Fechner's Distribution and Connections to Skew Brownian Motion

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Abstract This note investigates two aspects of Fechner's two-piece normal distribution: (1) connections with the mean-median-mode inequality and (strong) log-concavity; (2) connections with skew and oscillating Brownian motion processes. The developments here have been inspired by Wallis (Stat Sci 29:106–112, 2014) and rely on Chen and Zili (Sci China Math 58:97–108, 2015).

Keywords Fechner's law • Local time • Mean • Median • Mode • Oscillating Brownian motion • Pieced half normal • Quantiles • Skewed Brownian motion

Mathematics Subject Classification (2010). Primary 62E20; Secondary 62G20, 62D99, 62N01

1 Three Two-Piece Half-Normal Distributions

The standard Gaussian density ϕ and distribution function Φ are given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \exp(-z^2/2), \quad z \in \mathbb{R},$$

and

$$\Phi(z) = \int_{-\infty}^{z} \phi(x)dx = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)dx, \quad z \in \mathbb{R}.$$

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Now let $\sigma_+, \sigma_- > 0$ be two positive numbers with $\sigma_+ \neq \sigma_-$ in general, and consider the following three densities on $\mathbb{R}$:

$$
\begin{aligned}
f(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{2\sigma_-}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{2\sigma_+}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
g(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
h(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{2\sigma_+}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{2\sigma_-}{\sigma_+ + \sigma_-} \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0. \end{cases}
\end{aligned}
\tag{1.1}
$$

It is easily seen that f, g, and h differ only in the scaling of the two half normal densities $\phi(x/\sigma_\pm)/\sigma_\pm 1_{(0,\infty)}(x\text{sign}(x))$. Thus with $\theta \equiv \sigma_-/(\sigma_- + \sigma_+)$ we have

$$
\begin{aligned}
f(x;\sigma_+,\sigma_-) &\equiv \begin{cases} 2\theta \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ 2(1-\theta) \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
g(x;\sigma_+,\sigma_-) &\equiv \begin{cases} \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0; \end{cases} \\
h(x;\sigma_+,\sigma_-) &\equiv \begin{cases} 2(1-\theta) \cdot \frac{1}{\sigma_-}\phi(x/\sigma_-), & x < 0, \\ 2\theta \cdot \frac{1}{\sigma_+}\phi(x/\sigma_+), & x \geq 0. \end{cases}
\end{aligned}
$$

The density f is continuous on $\mathbb{R}$, while the densities g and h are discontinuous at 0. The density f is associated with [8] and "Fechner's Lagegesetz der Mittlewerte"; see [20, 25]. (Also see [9, 21, 22], and [23, Chap. 7] for further historical information about Fechner.) As noted by Wallis [25], this density (and the version thereof with an additional shift parameter) has been rediscovered repeatedly. It is interesting to note that the density f is *log-concave* (see e.g. [7]) and even *strongly log-concave* (see e.g. [28]).

The density g is the limit distribution of the median of i.i.d random variables with density p when when p is discontinuous at its median m, and then $\sigma_\pm^2 = 1/(4p(m\pm)^2)$ where $p(m\pm)$ denote the left and right limits of p at m respectively; see e.g. [27, pp. 343–354], [14, 15].

The density h is the marginal density of *oscillating Brownian motion*, see e.g. [13, p. 302]. This process, which is closely related to *skew Brownian motion* (see e.g. [3, 10, 12, 16, 19]), arises as the weak limit of random walk processes which are inhomogeneous in space: imagine letting the increment distributions change as the walk crosses through 0 with variance σ_+^2 for $x \geq 0$ and variance σ_-^2 for $x < 0$. See [13] for a first theorem of this type and [11] for further convergence results in this direction.

One point of interest here is the connection with the mean–median–mode inequality going back to Fechner and Pearson.

