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Uniform convergence in some limit theorems for multiple particle systems

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Abstract

For n particles diffusing throughout R (or R^d), let $\eta_{n,t}(A)$, $A \in \mathscr{B}$, $t \geqslant 0$, be the random measure that counts the number of particles in A at time t. It is shown that for some basic models (Brownian particles with or without branching and diffusion with a simple interaction) the processes $\{(\eta_{n,t}(\phi) - E\eta_{n,t}(\phi))/\sqrt{n} : t \in [0,M], \phi \in C_L^\alpha(R)\}$, $n \in \mathbb{N}$, converge in law uniformly in (t,ϕ). Previous results consider only convergence in law uniform in t but not in ϕ. The methods used are from empirical process theory. © 1997 Elsevier Science B.V.

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1. Introduction

Consider n particles starting at random i.i.d. locations $Y_1, \dots, Y_n$ and performing random motions (diffusion processes) $X_1(t), \dots, X_n(t)$ in R^d, with or without branching, with or without interactions. Under certain conditions, the random measures $\mathbb{P}_n(A) := n^{-1} \sum_{i=1}^n \delta_{X_i(t)}(A)$, $A \in \mathscr{B}$, which give the proportion of particles present in region A at time t, stabilize at a (deterministic) measure μ_t (e.g. at $\mu_t(A) = E\delta_{X_1(t)}(A)$ if the processes $X_i(t)$ are i.i.d.). Then, the limit in some weak sense of the random measures $\nu_{n,t} := n^{-1/2} \sum_{i=1}^n (\delta_{X_i(t)} - \mu_t)$ as $n \to \infty$ measures, if it exists, the fluctuation of $\mathbb{P}_n$ about equilibrium when the number of particles is practically infinite (Martin-Löff, 1976). Ito (1983) studies such a system for $X_i(t) = Y_i + B_i(t)$ where B_i are independent

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Brownian motions, independent of Y_i. Other authors consider more complicated processes, with the initial distribution $(Y_1, \ldots, Y_n)$ replaced by the points of a point process with intensity λ_n (Martin-Löff, 1976), where the particles may double or disappear at random branching times (Holley and Stroock, 1978, where they attribute such a model to Spitzer; Gorostiza, 1983; Walsh, 1986), and where interactions among the particles may be present (Holley and Stroock, 1979; Tanaka and Hitsuda, 1981; Adler, 1990). Usually, the weak convergence of the random measures $\nu_{n,t}$ is not strong enough to produce a limiting random measure, and the above-mentioned authors circumvent this problem by restricting themselves, when passing to the limit, to the action of $\nu_{n,t}$ only on the space $\mathscr{S}$ of rapidly decreasing functions (with rapidly decreasing derivatives of all orders). They prove that the processes $\nu_{n,t}$ converge weakly to a sample continuous distribution-valued Gaussian process G_t in the sense of weak convergence of probability measures on $C([0,1],\mathscr{S}')$ or $D([0,1],\mathscr{S}')$, where $\mathscr{S}'$ is the dual of $\mathscr{S}$. By a theorem of Mitoma (1983), $\nu_{n,t}$ converges to G_t in this sense (in the case of $C([0,1],\mathscr{S}')$) if: (i) $(\nu_{n,t_1}(\phi_1), \ldots, \nu_{n,t_k}(\phi_k)) \to_d (G_{t_1}(\phi_1), \ldots, G_{t_k}(\phi_k))$ for all $k<\infty$, $t_i \in [0,1]$, $\phi_i \in \mathscr{S}$; and (ii) *for each fixed* $\phi \in \mathscr{S}$, the sequence of processes $\{\nu_{n,t}(\phi) : t \in [0,1]\}_{n=1}^{\infty}$ is tight in $C[0,1]$, that is

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \Pr\left\{ \sup_{|t-s|<\delta} |\nu_{n,t}(\phi) - \nu_{n,s}(\phi)| > \varepsilon \right\} = 0 \tag{1.1}$$

for all $\varepsilon > 0$.

Motivated in part by modern empirical process theory, we may ask whether these limits in fact take place not only uniformly in $t \in [0,1]$, but also uniformly in $\phi \in \Phi$ for some reasonably large set of functions Φ. That is, whether the tightness condition (1.1) can be replaced by the stronger condition

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \Pr\left\{ \sup_{d((s,\phi),(t,\psi))<\delta,\, s,t\in[0,1],\, \phi,\psi\in\Phi} |\nu_{n,s}(\phi) - \nu_{n,t}(\psi)| > \varepsilon \right\} = 0 \tag{1.2}$$

for all $\varepsilon > 0$, or equivalently, whether the convergence $\nu_{n,t} \to_d G_t$ takes place in $l^\infty([0,1]\times\Phi)$. The type of distances we have in mind for (1.2) are e.g. $d((s,\phi),(t,\psi)) = |t-s| \vee \|\phi-\psi\|_\infty$ or distances involving also the derivatives, and Φ could be, for instance, the set of Hölder functions of order α with Hölder constant bounded by $M<\infty$. The interest of results of this kind is that: (1) they extend convergence of $\nu_{n,t}$ to larger classes of functions than $\mathscr{S}$, and the limit process is correspondingly extended too; (2) weak convergence of $\nu_{n,t}(\phi)$ is uniform in both t and ϕ simultaneously (as opposed to being uniform with respect to t only) and therefore we have convegence $EH(\nu_n) \to EH(G)$ for more functionals H; and finally (3) this convergence implies stronger continuity properties on the limit process: $G_t(\phi)$ is then sample continuous with respect to the distance d in (1.2).

