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Limit Theorems for the Ratio of the Empirical Distribution Function to the True Distribution Function

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Summary. We consider almost sure limit theorems for $\|\Gamma_n/I\|_{a_n}^1 \equiv \sup_{a_n \leqq t \leqq 1} (\Gamma_n(t)/t)$ and $\|I/\Gamma_n\|_{a_n}^1 = \sup_{a_n \leqq t \leqq 1} (t/\Gamma_n(t))$ where Γ_n is the empirical distribution function of a random sample of n uniform $(0,1)$ random variables and $a_n \downarrow 0$. It is shown that (1) if $na_n/\log_2 n \to \infty$ then both $\|\Gamma_n/I\|_{a_n}^1$ and $\|I/\Gamma_n\|_{a_n}^1$ converge to 1 a.s.; (2) if $na_n/\log_2 n = d > 0$ $(d>1)$ then $\|\Gamma_n/I\|_{a_n}^1$ $(\|I/\Gamma_n\|_{a_n}^1)$ has an almost surely finite limit superior which is the solution of a certain transcendental equation; and (3) if $na_n/\log_2 n \to 0$ then $\|\Gamma_n/I\|_{a_n}^1$ and $\|I/\Gamma_n\|_{a_n}^1$ have limit superior $+\infty$ almost surely. Similar results are established for the inverse function Γ_n^{-1}.

1. Introduction and Statement of Results

Let $\xi_1, \ldots, \xi_n$ be independent uniform $(0,1)$ rv's with empirical distribution Γ_n and ordered values $0 \equiv \xi_{n0} \leqq \xi_{n1} \leqq \cdots \leqq \xi_{nn} \leqq \xi_{n,n+1} \equiv 1$. Let Γ_n^{-1} denote the left continuous inverse of Γ_n; i.e. $\Gamma_n^{-1}(t) \equiv \inf\{s: \Gamma_n(s) \geqq t\}$. The true distribution function of the ξ's is the identity function on $[0,1]$ which we denote by I.

For functions f on $[0,1]$ we let f^+ and f^- denote the positive and negative parts respectively. Let $\|f\|_a^b$ denote $\sup_{a \leqq t \leqq b} |f(t)|$ and write $\|f\|$ if $a=0$ and $b=1$.

It is known from results of Kiefer (1972) and Robbins and Siegmund (1972) that

$$\limsup_{n\to\infty} \|\Gamma_n/I\| \geqq \limsup_{n\to\infty} (1/n\xi_{n1}) = \infty \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \|I/\Gamma_n\|_{\xi_{n1}}^1 \geqq \limsup_{n\to\infty} (n\xi_{n2}) = \infty \quad \text{a.s.}$$

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In fact,

$$\limsup_{n\to\infty} \log \|\Gamma_n/I\|/\log_2 n = 1 \quad \text{a.s.} \tag{1}$$

and

$$\limsup_{n\to\infty} \|I/\Gamma_n\|_{\xi_{n1}}^1/\log_2 n = 1 \quad \text{a.s.}; \tag{2}$$

see Shorack and Wellner (1978) for even more precise results. On the other hand, the well known Glivenko-Cantelli theorem implies that for any fixed $0<\varepsilon\leqq 1$

$$\lim_{n\to\infty} \|\Gamma_n/I\|_\varepsilon^1 = 1 \quad \text{a.s.}$$

and

$$\lim_{n\to\infty} \|I/\Gamma_n\|_\varepsilon^1 = 1 \quad \text{a.s.}$$

Our object in this paper is to provide answers to the following types of questions: for what sequences $a_n \to 0$ is it true that

$$\lim_{n\to\infty} \|\Gamma_n/I\|_{a_n}^1 = 1 \quad \text{a.s.,}$$

$$\lim_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 = 1 \quad \text{a.s.,}$$

and "how rapidly" does this convergence occur? Also, for what sequences $a_n \to 0$ do the random variables $\|\Gamma_n/I\|_{a_n}^1$ and $\|I/\Gamma_n\|_{a_n}^1$ have finite limit superiors a.s.? Further, for $a_n \to 0$ so rapidly that the limit superiors are a.s. infinite, "how fast" does this occur? We will also deal with similar problems concerning Γ_n^{-1}.

This class of problems is related to some problems posed by Csörgő and Révész (1975) concerning $\sup_{a_n \leqq t \leqq 1-a_n} |\Gamma_n(t)-t|/(t(1-t))^{1/2}$; these problems have recently been answered by Csáki (1977) and Shorack (1978). Our results are also closely related to the theorems of Kiefer (1972); in fact our theorems extend Kiefer's results for large value behavior of $\Gamma_n(a_n)/a_n$ and $a_n/\Gamma_n(a_n)$ to $\|\Gamma_n/I\|_{a_n}^1$ and $\|I/\Gamma_n\|_{a_n}^1$ respectively. The general theme that emerges is that behavior at the single point a_n determines the behavior of the ratios over the entire interval $[a_n, 1]$.

The work of McBride (1974) is related to the present study, but is in a somewhat different spirit. He obtains functional laws of the iterated logarithm for processes R_n defined by

$$R_n(t) \equiv n\Gamma_n(a_n t), \qquad 0 \leqq t < \infty$$

with $a_n \to 0$ at various rates. The present theorems do not seem to overlap with his. We should also mention the recent papers by Krumbholz (1976a, 1976b). Krumbholz treats statistics of the form

$$T_n(a,b,u,v) \equiv \|(\Gamma_n - I)/(u+vI)\|_a^b$$

where a, b, u, v are such that $u+vt>0$ for all $a \leqq t \leqq b$. The requirement that $u+vt>0$ puts the problems considered by Krumbholz in the "Gaussian domain"; note that all of his limiting distributions or approximating probabilities are obtained from the distributions of the appropriate functionals of a Brownian bridge process. On the other hand, in the cases that we shall deal with here, the limiting probabilities are in the "Poisson domain"; indeed the function h introduced below can be viewed roughly as a function which "interpolates" between "Gaussian behavior" ($h(1+\lambda) \sim \lambda^2/2$ as $\lambda \to 0$) and "Poisson behavior" ($\exp(-\lambda h(1/\lambda)) = e\lambda \exp(-\lambda)$).

