x \wedge a)$. This suggests the parametric model of (5.3) for the survival function: beta followed by exponential. By maximum likelihood methods, we fit the parameters as $\mu = 176.3$, $c = 0.8278$, and $a = 91.9$, yielding an asymptote of 100.2. By contrast, other plots (not shown) do not so clearly suggest a parametric model. This model suggests that an abrupt change in the mechanism of mortality occurs after an 'incubation period' of about 92 hours.

Examining Bjerkedal's Regimen 6.6 data somewhat analogously (see Figure 2 in Hall and Wellner (1980) and note from the text that the MRL curves downward eventually) the MRL plot suggests a linear segment followed by a function curving upward, flattening, and then proceeding downward. This can be fit by piecing together two beta distributions, consistent again with an abrupt change after an initial period.

Finally, study of the MRL plot of survival times (from date of diagnosis) of 43 leukemia patients, presented by Bryson and Siddiqui (1969), suggests a linear MRL tail from 1000 to 2500 days with slope $-\frac{1}{2}$; then, according to Proposition 10, this is consistent with a beta(1,1) = uniform distribution on this interval.

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