We show in this article that such strengthening is indeed possible in three simple cases; namely: Ito's (1983) case of independent Brownian particles (Section 2), Spitzer's case (Walsh, 1986) of independent branching Brownian particles (Section 3), and the example of McKean's (1967) case of particles undergoing interacting diffusions considered by Tanaka and Hitsuda (1981) (Section 4). Our intent here is not to be exhaustive, but to show that this program is possible by carrying it out in examples of show-case value. For simplicity of exposition we consider the diffusions to take values in R, but only trivial changes are required for diffusions in R^d, $d>1$. In the first two cases considered, the class Φ is the set of Hölder functions $\phi: R\to R$, $\|\phi\|_\infty \leqslant M$, $|\phi(y)-\phi(x)|\leqslant M|y-x|^\alpha$ for some $M<\infty$ and $\alpha>\frac{1}{2}$; whereas in the third case, we also assume the functions ϕ' to be uniformly bounded and have uniformly bounded δ-Hölder constant for some $\delta>0$. In the first two cases the distance d can be taken to be $|t-s|\vee\|\phi-\psi\|_\infty$, and in the third, $|t-s|\vee\|\phi-\psi\|_\infty\vee\|\phi'-\psi'\|_\infty$.

The methods, not surprisingly, are those of empirical processes. In each case the processes $X_k(t)$, or their more complicated counterparts, can be defined as coordinates in a large probability space, and the class of functions $\mathscr{F}=\{f_{t,\phi}: f_{t,\phi}(x)=\phi(x_t), t\in[0,1], \phi\in\Phi\}$ (or its more complicated counterpart) can be shown to be P-Donsker for the law P of X_1 by application of basic empirical process results. Similar schemes of proof apply to the three situations considered, although the details are different. The result for independent Brownian particles in Section 2 follows in fact from the limit theorem in Section 3 (it is the special case corresponding to $\tau=0$); however its separate proof, short and simple, is the model for the other two proofs, which are necessarily more complicated as they deal with more complex processes.

The type of convergence we will prove for our processes is as follows. Let T be an index set (usually a set of functions), and let $\mathbb{Z}_n(t)$, $\mathbb{Z}(t)$, $t\in T$, be processes indexed by t such that almost all their sample paths are bounded functions of $t\in T$, and such that the finite-dimensional distributions of $\mathbb{Z}(t)$ are those of a Radon measure on the space $(l^\infty(T),\|\cdot\|_\infty)$. Then we say that $\mathbb{Z}_n$ *converges weakly to* $\mathbb{Z}$ in $l^\infty(T)$, and write

$$\mathbb{Z}_n \rightsquigarrow \mathbb{Z} \quad \text{in } l^\infty(T)$$

if

$$E^*H(\mathbb{Z}_n)\to EH(\mathbb{Z})$$

for all $H: l^\infty(T)\to R$ bounded and continuous, where E^* denotes outer expectation. This definition is due, at its final stage, to Hoffmann-Jørgensen, and we refer to Van der Vaart and Wellner (1996) for further description and properties. Perhaps we should only mention that $\mathbb{Z}_n\rightsquigarrow\mathbb{Z}$ in $l^\infty(T)$ if and only if: (i) the finite dimensional distributions of $\mathbb{Z}_n$ converge to those of $\mathbb{Z}$, and (ii) there is a distance d on T for which (T,d) is totally bounded and such that

$$\lim_{\delta\to 0}\limsup_{n\to\infty}\Pr\left\{\sup_{d(s,t)\leqslant\delta}|\mathbb{Z}_n(s)-\mathbb{Z}_n(t)|>\varepsilon\right\}=0$$

for all $\varepsilon>0$; see e.g. Van der Vaart and Wellner (1996, Theorem 1.5.7, p. 37). If this is the case, $\mathbb{Z}$ is sample continuous on T with respect to d. If $\mathbb{Z}$ is centered Gaussian, $d^2(s,t)=E(\mathbb{Z}(s)-\mathbb{Z}(t))^2$ is a choice that always works, and so does any distance d' dominating d for which (T,d') is totally bounded.