Because our results depend heavily on several basic exponential inequalities, it seems worthwhile to begin with these. To state the inequalities we introduce the convex function h defined for all $x \geqq 0$ by

$$h(x) = x(\log x - 1) + 1, \qquad x > 0,$$

and $h(0)=1$; h is non-negative, attains its minimum value 0 at $x=1$, is strictly decreasing for $0 \leqq x < 1$, and is strictly increasing for $1 < x < \infty$. h is related to the function h_1 of Bennett (1962) and Hoeffding (1963); for $0 \leqq x < \infty$, $x \neq 1$, $h(x) = (x-1)\, h_1(x-1)$ where $h_1(x) = (1+x^{-1}) \log(1+x) - 1$.

Lemma 1. *For all* $\lambda \geqq 1$ *and* $0 \leqq a \leqq 1$

(i) $P(\|\Gamma_n/I\|_a^1 \geqq \lambda) \leqq \exp(-nah(\lambda))$

and

(ii) $P(\|I/\Gamma_n\|_a^1 \geqq \lambda) = P(\sup_{a \leqq t \leqq 1} (-\Gamma_n(t)/t) \geqq -1/\lambda) \leqq \exp(-nah(1/\lambda))$.

These inequalities have several useful equivalent expressions. Lemma 2 restates Lemma 1 in terms of Γ_n^{-1}; Lemmas 3 and 4 restate Lemmas 1 and 2 in terms of $(\Gamma_n/I - 1)^{\pm}$ and $(\Gamma_n^{-1}/I - 1)^{\pm}$ respectively.

Lemma 2. *For all* $\lambda \geqq 1$ *and* $0 \leqq b \leqq 1$

(i) $P(\|I/\Gamma_n^{-1}\|_b^1 \geqq \lambda) \leqq \exp(-nb\lambda^{-1} h(\lambda)) = \exp(-nbf(1/\lambda))$

and

(ii) $P(\|\Gamma_n^{-1}/I\|_b^1 \geqq \lambda) \leqq \exp(-nb\lambda h(1/\lambda)) = \exp(-nbf(\lambda))$

where $f(\lambda) \equiv \lambda h(1/\lambda) = \lambda + \log(1/\lambda) - 1$.

Remark 1. The special case $b = n^{-1}$ of Lemma 2 yields the interesting inequalities

(i) $P(\|\Gamma_n/I\| \geqq \lambda) = P(\|I/\Gamma_n^{-1}\|_{1/n}^1 \geqq \lambda) \leqq e^{-f(1/\lambda)} \leqq e\lambda^{-1}$

and

(ii) $P(\|I/\Gamma_n\|_{\xi_{n1}}^1 \geqq \lambda) = P(\|\Gamma_n^{-1}/I\|_{1/n}^1 \geqq \lambda) \leqq e^{-f(\lambda)} = e\lambda e^{-\lambda}$

for all $\lambda \geqq 1$. Of course it is well known (Daniels, 1945; Robbins, 1954) that the probability in (i) equals $1/\lambda$ for $\lambda \geqq 1$ and all $n \geqq 1$, so (i) yields nothing new. But (ii) improves the constant in (2) of Shorack and Wellner (1978), replacing 16 by e, and has a simpler proof than the proof of (2) given there (which relied on results of Chang (1964) concerning the exact distribution of $\|I/\Gamma_n\|_{\xi_{n1}}^{1}$).

Lemma 3. *For all* $\lambda \geqq 0$ *and* $0 \leqq a \leqq 1$,

$$P\left(\left\|\left(\frac{\Gamma_n}{I}-1\right)^{\pm}\right\|_a^1 \geqq \lambda\right) \leqq \exp(-nah(1 \mp \lambda)).$$

Lemma 4. *For all* $\lambda \geqq 0$ *and* $0 \leqq b \leqq 1$,

$$P\left(\left\|\left(\frac{\Gamma_n^{-1}}{I}-1\right)^{\pm}\right\|_b^1 \geqq \lambda\right) \leqq \exp(-nbf(1 \pm \lambda)).$$

Remark 2. The inequalities of part (i) of Lemma 1 and the "+" part of Lemma 3 can be improved slightly. By using the methods of Bennett (1962) and Hoeffding (1963), it can be shown that

$$P(\|\Gamma_n/I\|_a^1 \geqq \lambda) \leqq \exp(-nah(\lambda)/(1-a))$$

and

$$P\left(\left\|\left(\frac{\Gamma_n}{I}-1\right)^{+}\right\|_a^1 \geqq \lambda\right) \leqq \exp(-nah(1+\lambda)/(1-a)).$$

Of course there are corresponding improvements of (i) of Lemma 2 and the "−" part of Lemma 4. The inequalities of our lemmas have simpler proofs (because of the step which uses $1-x \leqq e^{-x}$); since we are interested in the case $a \to 0$ the easier, although slightly less precise, inequalities will suffice.

Remark 3. The inequalities (ii) of Lemmas 1 and 2, the "−" part of Lemma 3, and the "+" part of Lemma 4 are tighter than any of the Bennett-Hoeffding inequalities for small values of a and are apparently new. (T.L. Lai (1975) has proved special cases of these inequalities and used them to deal with sequential rank tests; our Lemmas 1 and 2 imply Lai's (2.4) and (2.5) with precise values for the constants.) In particular, the "−" inequality of Lemma 3 implies the following inequality for the Binomial random variable $n\Gamma_n(a)$:

$$P(\Gamma_n(a)-a \leqq -\lambda) \leqq \exp\left(-nah\left(1-\frac{\lambda}{a}\right)\right) \equiv W.$$

This should be compared to the bound implied by one of Hoeffding's (1963, Theorem 1, page 15) inequalities, which yields

$$P(\Gamma_n(a)-a \leqq -\lambda) \leqq \exp(-n\lambda^2/2a(1-a)) \equiv H.$$

By means of the series expansion $h(1+x)=\sum_{n=2}^{\infty}(-1)^n n^{-1}(n-1)^{-1}x^n$, valid for $|x| \leqq 1$, it is easily shown that $H \leqq W$ if $\lambda \leqq a^2$, while $W \leqq H$ if $\lambda \geqq 3a^2$. I am indebted to G.R. Shorack for this comparison and for several helpful comments on the inequalities presented here.

The inequalities of Lemmas 3 and 4 provide an easy proof of the convergence in probability of the ratios Γ_n/I and Γ_n^{-1}/I to 1; the part of the theorem concerning Γ_n is due to Chang (1964).

Theorem 0. *If* $na_n \to \infty$ *then* $\left\|\frac{\Gamma_n}{I}-1\right\|_{a_n}^{1} \to_P 0$. *If* $nb_n \to \infty$ *then* $\left\|\frac{\Gamma_n^{-1}}{I}-1\right\|_{b_n}^{1} \to_P 0$.