Fechner proved that for the density f with $\sigma_- \geq \sigma_+$ the inequality

$$\text{mean} \leq \text{median} \leq \text{mode} \tag{1.2}$$

holds true, and that strict inequalities hold when $\sigma_- > \sigma_+$. Fechner did this by examining the ratio $(\text{Med} - \text{Mode})/(\text{Mean} - \text{Mode})$ and considering the limits as $\sigma_+ \nearrow \sigma_-$ and as $\sigma_+ \searrow 0$ for fixed σ_+. In our notation this ratio becomes (see Table 1)

$$\begin{aligned}\frac{\text{Med} - \text{Mode}}{\text{Mean} - \text{Mode}} &= \frac{\sigma_- \Phi^{-1}\left(\frac{\sigma_+ + \sigma_-}{4\sigma_-}\right)}{\sqrt{2/\pi}(\sigma_+ - \sigma_-)} \\ &\to \begin{cases} \pi/4, & \text{as } \sigma_+ \nearrow \sigma_-, \\ \sqrt{\pi/4}\Phi^{-1}(3/4), & \text{as } \sigma_+ \to 0, \end{cases} \\ &= \left\{\begin{matrix} 0.785398\ldots, \\ 0.845348\ldots \end{matrix}\right\} < 1.\end{aligned}$$

Apparently the phenomena of the inequalities in (1.2) was observed (but not proved) by Pearson [17] in connection with his Type III curves.

The inequalities in (1.2) are illustrated in Fig. 1.

As a result of the series of papers [2, 6, 20, 24], and counterexamples (see e.g. [1]), this phenomena is now well-understood. In particular, from [6], for distributions F with median $m = 0$ (so that, with $X \sim F$, $P(X \leq m) \geq 1/2$ and $P(X \geq m) \geq 1/2$) and $\mu = E(X)$ assumed finite, if $X^+ = \max\{X, 0\}$ and

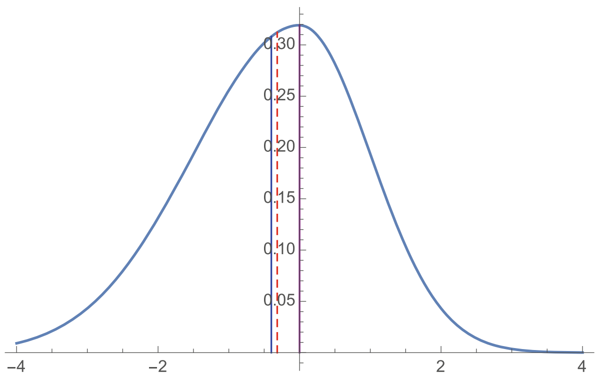

Fig. 1 Fechner's density $f(x; \sigma_-, \sigma_+)$ with $\sigma_- = 3/2$, $\sigma_+ = 1$; mean (*solid line*), median (*dashed line*)

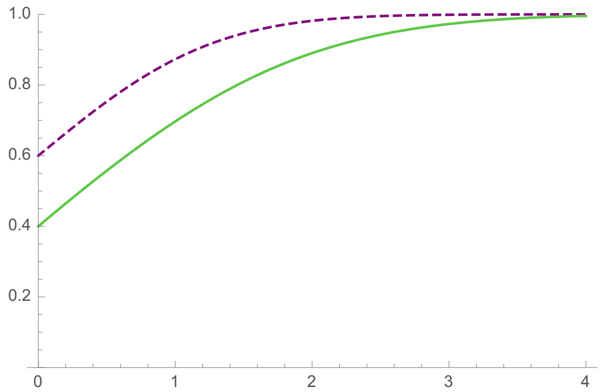

Fig. 2 Fechner stochastic order plot: F_+, *dashed curve*, F_-, *solid curve*; $\sigma_- = 3/2$, $\sigma_+ = 1$

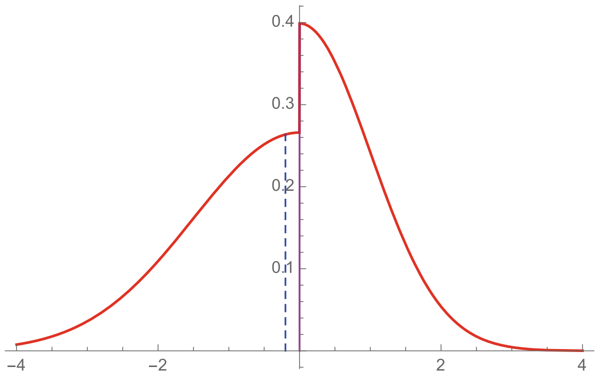

Fig. 3 Quantile limit density $g(x; \sigma_-, \sigma_+)$ with $\sigma_- = 3/2$, $\sigma_+ = 1$, mean at *dashed line*

$X^- \equiv -\min\{0, X\}$ satisfy $X^- >_s X^+$, then there is at least one mode M such that $\mu \le 0 \le M$. This is illustrated in Fig. 2.