The following theorems show that $na_n \to \infty$ (or $nb_n \to \infty$) is not sufficient for the almost sure version of Theorem 0.

Theorem 1. *If* $a_n \downarrow$ *and* $c_n \equiv na_n/\log_2 n \to \infty$ *then*

(i) $\lim_{n\to\infty} \|\Gamma_n/I\|_{a_n}^1 = 1$ a.s.

and

(ii) $\lim_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 = 1$ a.s.

The next theorem strengthens the conclusion of Theorem 1 under an additional monotonicity assumption; the corollary which follows is a direct consequence of either Theorem 1 or Theorem 1S.

Theorem 1S. *If* $a_n \downarrow$ *and* $c_n \equiv na_n/\log_2 n \uparrow \infty$ *then*

$$\limsup_{n\to\infty} (na_n/2\log_2 n)^{1/2} \left\|\left(\frac{\Gamma_n}{I}-1\right)^{\pm}\right\|_{a_n}^{1} = 1 \quad \text{a.s.}$$

Corollary 1. *If* $a_n \downarrow$ *and* $c_n \equiv na_n/\log_2 n \to \infty$ *then*

$$\lim_{n\to\infty} \left\|\frac{\Gamma_n}{I}-1\right\|_{a_n}^{1} = 0 \quad \text{a.s.}$$

If $c_n = na_n/\log_2 n = d > 0$ then $\|\Gamma_n/I\|_{a_n}^1$ and $\|I/\Gamma_n\|_{a_n}^1$ do not have almost sure limits; they do have limit superiors which are finite in most cases and related to the function h.

Theorem 2. *If* $na_n/\log_2 n = d > 0$ *then*

(i) $\limsup_{n\to\infty} \|\Gamma_n/I\|_{a_n}^1 = \beta'_d$ a.s.

where $\beta'_d > 1$ *is the solution of* $h(\beta'_d) = 1/d$; *and*

(ii) $\limsup_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 = 1/\beta''_d$ a.s.

where $\beta''_d < 1$ *is the solution of* $h(\beta''_d) = 1/d$ *for* $1 < d < \infty$ *and* $\beta''_d \equiv 0$ *for* $0 < d < 1$.

Remark 4. As $d \uparrow \infty$, $\beta'_d \downarrow 1$ and $\beta''_d \uparrow 1$; as $d \downarrow 0$, $\beta'_d \uparrow \infty$; as $d \downarrow 1$, $\beta''_d \downarrow 0$. The equations involving h which define β'_d and β''_d in (i) and (ii) above are just the "fundamental equation" (2.24) of Kiefer (1972).

Remark 5. The condition $c_n = na_n/\log_2 n = d$ of Theorem 2 can, in most cases, be replaced by the condition $c_n \to d$; one notable exception is (ii) of the theorem when $d=1$. For simplicity we avoid these complications.

Remark 6. For any fixed integer n $\|I/\Gamma_n\|_{a_n}^1$ is an extended-valued random variable; it equals $+\infty$ when $\xi_{n1} > a_n$ which occurs with probability $(1-a_n)^n$.

Corollary 2. *If* $c_n = na_n/\log_2 n = d > 0$ *then*

(i) $$\limsup_{n\to\infty} \left\| \left(\frac{\Gamma_n}{I} - 1\right)^+ \right\|_{a_n}^1 = \beta_d' - 1 \quad a.s.,$$

(ii) $$\limsup_{n\to\infty} \left\| \left(\frac{\Gamma_n}{I} - 1\right)^- \right\|_{a_n}^1 = 1 - \beta_d'' \quad a.s.,$$

and

(iii) $$\limsup_{n\to\infty} \left\| \frac{\Gamma_n}{I} - 1 \right\|_{a_n}^1 = \beta_d' - 1 \quad a.s.$$

The next theorem deals with $\|\Gamma_n/I\|_{a_n}^1$ when $c_n \to 0$ and also with $\|I/\Gamma_n\|_{a_n}^1$ when $c_n \downarrow 1$. Our results for the first ratio are fairly complete; (ia), (ib), and (ic) below tell how fast $\|\Gamma_n/I\|_{a_n}^1$ blows up when $c_n \to 0$ at various rates. Our results for $\|I/\Gamma_n\|_{a_n}^1$ are much less complete however; at present we can only give a condition which implies that the latter ratio has $+\infty$ as its upper limit w.p. 1.

Theorem 3. (ia) *If* $c_n \equiv na_n/\log_2 n \downarrow 0$ *and* $\log_2 n/\log c_n^{-1} \to \infty$ *then*

$$\limsup_{n\to\infty} c_n \log c_n^{-1} \|\Gamma_n/I\|_{a_n}^1 = 1 \quad \text{a.s.}$$

(ib) *If* $c_n \to 0$, $na_n \downarrow$, $\sum_1^\infty n^{-1}(na_n)^r = \infty$ *for some fixed integer* r, $1 \leqq r < \infty$, *and* $\lim_{n\to\infty} \log_2 n/\log c_n^{-1} = \rho$, $r \leqq \rho < r+1$, *then*

$$r \leqq \limsup_{n\to\infty} na_n \|\Gamma_n/I\|_{a_n}^1 \leqq \rho \quad a.s.$$

(ic) *If* $c_n \to 0$ *and* $\sum_1^\infty a_n < \infty$ *then*

$$\limsup_{n\to\infty} \log \|\Gamma_n/I\|_{a_n}^1/\log_2 n = 1 \quad a.s.$$

(ii) *If* $na_n \uparrow$, $c_n \downarrow 1$, *and* $\sum_1^\infty n^{-1}(na_n)^k \exp(-na_n) = \infty$ *for some* $k \geqq 2$, *then*

$$\limsup_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 = +\infty \quad a.s.$$

Remark 7. $\sum n^{-1}(na_n)^r = \infty$ and $\lim_{n\to\infty} \log_2 n/\log c_n^{-1}$ exists together imply that the latter limit is $\geqq r$; also, $\lim_{n\to\infty} \log_2 n/\log c_n^{-1} < r+1$ implies $\sum n^{-1}(na_n)^{r+1} < \infty$. Thus the conditions of (ib) are slightly redundant.

Lemma 5. *If $a_n \downarrow$ then, for any $\lambda > 1$ and integers $n' \leqq n''$, $[\|\Gamma_n/I\|_{a_n}^1 \geqq \lambda$ for some*

$$n' \leqq n \leqq n''] \subset \left[\|\Gamma_{n''}/I\|_{a_{n''}}^1 \geqq \frac{n'}{n''}\lambda\right].$$

Proof. Easy, using the monotonicity of $n\Gamma_n$ and a_n. □

Lemma 6. *If $a_n \downarrow$ then, for any $-1 \leqq \lambda < 0$ and integers $n' \leqq n''$,*

$$[\sup_{a_n \leqq t \leqq 1} (-\Gamma_n(t)/t) \geqq \lambda \text{ for some } n' \leqq n \leqq n''] \subset \left[\sup_{a_{n''} \leqq t \leqq 1} (-\Gamma_{n'}(t)/t) \geqq \frac{n''}{n'}\lambda\right].$$

Proof. Also easy using $a_n \downarrow$, $-n\Gamma_n(t) \downarrow$ for each fixed t, and $\lambda < 0$. □

With Lemmas 5 and 6 in hand we are ready to prove the main theorems.

Proof of Theorem 1. Since $\|\Gamma_n/I\|_{a_n}^1 \geqq \Gamma_n(1) = 1$, to prove (i) of the theorem it suffices to show that $a_n \downarrow$ and $n a_n/\log_2 n \to \infty$ imply

$$\limsup_{n\to\infty} \|\Gamma_n/I\|_{a_n}^1 \leqq 1 \quad a.s. \tag{3}$$

To show this, fix $\lambda > 1$ and $\alpha > 1$ so that $\lambda/\alpha > 1$. Then $\varepsilon \equiv h(\lambda/\alpha) > 0$. Set $n_k = [\alpha^k]$ for $k \geqq 1$. To prove (3) it suffices by Lemma 5 and Borel-Cantelli, to show that $\sum_k P(B_k) < \infty$ where

$$B_k \equiv [\|\Gamma_{n_{k+1}}/I\|_{a_{n_{k+1}}}^1 \geqq (n_k/n_{k+1})\lambda].$$

Now Lemma 1(i) implies that

$$\begin{aligned} P(B_k) &\leqq e^{-\varepsilon c_{n_{k+1}} \log_2 n_{k+1}} \\ &\leqq e^{-\tau \log_2 n_{k+1}} \\ &\leqq \text{constant} \cdot k^{-\tau} \end{aligned}$$

with $\tau > 1$ for k sufficiently large since $c_n = n a_n/\log_2 n \to \infty$. Hence $\sum_k P(B_k) < \infty$ and (3) is proved.

Since $\|I/\Gamma_n\|_{a_n}^1 \geqq 1/\Gamma_n(1) = 1$, to prove (ii) of the theorem it suffices to show that if $a_n \downarrow$ and $n a_n/\log_2 n \to \infty$ then

$$\limsup_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 \leqq 1 \quad \text{a.s.,}$$

or, equivalently, that

$$\limsup_{n\to\infty} \sup_{a_n \leqq t \leqq 1} (-\Gamma_n(t)/t) \leqq -1 \quad \text{a.s.} \tag{4}$$

To prove this, fix $\lambda > -1$ and $\alpha > 1$ so that $-\alpha\lambda < 1$. Then $\varepsilon \equiv h(-\alpha\lambda)/\alpha > 0$. Again set $n_k = [\alpha^k]$, $k \geqq 1$. To prove (4) it suffices, by Lemma 6 and Borel-Cantelli, to show that $\sum_k P(C_k) < \infty$ where

$$C_k \equiv \left[\sup_{a_{n_{k+1}} \leqq t \leqq 1} (-\Gamma_{n_k}(t)/t) \geqq \frac{n_{k+1}}{n_k}\lambda\right].$$

Now Lemma 1(ii) implies that

$$\begin{aligned} P(C_k) &\leqq \exp(-n_k a_{n_{k+1}} h(-\alpha\lambda)) \\ &\leqq \exp(-\varepsilon c_{n_{k+1}} \log_2 n_{k-1}) \\ &\leqq \exp(-\tau \log_2 n_{k+1}) \\ &\leqq \text{constant} \cdot k^{-\tau} \end{aligned}$$

with $\tau > 1$ for k sufficiently large since $c_n = n a_n / \log_2 n \to \infty$. Hence $\sum_k P(C_k) < \infty$ and (4) holds. □

Proof of Theorem 1S. If $a_n \downarrow$ and $c_n \uparrow$ then Theorem 5 of Kiefer (1972) implies that

$$\begin{aligned} &\limsup_{n\to\infty} (n a_n/2\log_2 n)^{1/2} \left\| \left(\frac{\Gamma_n}{I} - 1\right)^{\pm} \right\|_{a_n}^{1} \\ &\quad \geqq \limsup_{n\to\infty} \pm (n a_n/2\log_2 n)^{1/2} (\Gamma_n(a_n)/a_n - 1) \\ &\quad = \limsup_{n\to\infty} \pm (2 n a_n \log_2 n)^{-1/2} (n \Gamma_n(a_n) - n a_n) \\ &\quad = 1 \quad \text{a.s.} \end{aligned}$$

and thus it remains only to prove the reverse inequalities. These are easily proved by use of Lemmas 3, 5 and 6, monotonicity of c_n, $h(1+\lambda) \sim \frac{1}{2}\lambda^2$ as $\lambda \to 0$, and standard arguments. We omit the details. □

Proof of Theorem 2. Since Theorem 3 of Kiefer (1972) implies that

$$\limsup_{n\to\infty} \|\Gamma_n/I\|_{a_n}^{1} \geqq \limsup_{n\to\infty} n\Gamma_n(a_n)/d \log_2 n = \beta_d' \quad \text{a.s.}$$

when $n a_n / \log_2 n = d > 0$, to prove (i) of Theorem 2 it suffices to show that

$$\limsup_{n\to\infty} \|\Gamma_n/I\|_{a_n}^{1} \leqq \beta_d' \quad \text{a.s.} \tag{5}$$

This will follow from Lemma 5 and Borel-Cantelli if we show that $\sum_k P(D_k) < \infty$ where

$$D_k \equiv [\|\Gamma_{n_{k+1}}/I\|_{a_{n_{k+1}}}^{1} \geqq (n_k/n_{k+1})(\beta_d' + \varepsilon)],$$

$\varepsilon > 0$, $n_k = [\alpha^k]$, and $\alpha > 1$ is chosen so that $(\beta_d' + \varepsilon) > \alpha\beta_d'$. Then, by Lemma 1(i)

$$\begin{aligned} P(D_k) &\leqq \exp(-dh((\beta_d' + \varepsilon)/\alpha) \log_2 n_{k+1}) \\ &= \exp(-\tau \log_2 n_{k+1}) \\ &\leqq \text{constant} \cdot k^{-\tau} \end{aligned}$$

with $\tau \equiv dh((\beta_d' + \varepsilon)/\alpha) > 1$. Hence $\sum_k P(D_k) < \infty$ and (15) holds.