Here we note that while the densities g and h also have mode at 0, the density g has median 0 and mean < 0 (when $\sigma_- > \sigma_+$), the density h has mean 0 and median > 0. Thus g gives an example of a density in which the equality median $=$ mode occurs, while h gives an example of a density for which the median fails to fall between the mean and mode, and thus, necessarily, X^- fails to be stochastically larger (or smaller) than X^+. These facts are illustrated in Figs. 3, 4, and 5, 6, respectively.

Finally, Fig. 7 gives a plot of all three of these densities together, all with $\sigma_- = 3/2$, $\sigma_+ = 1$.

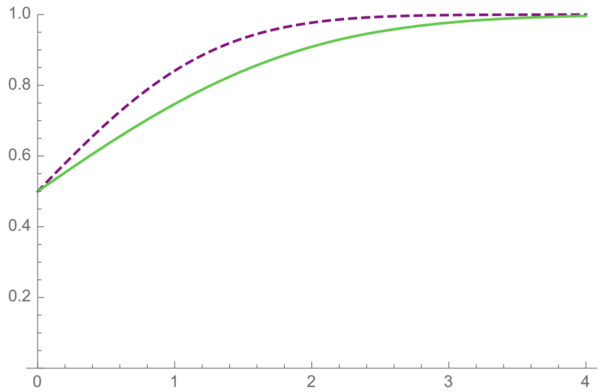

Fig. 4 Quantile stochastic order plot: G_+, *dashed curve*, G_- *solid curve*; $\sigma_- = 3/2$, $\sigma_+ = 1$

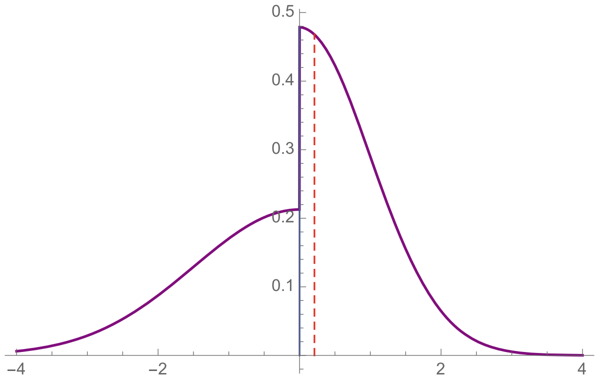

Fig. 5 Oscillating Brownian motion limit density $h(x; \sigma_-, \sigma_+)$ with $\sigma_- = 3/2$, $\sigma_+ = 1$, median at *dashed line*

2 Summary of the Properties of f, g, and h

Table 1 summarizes some of the properties of the densities f, g, and h. The formulas for the median are given only for the case that $\sigma_- > \sigma_+$.

In addition, the variances are given as follows:

$$Var_f(X) = \left(1 - \frac{2}{\pi}\right)(\sigma_+ - \sigma_-)^2 + \sigma_+\sigma_-,$$

$$Var_g(X) = \frac{1}{2}\left(1 - \frac{1}{\pi}\right)(\sigma_+ - \sigma_-)^2 + \sigma_+\sigma_-,$$

$$Var_h(X) = \sigma_+\sigma_-.$$

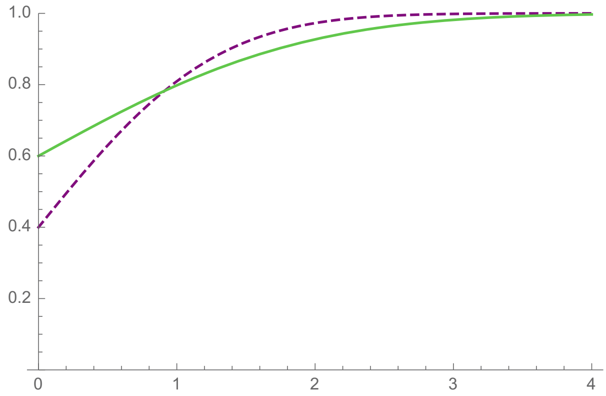

Fig. 6 Oscillating BM limit stochastic order plot: H_+, *dashed curve*, H_- *solid curve*; $\sigma_- = 3/2$, $\sigma_+ = 1$

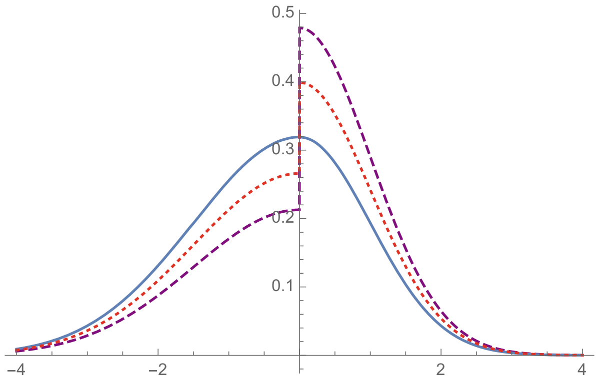

Fig. 7 The three densities f (*solid*), g (*dotted*), and h *dashed*; $\sigma_- = 3/2$, $\sigma_+ = 1$

Table 1 The mode, median, and mean of three (marginal) densities: Fechner, (nonstandard) quantile limit, and oscillating Brownian motion, as functions of σ_+ and σ_-

	Fechner	Quantile limit	Osc BM limit
Symbol	f	g	h
Mode	0	0	0
Median	$\sigma_- \Phi^{-1}\left(\frac{\sigma_+ + \sigma_-}{4\sigma_-}\right)$	0	$\sigma_+ \Phi^{-1}\left(1 - \left(\frac{4\sigma_-}{\sigma_+ + \sigma_-}\right)^{-1}\right)$
Mean	$\sqrt{\frac{2}{\pi}}(\sigma_+ - \sigma_-)$	$\frac{1}{\sqrt{2\pi}}(\sigma_+ - \sigma_-)$	0
$P(X > 0)$	$\frac{\sigma_+}{\sigma_+ + \sigma_-} = 1 - \theta$	$1/2$	$\frac{\sigma_-}{\sigma_+ + \sigma_-} = \theta$

3 Questions

We know that skew Brownian motion was studied by Walsh [26] because it provides an example of a diffusion process with discontinuous local time. We know that oscillating Brownian motion with $\sigma_+ \neq \sigma_-$ (or $q \neq p$ and $\alpha = 0$ in the notation of following sections) has both discontinuous marginal (which are scaled versions of the density h), and discontinuous local time. What are the properties of processes (if any) related to the densities f and g?

- Does Fechner's density f arise as the marginal density of a diffusion process in $\mathbb{R}$?
- Does the median zero density g arise as the marginal density of a diffusion process?
- What are the continuity properties of the marginal densities of the processes connected to the densities f and g?
- What are the continuity properties of the corresponding local time processes?

We will give answers to these questions in the next two sections.

4 A General Three-Parameter Mixture Family

Of course it is clear that f, g, and h as defined in Sect. 1 are special cases of the following mixture family: For $\theta \in [0, 1]$ and $\sigma_+, \sigma_- > 0$, let

$$q(x; \sigma_+, \sigma_-, \theta) = \theta \frac{2}{\sigma_-} \phi \left(\frac{x}{\sigma_-} \right) 1_{(-\infty,0)}(x) + (1-\theta) \frac{2}{\sigma_+} \phi \left(\frac{x}{\sigma_+} \right) 1_{[0,\infty)}(x).$$

Then

$$q(x; \sigma_+, \sigma_-, \theta) = \begin{cases} f(x; \sigma_+, \sigma_-), \text{ if } \theta = \theta_f \equiv \frac{\sigma_-}{\sigma_+ + \sigma_-}, \\ g(x; \sigma_+, \sigma_-), \text{ if } \theta = \theta_g \equiv 1/2, \\ h(x; \sigma_+, \sigma_-), \text{ if } \theta = \theta_h \equiv \frac{\sigma_+}{\sigma_+ + \sigma_-}. \end{cases}$$

For this three-parameter family, with $X \sim q$,

$$E_q X = \sqrt{\frac{2}{\pi}} \left\{ (1-\theta)\sigma_+ - \theta\sigma_- \right\},$$

$$\text{median}(X) = \begin{cases} \sigma_- \Phi^{-1}\left(\frac{1}{4\theta}\right), & \text{if } \theta \geq 1/2, \\ \sigma_+ \Phi^{-1}\left(1 - \frac{1}{4(1-\theta)}\right), & \text{if } \theta < 1/2, \end{cases}$$