Similarly, Theorem 4 of Kiefer (1972) implies, for $1 < d < \infty$,

$$\limsup_{n\to\infty} \|I/\Gamma_n\|_{a_n}^{1} \geqq \limsup_{n\to\infty} (n\Gamma_n(a_n)/d \log_2 n)^{-1} = 1/\beta_d'' \quad \text{a.s.} \tag{6}$$

In case $0<d\leqq 1$, (3.13) of Kiefer implies that (6) continues to hold with $\beta_d''=0$ so $1/\beta_d''=\infty$. For $0<d\leqq 1$ this completes the proof. To complete the proof of (ii) of Theorem 2 when $1<d<\infty$ it therefore suffices to show that, in this case,

$$\limsup_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 \leqq 1/\beta_d'' \quad \text{a.s.}$$

or equivalently, that

$$\limsup_{n\to\infty} \sup_{a_n\leqq t\leqq 1} (-\Gamma_n(t)/t) \leqq -\beta_d'' \quad \text{a.s.} \tag{7}$$

By Lemma 6 and Borel-Cantelli, (7) will be proved if we show that $\sum_k P(E_k)<\infty$ where

$$E_k \equiv [\sup_{a_{n_{k+1}}\leqq t\leqq 1} (-\Gamma_{n_k}(t)/t) \geqq (n_{k+1}/n_k)(-\beta_d''+\varepsilon)],$$

$\varepsilon>0$, $n_k=[\alpha^k]$, and $\alpha>1$ is chosen so that $\tau\equiv\alpha^{-1} dh(\alpha(\beta_d''-\varepsilon))>1$. But by Lemma 1(ii),

$$\begin{aligned} P(E_k) &\leqq \exp(-(n_k/n_{k+1})\, dh(\alpha(\beta_d''-\varepsilon)) \log_2 n_{k+1}) \\ &= \exp(-\alpha^{-1} dh(\alpha(\beta_d''-\varepsilon)) \log_2 n_{k+1}) \\ &= \exp(-\tau \log_2 n_{k+1}) \\ &\leqq \text{constant} \cdot k^{-\tau}, \end{aligned}$$

so $\sum_k P(E_k)<\infty$, (7) holds, and Theorem 2 is proved. □

Proof of Corollaries 1 and 2. (i) and (ii) of both corollaries follow easily from the theorems together with the observation that

$$\left\| \left(\frac{\Gamma_n}{I}-1\right)^+ \right\|_{a_n}^1 = \|\Gamma_n/I\|_{a_n}^1 - 1$$

and

$$\left\| \left(\frac{\Gamma_n}{I}-1\right)^- \right\|_{a_n}^1 = 1 + \sup_{a_n\leqq t\leqq 1} (-\Gamma_n(t)/t).$$

(iii) of Corollary 2 follows from (i) and (ii) together with the inequality $\beta_d'-1>1-\beta_d''$ for all $0<d<\infty$ (since $h'(x)=\log x$ and hence, for $0<x<1$, $-h'(1-x)=-\log(1-x)>\log(1+x)=h'(1+x)$. □

Proof of Theorem 3. (ia) From Theorem 3 of Kiefer (1972), (3.9) in particular, it follows that

$$\begin{aligned} \limsup_{n\to\infty} c_n \log c_n^{-1} \|\Gamma_n/I\|_{a_n}^1 &\geqq \limsup_{n\to\infty} c_n \log c_n^{-1}\, \Gamma_n(a_n)/a_n \\ &= \limsup_{n\to\infty} (\log c_n^{-1}/\log_2 n)\, n\Gamma_n(a_n) \\ &= 1 \quad \text{a.s.} \end{aligned}$$

are direct translations of Theorems 1 and 2; Theorem 6 requires a separate proof.

Theorem 4. *If* $b_n \downarrow$ *and* $n b_n/\log_2 n \to \infty$ *then*

(i) $\lim_{n\to\infty} \|\Gamma_n^{-1}/I\|_{b_n}^{1} = 1$ *a.s.*

and

(ii) $\lim_{n\to\infty} \|I/\Gamma_n^{-1}\|_{b_n}^{1} = 1$ *a.s.*

The following theorem is an analogue of Theorem 1S. It sharpens the conclusion of Theorem 4 and again requires an additional monotonicity assumption.

Theorem 4S. *If* $b_n \downarrow$ *and* $n b_n/\log_2 n \uparrow \infty$ *then*

$$\limsup_{n\to\infty} (n b_n/\log_2 n)^{1/2} \left\| \left(\frac{\Gamma_n^{-1}}{I} - 1\right)^{\pm} \right\|_{b_n}^{1} = 1 \quad a.s.$$

The following corollary is an immediate consequence of either of the preceding two theorems.

Corollary 4. *If* $b_n \downarrow$ *and* $n b_n/\log_2 n \to \infty$ *then*

$$\lim_{n\to\infty} \left\| \frac{\Gamma_n^{-1}}{I} - 1 \right\|_{b_n}^{1} = 0 \quad a.s.$$

Corresponding to Theorem 2 we have the following theorem for Γ_n^{-1}.

Theorem 5. *If* $n b_n/\log_2 n = v$, $0 < v < \infty$, *then*

(i) $\limsup_{n\to\infty} \|\Gamma_n^{-1}/I\|_{b_n}^{1} = 1/\gamma_v''$ a.s.

where $0 < \gamma_v'' < 1$ *is the solution of* $h(\gamma_v'') = v^{-1}\gamma_v''$; *and*

(ii) $\limsup_{n\to\infty} \|I/\Gamma_n^{-1}\|_{b_n}^{1} = \gamma_v'$ a.s.

where $1 < \gamma_v' < \infty$ *is the solution of* $h(\gamma_v') = v^{-1}\gamma_v'$.

Remark 9. As $v \uparrow \infty$, $\gamma_v' \downarrow 1$ and $\gamma_v'' \uparrow 1$; as $v \downarrow 0$, $\gamma_v' \uparrow \infty$ and $\gamma_v'' \downarrow 0$. The equations involving h in the statement of Theorem 5 may be obtained by rewriting the two equations of (3.16) of Kiefer (1972). We prefer to define the quantities c_v' and c_v'' separately and solve only one equation (for γ' or γ'') rather than two.

Theorem 5 has the following corollary which is an analogue of Corollary 2.