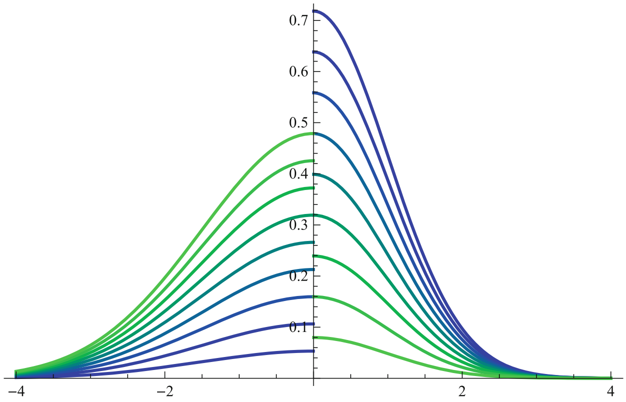

Fig. 8 The densities $q(\cdot; 3/2, 1, \theta)$ for $\theta \in \{\{.1, .2, \ldots, .9\}$

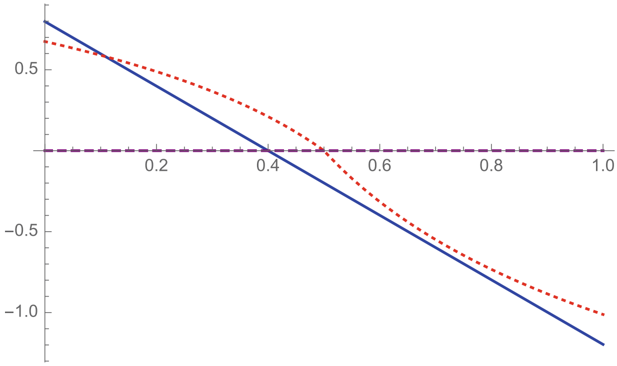

Fig. 9 Mean (*solid*), median (*dotted*), and mode (*dashed*) of the densities $q(\cdot; 3/2, 1, \theta)$ for $\theta \in (0, 1)$

$$Var_q(X) = (1-\theta)\sigma_+^2 + \theta\sigma_-^2 - ((1-\theta)\sigma_+ - \theta\sigma_-)^2 \frac{2}{\pi},$$
$$P_q(X > 0) = 1 - \theta.$$

Figure 8 shows the densities $q(\cdot; 3/2, 1, \theta)$ with $\theta \in \{.1, .2, \ldots, .9\}$

In Fig. 9 we see that the mean median and mode follow the inequality (1.2) for $\theta \geq 1/2$, and the reverse inequalities

$$\text{mode} \leq \text{median} \leq \text{mean} \tag{4.1}$$

for $\theta \leq .108389\ldots$, but that such inequalities fail for $\theta \in (.108389, .5)$.

5 Skew and Oscillating Brownian Motion Connections

How do these various densities connect with processes? From [19, Exercise 1.16, p. 82], we see that $q(\cdot; t, t, \theta)$ is the marginal density of *skew Brownian motion with parameter* $1-\theta$ at time t starting from 0 at $t = 0$. This process is denoted by $X_t^{1-\theta}$ in [19]. Moreover, from [19, Exercise 2.24, p. 401], $X_t^{1-\theta} = r_{1-\theta}(Y_t^{1-\theta})$ where $r_{1-\theta}(x) = (x/\theta)1_{[0,\infty)}(x) + (x/(1-\theta))1_{(-\infty,0)}(x)$. Equivalently, $Y_t^{1-\theta} = s_{1-\theta}(X_t^{1-\theta})$ where

$$s_{1-\theta}(x) = \theta x 1_{[0,\infty)}(x) + (1-\theta)x1_{(-\infty,0)}(x).$$

Thus $Y_t^{1-\theta}$ has marginal density $h(\cdot/t, \theta, 1-\theta) = q(\cdot/t; \theta, 1-\theta, \theta)$, and it becomes clear that $Y_t^{1-\theta}$ is *oscillating Brownian motion* with $\sigma_+ = \theta, \sigma_- = 1-\theta$.

Now consider $Z_t^{1-\theta} \equiv v_\theta(X_t^{1-\theta})$ where

$$v_\theta(x) = (1-\theta)x1_{[0,\infty)}(x) + \theta x 1_{(-\infty,0)}(x).$$

Then $Z_t^{1-\theta}$ has marginal density $f(\cdot/t, 1-\theta, \theta) = q(\cdot/t; 1-\theta, \theta, \theta)$. This is Fechner's density, and hence we call the process $Z_t^{1-\theta}$ the *Fechner process*.