Corollary 5. *If* $n b_n/\log_2 n = v > 0$ *then*

(i) $\limsup_{n\to\infty} \left\| \left(\frac{\Gamma_n^{-1}}{I} - 1\right)^{+} \right\|_{b_n}^{1} = \frac{1}{\gamma_v''} - 1$ *a.s.*,

(ii) $\limsup_{n\to\infty} \left\| \left(\frac{\Gamma_n^{-1}}{I} - 1\right)^{-} \right\|_{b_n}^{1} = 1 - \frac{1}{\gamma_v'}$ *a.s.*,

and

(iii) $\limsup_{n\to\infty} \left\| \left(\frac{\Gamma_n^{-1}}{I} - 1\right)^{-} \right\|_{b_n}^{1} = \frac{1}{\gamma_v''} - 1$ *a.s.*

Theorem 6 is analogous to Theorem 3; it handles the case of b_n's converging to zero more rapidly; since Γ_n^{-1} equals ξ_{n1} on $[0, n^{-1}]$ we treat only $b_n \geqq n^{-1}$.

Theorem 6. (i) *If* $b_n \geqq n^{-1}$, $b_n \downarrow$, *and* $n b_n/\log_2 n \downarrow 0$, *then*

$$\limsup_{n\to\infty} (n b_n/\log_2 n) \|\Gamma_n^{-1}/I\|_{b_n}^{1} = 1 \quad a.s.$$

(ii) *If* $b_n \downarrow$, $n b_n/\log_2 n \downarrow 0$, *and* $n b_n \uparrow$, *then*

$$\limsup_{n\to\infty} (n b_n/\log_2 n) \log \|I/\Gamma_n^{-1}\|_{b_n}^{1} = 1 \quad a.s.$$

Remark 10. When $b_n = n^{-1}$ Theorem 6 yields weak forms of the law of the iterated logarithm for $\|\Gamma_n/I\| = \|I/\Gamma_n^{-1}\|_{1/n}^{1}$ and $\|I/\Gamma_n\|_{\xi_{n1}}^{1} = \|\Gamma_n^{-1}/I\|_{1/n}^{1} \vee 1$ given by (1) and (2). This is to be expected in view of Remark 1. As mentioned before, strong forms of (1) and (2) have been established by Shorack and Wellner (1978).

Our final theorem is in a slightly different spirit than the preceding theorems; a version of it was first proved by James (1971). It improves the almost sure "nearly linear" bounds (lower bounds for Γ_n, upper bounds for Γ_n^{-1}) of Wellner (1977a) where $\phi(t) = t^{-(\gamma-1)}$, $1 < \gamma < 2$, was treated.

Theorem 7. *If* $\phi(t) = \log_2(e^e/t)$ *for* $0 < t < 1$, *then*

(i) $\lim_{n\to\infty} \left\| \frac{(I/\phi)}{\Gamma_n} \right\|_{\xi_{n1}}^{1} = 1$ *a.s.*

and

(ii) $\lim_{n\to\infty} \left\| \frac{\Gamma_n^{-1}}{I\phi} \right\|_{1/n}^{1} = 1$ *a.s.*

Thus we have the following corollary:

Corollary 7. *If* $\phi(t) = \log_2(e^e/t)$ *for* $0 < t < 1$ *and* $\tau > 1$, *then for all* ω *in a set with probability one there is an* $N = N(\omega, \tau)$ *such that* $n \geqq N$ *implies*

(i) $\Gamma_n(t) \geqq t/\tau\,\phi(t)$ *for* $\xi_{n1} \leqq t \leqq 1$

and

(ii) $\Gamma_n^{-1}(t) \leqq \tau\, t\, \phi(t)$ *for* $n^{-1} \leqq t \leqq 1$.

2. Proofs

We begin with proofs of the basic exponential inequalities.

Proof of Lemma 1. Since $\{\Gamma_n(t)/t, 0<t\leqq 1\}$ is a reverse martingale, for $r>0$

$$\begin{aligned} P(\|\Gamma_n/I\|_a^1 \geqq \lambda) &= P(\|e^{r(\Gamma_n/I)}\|_a^1 \geqq e^{r\lambda}) \\ &\leqq e^{-r\lambda} E\exp((r/n\,a)\, n\,\Gamma_n(a)) \\ &= e^{-r\lambda}\{1-a(1-e^{r/na})\}^n \\ &\leqq e^{-r\lambda} e^{-na(1-e^{r/na})} \end{aligned}$$

since $1-x\leqq e^{-x}$. Choosing $r=na\log\lambda$ yields (i) of Lemma 1. To prove (ii) of Lemma 1, first note that the two events

$$[\|I/\Gamma_n\|_a^1 \geqq \lambda] \quad \text{and} \quad [\sup_{a\leqq t\leqq 1}(-\Gamma_n(t)/t)\geqq -1/\lambda]$$

are equal. Then, since $\{-\Gamma_n(t)/t, 0<t\leqq 1\}$ is a reverse martingale, for $r>0$

$$\begin{aligned} P(\|I/\Gamma_n\|_a^1 \geqq \lambda) &= P(\sup_{a\leqq t\leqq 1}(-\Gamma_n(t)/t)\geqq -1/\lambda) \\ &= P(\|e^{r(-\Gamma_n/I)}\|_a^1 \geqq e^{-r/\lambda}) \\ &\leqq e^{r/\lambda} E\exp((r/n\,a)(-n\,\Gamma_n(a))) \\ &= e^{r/\lambda}\{1-a(1-e^{-r/na})\}^n \\ &\leqq e^{r/\lambda} e^{-na(1-e^{-r/na})} \quad \text{since } 1-x\leqq e^{-x} \\ &= e^{-nah(1/\lambda)} \end{aligned}$$

by choosing $r=-n\,a\log(1/\lambda)>0$ for $\lambda>1$. □

Proof of Lemma 2. Lemma 2 follows from Lemma 1 and the following identities:

$$[\|I/\Gamma_n^{-1}\|_b^1 \geqq \lambda] = [\|\Gamma_n/I\|_{b/\lambda}^1 \geqq \lambda]$$

and

$$[\|\Gamma_n^{-1}/I\|_b^1 \geqq \lambda] = [\|I/\Gamma_n\|_{\lambda b}^1 \geqq \lambda]. \quad \square$$

Proof of Lemmas 3 and 4. Lemma 3 is a consequence of Lemma 1 and the following set equalities:

$$\left[\left\|\left(\frac{\Gamma_n}{I}-1\right)^+\right\|_a^1 \geqq \lambda\right] = [\|\Gamma_n/I\|_a^1 \geqq 1+\lambda]$$

and

$$\left[\left\|\left(\frac{\Gamma_n}{I}-1\right)^-\right\|_a^1 \geqq \lambda\right] = [\|I/\Gamma_n\|_a^1 \geqq 1/(1-\lambda)].$$

Lemma 4 follows from Lemma 2 by way of similar identities. □

Theorem 0 follows immediately from Lemmas 3 and 4 upon noting that all of the four quantities $h(1\pm\varepsilon)$, $f(1\pm\varepsilon)$ are strictly positive for every $0<\varepsilon<1$.