6 More on the Fechner Process

Chen and Zili [5] study the following stochastic differential equation:

$$\begin{cases} dY_t^x = \left(p1_{\{Y_t^x \le 0\}} + q1_{\{0<Y_t^x\le a\}} + r1_{\{a<Y_t^x\}}\right) dB_t + \frac{\alpha}{2} dL_t^0(Y^x) + \frac{\beta}{2} dL_t^a(Y^x), \\ Y_0 = x \in \mathbb{R}, \end{cases}$$

where $\alpha, \beta \in (-\infty, 1)$, B is a one-dimensional standard Brownian motion, and for $w \in \mathbb{R}$, $L_t^w(Y^x)$ is the semimartingale local time for Y^x at level w; that is,

$$L_t^w(Y^x) = \lim_{\epsilon\to 0} \frac{1}{\epsilon} \int_0^t 1_{[w,w+\epsilon]}(Y_s^x) d\langle Y^x\rangle_s.$$

Here $\langle Y^x\rangle$ denotes the predictable quadratic variation process of Y. They note that in the special case $p = q = r = 1$ and $\beta = 0$, Y_t^x is a *skew Brownian motion* with skew parameter $1/(2-\alpha)$; and in the special case when $p = q = r = 1$ the process Y_t^x is a *double-skewed Brownian motion.* Another special case of interest is $p \ne q = r$, $\alpha = 0$, and $\beta = 0$, which corresponds to *oscillating Brownian motion* in the terminology of [13]. In the special case of $r = q$ and $\beta = 0$, Y_t^x is a *skewed oscillating Brownian motion process*, to use a combination of the terminology of [5, 13]. For further developments and applications of processes defined by the stochastic differential equation in the last display, see [18].

We are interested in a particular member of this class of processes, namely the *Fechner process* having continuous marginal densities.

Chen and Zili [5] show that the resulting SDE in this latter case, namely

$$\begin{cases} dX_t^x = \big(p1_{\{Y_t^x \le 0\}} + q1_{\{0<Y_t^x\}}\big)\, dB_t + \frac{\alpha}{2} dL_t^0(X^x)), \\ X_0^x = x \in \mathbb{R}, \end{cases} \tag{6.1}$$

has a unique strong solution, and that moreover the transition density of the diffusion X^x is given by

$$\begin{aligned} p_t^X(x,y) &= \frac{1}{\sqrt{2\pi t}} \left(\frac{1_{\{y\le 0\}}}{p} + \frac{1_{\{y>0\}}}{q} \right) \times \left\{ \exp\left(-\frac{(f(x) - f(y))^2}{2t} \right) \right. \\ &\quad \left. + \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \text{sign}(y) \exp\left(-\frac{(|f(x)| + |f(y)|)^2}{2t} \right) \right\} \end{aligned} \tag{6.2}$$

where $f(y) \equiv (y/p)1_{[y\le 0]} + (y/q)1_{[y>0]}$. This implies that the transition density $p_t^X(0,y)$ is given by

$$\begin{aligned} p_t^X(0,y) &= \frac{1}{\sqrt{2\pi t}} \left(\frac{1_{\{y\le 0\}}}{p} + \frac{1_{\{y>0\}}}{q} \right) \times \left\{ \exp\left(-\frac{f(y)^2}{2t} \right) \right. \\ &\quad \left. + \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \text{sign}(y) \exp\left(-\frac{f(y)^2}{2t} \right) \right\} \\ &= \frac{1}{\sqrt{2\pi t}} \left(\frac{1_{\{y\le 0\}}}{p} + \frac{1_{\{y>0\}}}{q} \right) \times \left\{ 1 + \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \text{sign}(y) \right\} \exp\left(-\frac{f(y)^2}{2t} \right) \\ &= \begin{cases} \frac{1}{\sqrt{2\pi t}} \cdot \frac{1}{p} \left(1 - \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \right) \cdot \exp\left(-\frac{f(y)^2}{2t} \right), \ y \le 0, \\ \frac{1}{\sqrt{2\pi t}} \cdot \frac{1}{q} \left(1 + \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \right) \cdot \exp\left(-\frac{f(y)^2}{2t} \right), \ y > 0. \end{cases} \end{aligned} \tag{6.3}$$