Our proofs of the remaining theorems require the following preliminary lemmas.

Remark 8. Under the stated conditions (ib) implies the less precise result

$$\limsup_{n\to\infty} \log \|\Gamma_n/I\|_{a_n}^1/\log_2 n = 1/\rho \quad \text{a.s.}$$

The following examples may help in understanding the various parts of Theorem 3.

Example 1. Let $a_n = n^{-1}$. Then $c_n = (\log_2 n)^{-1} \to 0$ and $\log_2 n/\log c_n^{-1} = \log_2 n/\log_3 n \to \infty$; hence (ia) of Theorem 3 applies and yields

$$\limsup_{n\to\infty} (\log_3 n/\log_2 n) \|\Gamma_n/I\|_{1/n}^1 = 1 \quad \text{a.s.}$$

Example 2. Let $a_n = n^{-1}(\log_2 n)(\log n)^{-1/\rho}$, $\rho \geqq 1$. Then $c_n = (\log n)^{-1/\rho} \to 0$, $\sum_1^\infty n^{-1}(n a_n)^{[\rho]} = \infty$ and $\log_2 n/\log c_n^{-1} = \rho$; hence (ib) of Theorem 3 applies and yields

$$[\rho] \leqq \limsup_{n\to\infty} (\log_2 n/(\log n)^{1/\rho}) \|\Gamma_n/I\|_{a_n}^1 \leqq \rho \quad \text{a.s.}$$

Example 3. Let $a_n = n^{-1}(\log n)^{-1}$. Then $c_n = (\log n \cdot \log_2 n)^{-1} \to 0$ and $\log_2 n/\log c_n^{-1} = \log_2 n/(\log_2 n + \log_3 n) \to 1 \equiv \rho$; hence (ib) of Theorem 3. again applies and yields

$$\limsup_{n\to\infty} (\log n)^{-1} \|\Gamma_n/I\|_{a_n}^1 = 1 \quad \text{a.s.}$$

Example 4. Let $a_n = (n \log n)^{-1} (\log_2 n)^{-\tau}$, $\tau \geqq 1$. Then

$$c_n = (\log n)^{-1} (\log_2 n)^{-(\tau+1)} \to 0 \quad \text{and} \quad \sum_1^\infty a_n < \infty;$$

hence (ic) of Theorem 3 applies and yields

$$\limsup_{n\to\infty} \log \|\Gamma_n/I\|_{a_n}^1/\log_2 n = 1 \quad \text{a.s.}$$

In this domain $\xi_{n1} \geqq a_n$ eventually w.p. 1 and $\|\Gamma_n/I\|_{a_n}^1$ behaves the same as $\|\Gamma_n/I\|$.

Example 5. Let $a_n = n^{-1}(\log_2 n + \gamma \log_3 n)$, $\gamma > 0$. Then $n a_n \uparrow$, $c_n = (\log_2 n + \gamma \log_3 n)/\log_2 n \downarrow 1$, and, if $k+1 > \gamma$, $\sum_1^\infty n^{-1}(n a_n)^k e^{-n a_n} = \infty$; hence (ii) of Theorem 3 applies and yields

$$\limsup_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 = \infty \quad \text{a.s.}$$

I do not know if it is possible to have $c_n \downarrow 1$ slowly and still maintain $\limsup_{n\to\infty} \|I/\Gamma_n\|_{a_n}^1 < \infty$ a.s.; in this connection see the discussion on page 237 of Kiefer (1972).

Now we translate our results for Γ_n into similar theorems for Γ_n^{-1}. The following theorems are related to Theorem 6 of Kiefer (1972). Theorems 4 and 5 when $\log_2 n/\log c_n^{-1} \to \infty$. Thus it remains to prove the reverse inequality. Set $\lambda_n = (c_n \log c_n^{-1})^{-1}$; let $\varepsilon > 0$, choose $1 < \alpha < 1+\varepsilon$, and let $n_k = [\alpha^k]$, $k \geqq 1$. Then, using $\lambda_n \uparrow$ and Lemma 5,

$$\begin{aligned}&[\|\Gamma_n/I\|_{a_n}^1 \geqq \lambda_n(1+\varepsilon) \text{ for some } n_k \leqq n \leqq n_{k+1}]\\ &\quad \subset [\|\Gamma_{n_{k+1}}/I\|_{a_{n_{k+1}}}^1 \geqq (n_k/n_{k+1})\, \lambda_{n_k}(1+\varepsilon)] \equiv F_k.\end{aligned}$$

Thus to show that

$$\limsup_{n\to\infty} \lambda_n^{-1} \|\Gamma_n/I\|_{a_n}^1 \leqq 1 \quad \text{a.s.}$$

it suffices to show that $\sum_k P(F_k) < \infty$. But, by Lemma 1(i) and $h(x) \sim x \log x$ as $x \to \infty$, if $1 < \tau < \alpha^{-1}(1+\varepsilon)$, then, for k sufficiently large,

$$\begin{aligned}P(F_k) &\leqq \exp(-n_{k+1} a_{n_{k+1}} h(\alpha^{-1}(1+\varepsilon)\, \lambda_{n_k}))\\ &\leqq \exp(-\tau \log_2 n_k)\\ &\leqq \text{constant} \cdot k^{-\tau},\end{aligned}$$

and this completes the proof of (ia).

(ib) From Theorem 1 of Kiefer (1972), $n a_n \downarrow$ and $\sum n^{-1}(n a_n)^r = \infty$ imply that $P(\xi_{nr} \leqq a_n \text{ i.o.}) = 1$. Hence

$$n a_n \|\Gamma_n/I\|_{a_n}^1 \geqq n \Gamma_n(a_n) \geqq r$$

occurs infinitely often w.p. 1; to complete the proof it suffices to show that

$$\limsup_{n\to\infty} n a_n \|\Gamma_n/I\|_{a_n}^1 \leqq \rho \quad \text{a.s.} \tag{8}$$

with $\rho = \lim_{n\to\infty} \log_2 n/\log c_n^{-1}$. Let $\varepsilon > 0$, choose $1 < \alpha < 1+\varepsilon$, and set $n_k = [\alpha^k]$, $k \geqq 1$. Then, using $n a_n \downarrow$ and Lemma 5,

$$\begin{aligned}&[\|\Gamma_n/I\|_{a_n}^1 \geqq (1+\varepsilon)\, \rho/n a_n \text{ for some } n_k \leqq n \leqq n_{k+1}]\\ &\quad \subset [\|\Gamma_{n_{k+1}}/I\|_{a_{n_{k+1}}}^1 \geqq (n_k/n_{k+1})\, \rho(1+\varepsilon)/n_k a_{n_k}]\\ &\quad \equiv G_k.\end{aligned}$$

Thus to prove (8) it suffices to show that $\sum_k P(G_k) < \infty$. But by Lemma 1(i), $h(x) \sim x \log x$ as $x \to \infty$, and $\lim_{n\to\infty} \log_2 n/\log c_n^{-1} = \rho$, if $1 < \tau < \alpha^{-1}(1+\varepsilon)$ then for k sufficiently large

$$\begin{aligned}P(G_k) &\leqq \exp(-n_{k+1} a_{n_{k+1}} h(\alpha^{-1}(1+\varepsilon)\, \rho/n_k a_{n_k}))\\ &\leqq \exp(-\tau \log_2 n_k)\\ &\leqq \text{constant} \cdot k^{-\tau},\end{aligned}$$

and this completes the proof of (ib).

(ic) If $\sum a_n < \infty$, $\xi_{n1} \geqq a_n$ for all sufficiently large n w.p. 1 since $P(\xi_{n1} < a_n$ i.o.$) = 0$ by Theorem 1 of Kiefer (1972). Hence

$$\limsup_{n\to\infty} \log \|\Gamma_n/I\|_{a_n}^1/\log_2 n \geqq \limsup_{n\to\infty} \log(1/n\,\xi_{n1})/\log_2 n = 1 \quad \text{a.s.}$$

On the other hand, $\|\Gamma_n/I\|_{a_n}^1 \leqq \|\Gamma_n/I\|$ and hence (1) implies that

$$\limsup_{n\to\infty} \log \|\Gamma_n/I\|_{a_n}^1 \leqq \limsup_{n\to\infty} \log \|\Gamma_n/I\|/\log_2 n = 1 \quad \text{a.s.}$$

which completes the proof.

(ii) By (ii) of Theorem 1 of Shorack and Wellner (1978), the given conditions imply that $P(n\,\xi_{nk} \geqq n\,a_n$ i.o.$) = 1$. Thus for all ω in a set with probability one, there exists a subsequence $n' = n'(\omega)$ such that $\xi_{n'k} \geqq a_{n'}$. Therefore, for this same subsequence n', we have

$$\|I/\Gamma_{n'}\|_{a_{n'}}^1 \geqq n'\,\xi_{n'k}/(k-1) \geqq n'\,a_{n'}/(k-1) = (c_{n'}/k-1)\log_2 n' \to \infty. \quad \square$$

Proofs of Theorems 4, 4S, 5, and 6. The proofs of these theorems are similar to the proofs of Theorems 1, 1S, 2, and 3, so we will omit most of the details.

Theorems 4 and 5 are easily proved either by way of Theorems 1 and 2 or by direct use of Lemma 2 and (3.18) of Kiefer (1972). Theorem 4S is proved by using (3.17) of Kiefer (1972) to show that the limit superiors are $\geqq 1$, and using Lemmas 4, 5, and 6, monotonicity of $n b_n/\log_2 n$, and $f(1+\lambda) \sim \frac{1}{2}\lambda^2$ as $\lambda \to 0$ to prove the reverse inequalities. Corollaries 4 and 5 follow directly from Theorems 4 and 5; that $\frac{1}{\gamma_v''} - 1 > 1 - \frac{1}{\gamma_v'}$ for all $0 < v < \infty$ may be easily seen by writing $h(\gamma) = v^{-1}\gamma$ as $f(1/\gamma) = v^{-1}$ and noting that $-f'(1-x) = x/(1-x) > x/(1+x) = f'(1+x)$ for $0 < x < 1$. Theorem 6 is proved by use of Lemmas 2, 5, and 6, $f(x) \sim x$ and $f(1/x) \sim \log x$ as $x \to \infty$, and (3.19) and (3.20) of Kiefer (1972). $\square$

Proof of Theorem 7. To prove (i), first note that

$$\left\|\frac{(I/\phi)}{\Gamma_n}\right\|_{\xi_{n1}}^1 \geqq \frac{1/\phi(1)}{\Gamma_n(1)} = 1$$

and hence it suffices to show that

$$\limsup_{n\to\infty} \left\|\frac{I/\phi)}{\Gamma_n}\right\|_{\xi_{n1}}^1 \leqq 1 \quad \text{a.s.} \tag{12}$$

Now, letting $a_n = n^{-1}\log n$, we have

$$\left\|\frac{(I/\phi)}{\Gamma_n}\right\|_{\xi_{n1}}^1 = \left\|\frac{(I/\phi)}{\Gamma_n}\right\|_{\xi_{n1}}^{a_n} \vee \left\|\frac{(I/\phi)}{\Gamma_n}\right\|_{a_n}^1;$$

where

$$\left\|\frac{(I/\phi)}{\Gamma_n}\right\|_{\xi_{n1}}^{a_n} \leqq \|I/\Gamma_n\|_{\xi_{n1}}^{a_n} \cdot \phi(a_n)^{-1} \leqq (\|I/\Gamma_n\|_{\xi_{n1}}^1/\log_2 n)(\log_2 n/\phi(a_n)).$$

Hence, by (2) and $\phi(a_n) \sim \log_2 n$ as $n \to \infty$,

$$\limsup_{n\to\infty} \left\| \frac{(I/\phi)}{\Gamma_n} \right\|_{\xi_{n1}}^{a_n} \leqq 1 \quad \text{a.s.}$$

On the other hand, since $1/\phi \leqq 1$ on $(0, 1]$,

$$\left\| \frac{(I/\phi)}{\Gamma_n} \right\|_{a_n}^{1} \leqq \|I/\Gamma_n\|_{a_n}^{1}$$

and, by Theorem 1 this last quantity converges to 1 a.s. as $n \to \infty$. Thus (12) holds and the proof of (i) of Theorem 7 is complete. The proof of (ii) is similar to that of (i) and we omit it. □

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