This family of marginal densities for the process $X_t^0 \equiv X_t$ is continuous at 0 if

$$\frac{1}{p} \left(1 - \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \right) = \frac{1}{q} \left(1 + \frac{p+q(\alpha-1)}{p-q(\alpha-1)} \right).$$

and this is easily seen to hold if and only if

$$1 - \alpha = \frac{p^2}{q^2}, \qquad \text{or if} \qquad \alpha = 1 - \frac{p^2}{q^2} \in (-\infty, 1). \tag{6.4}$$

Then

$$p_t^X(0,y) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \cdot \frac{2}{p+q} \cdot \exp\left(-\frac{y^2}{2p^2 t}\right), \ y \le 0, \\ \frac{1}{\sqrt{2\pi t}} \cdot \frac{2}{p+q} \cdot \exp\left(-\frac{y^2}{2q^2 t}\right), \ y > 0. \end{cases}$$
$$= f(y/\sqrt{t}; p, q)/\sqrt{t}$$

where $f(\cdot;\cdot,\cdot)$ is Fechner's density as given in (1.1). Again, note that f is a continuous function of its first (and all) arguments. Furthermore, the transition density $p_t^X(x,y)$ is now given by

$$p_t^X(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi t}} \frac{1}{q} \left\{\exp\left(-\frac{(x-y)^2}{2q^2 t}\right) + \frac{1-p/q}{1+p/q} \exp\left(-\frac{(x+y)^2}{2q^2 t}\right)\right\}, & x > 0, \ y > 0, \\ \frac{1}{\sqrt{2\pi t}} \frac{1}{p} \left\{\exp\left(-\frac{(x-y)^2}{2p^2 t}\right) - \frac{1-p/q}{1+p/q} \exp\left(-\frac{(|x|+|y|)^2}{2p^2 t}\right)\right\}, & x \le 0, \ y \le 0, \\ \frac{1}{\sqrt{2\pi t}} \frac{1}{q} \left\{\exp\left(-\frac{(\frac{x}{p}-\frac{y}{q})^2}{2t}\right) + \frac{1-p/q}{1+p/q} \exp\left(-\frac{(\frac{|x|}{p}+\frac{|y|}{q})^2}{2t}\right)\right\}, & x \le 0, \ y > 0, \\ \frac{1}{\sqrt{2\pi t}} \frac{1}{p} \left\{\exp\left(-\frac{(\frac{x}{q}-\frac{y}{p})^2}{2t}\right) - \frac{1-p/q}{1+p/q} \exp\left(-\frac{(\frac{|x|}{q}+\frac{|y|}{p})^2}{2t}\right)\right\}, & x > 0, \ y \le 0, \end{cases}$$

which is jointly continuous as a function of (x,y). See Fig. 10. In general the transition densities of skewed oscillating Brownian motion given in (6.2) are discontinuous; see Fig. 11.

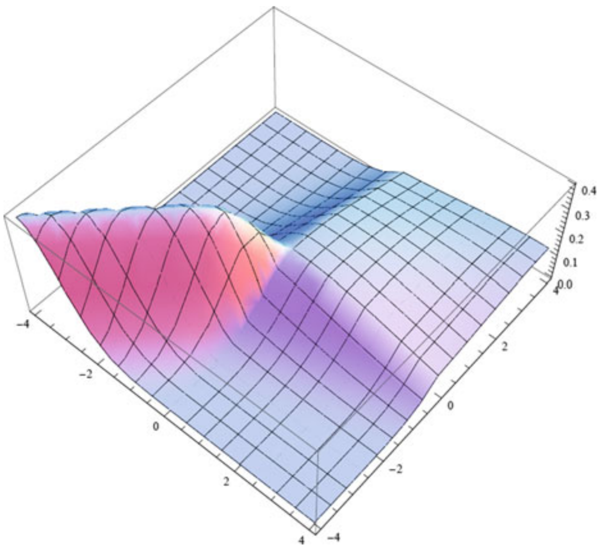

Fig. 10 Fechner process transition density $p_1^X(x,y)$ with $p = 1$ and $q = 3$

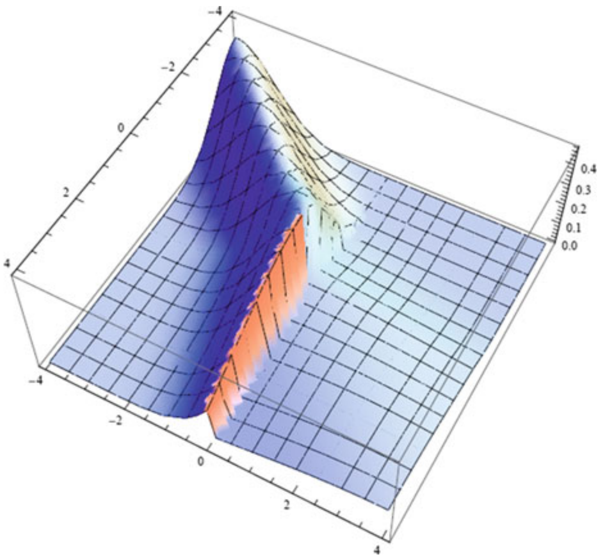

Fig. 11 Skewed oscillating Brownian motion process transition density $p_1^X(x, y)$ with $p = 1$, $q = 3$, and $\alpha = 1/2$

Question: With α related to p and q as in (6.4), does the process X_t^x have a jointly continuous local time process $L_t^w(X^x)$? (In particular is it continuous in w?)

The answer is *no* as shown by Chen [4]. Moreover, Chen [4] shows that the local time process $L_t^w(X^x)$ is jointly continuous only when $\alpha = 1 - p/q$.

Here is the proof of the two assertions from [4]. Define

$$f(y) = \begin{cases} y/p, \text{ for } y \leq 0, \\ y/q, \text{ for } y > 0. \end{cases}$$

By Chen and Zili [5, Eq. (2.9)]

$$L_t^0(X^x) = \frac{2}{2-\alpha} \widehat{L}_t^0(X^x), \tag{6.5}$$

where $\widehat{L}_t^0(X^x)$ is the symmetric local time of X^x at 0. From the proof of [5], Corollary 2.3, we see that $Z^{f(x)} \equiv f(X^x)$ is a skew driven Brownian motion driven by B starting from $f(x)$:

$$dZ_t^{f(x)} = dB_t + \frac{1}{2}\left(\frac{q(\alpha - 1)}{p} + 1\right) dL_t^0(Z^{f(x)}).$$

By use of (6.5) we can rewrite the last display in term by symmetric semimartingale local time:

$$dZ_t^{f(x)} = dB_t + \frac{1}{2}\left(\frac{p+q(\alpha-1)}{p-q(\alpha-1)}\right) d\widehat{L}_t^0(Z^{f(x)}).$$

By the same computation as for (2.5) of [5], it follows that $L_t^0(X^x) = qL_t^0(Z^{f(x)})$, and hence that

$$\widehat{L}_t^0(X^x) = \frac{(2-\alpha)q}{p+q(1-\alpha)}\widehat{L}_t^0(Z^{f(x)}). \tag{6.6}$$

Since Z is a skew Brownian motion, it follows from [3, Theorem 1.2], that unless $p+q(\alpha-1)=0$ (i.e. unless $\alpha = 1-(p/q)$), the process

$$w \mapsto w + \frac{p+q(\alpha-1)}{p-q(\alpha-1)}\widehat{L}_T^0(Z^w)$$

is a discontinuous homogeneous Markov process, where $T = \inf\{t > 0 : \widehat{L}_t^0(Z^0) = 1\}$. Thus, unless $\alpha = 1-(p/q)$, by (6.6) we have $x \mapsto \widehat{L}_T^0(X^x)$ is discontinuous, and so in view of (6.5), $x \mapsto L_T^0(X^x)$ is discontinuous. For the Fechner process, $L_t^0(X^x)$ cannot be jointly continuous in (t,x), nor is it continuous in x.

When $\alpha = 1-p/q$ we see that the factors

$$\left(1 \pm \frac{p+q(\alpha-1)}{p-q(\alpha-1)}\right) = 1,$$

and hence the marginal density $p_t^X(0,y)$ in (6.3) reduces the form of g given in (1.1).

Summarizing the discussion above leads to the following proposition:

Proposition *Let $X_t^x \equiv X_t^x(p,q,\alpha)$ denote the (strong) solution of the stochastic differential equation (6.1).*

(a) For $\alpha = 1-(p/q)^2$, X_t^x has continuous transition densities and marginal densities for $x=0$ which are scaled versions of the Fechner density f given in (1.1). On the other hand, the local time process $L_t^x(X^x)$ is discontinuous (at $x=0$).

(b) For $\alpha = 1-p/q$, X_t^x has discontinuous transition densities and marginal densities for $x=0$ which are scaled versions of the median zero density g given in (1.1). On the other hand, the local time process $L_t^x(X^x)$ is continuous.

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