2. The central limit theorem for Ito's model

Let $B_1, B_2, \ldots$ be independent standard Brownian motions, and let $Y_1, Y_2, \ldots$ be independent and identically distributed with distribution μ on R, and independent of $B_1, B_2, \ldots$. Then the processes $X_i \equiv \{X_i(t) : 0 \leqslant t < \infty\}$, $i = 1, 2, \ldots$, defined by

$$X_i(t) = Y_i + B_i(t), \quad t \geqslant 0, \ i = 1, 2, \ldots, \tag{2.1}$$

are independent and identically distributed Brownian motion processes with initial distribution μ. Let $\Phi_\alpha \equiv C_1^\alpha(R)$ be the collection of Hölder functions of index α, $0 < \alpha \leqslant 1$:

$$\Phi_\alpha = \{\phi : R \to R \mid |\phi(x)| \leqslant 1, \ |\phi(y) - \phi(x)| \leqslant |y - x|^\alpha, \ x, y \in R\}. \tag{2.2}$$

For $\phi \in \Phi_\alpha$ and $t \in [0, 1]$, consider the processes

$$\begin{aligned} \mathbb{Z}_n(t, \phi) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \{\phi(X_i(t)) - E\phi(X_i(t))\} \\ &= \mathbb{G}_n(f_{t,\phi}), \end{aligned} \tag{2.3}$$

where

$$f_{t,\phi}(x) = \phi(x(t)) \quad \text{for } x \in C[0, 1], \ t \in [0, 1], \ \phi \in \Phi_\alpha, \tag{2.4}$$

and $\mathbb{G}_n$ is the empirical process of $X_1, \ldots, X_n$:

$$\mathbb{P}_n = \frac{1}{n} \sum_{i=1}^n \delta_{X_i} \quad \text{and} \quad \mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - P), \tag{2.5}$$

where P, a Borel probability measure on $C[0, 1]$, is the law of the sample continuous process X_1. Let

$$\mathscr{F}_\alpha = \{f_{t,\phi} : \phi \in \Phi_\alpha, \ t \in [0, 1]\}, \tag{2.6}$$

and, for $f, g \in \mathscr{F}_\alpha$, let $\rho_P^2(f, g) \equiv Var_P(f(X_1) - g(X_1))$.

We want to use Ossiander's bracketing theorem to show that the class of functions $\mathscr{F}_\alpha$ is P-Donsker. To this end, we will first use an observation of Van der Vaart (1994) (see also Van der Vaart and Wellner (1996), corollary 2.7.4, page 158) to construct for every $\varepsilon > 0$ a "small" set of brackets for Φ_α of $L_2(Q_t)$-size less than ε for all the probability measures $Q_t = \mu * \mathrm{N}(0, t) = \mathscr{L}(X_1(t))$, $t \in [0, 1]$. We recall the definition of a bracket $[\phi, \psi]$ of a collection of functions $\Psi_\alpha \subset L_2(Q)$ for a law $Q : [\phi, \psi] = \{f \in \Psi_\alpha : \phi \leqslant f \leqslant \psi \ \text{pointwise}\}$, and the $L_2(Q)$ size of the bracket is $\sqrt{Q(\psi - \phi)^2}$; see e.g. Dudley (1984) or Van der Vaart and Wellner (1996), page 83. This requires that the function f in Ψ_α be in $L_2(Q)$; in our case $\Psi_\alpha = \mathscr{F}_\alpha$ consists of bounded functions, in particular showing that the finite dimensional distributions of $\mathbb{G}_n$ converge in law to the Gaussian process $G_P(f_{t,\phi})$ with the covariance of $f_{t,\phi}(X_1)$.

In Van der Vaart's (1994) notation, let $I_j = [j, j+1)$, $j \in \{\ldots, -2, -1, 0, 1, 2, \ldots\} \equiv \mathbb{Z}$, and let $a_j \in (0, \infty]$. Let $C_1^\alpha(I_j)$ denote the collection of Hölder functions of index $\alpha > 0$ on I_j. Let $f_{j1}, \ldots, f_{jp_j} \in C_1^\alpha(I_j)$ be an εa_j-net of $C_1^\alpha(I_j)$ for the uniform norm, $j \in \mathbb{Z}$. Van der Vaart (1994), example after Theorem 2.1 (or proof of Corollary 2.7.4 in Van der Vaart and Wellner, 1996, p. 158), observes that, for each $\varepsilon>0$, the cardinality J_ε of the set of brackets

$$\left[\sum_{j=-\infty}^{\infty}(f_{j,i_j}-\varepsilon a_j)1_{I_j},\ \sum_{j=-\infty}^{\infty}(f_{j,i_j}+\varepsilon a_j)1_{I_j}\right],\quad i_j\in\{1,\dots,p_j\}\quad \text{for each } j\in\mathbb{Z}, \tag{2.7}$$

satisfies

$$\log J_\varepsilon \leqslant K\left(\frac{1}{\varepsilon}\right)^V \sum_j \frac{1}{a_j^V} \tag{2.8}$$

for any $V\geqslant 1/\alpha$ and a constant K depending on V, and that the $L_2(Q_t)$ size of these brackets is (obviously)

$$2\varepsilon\left(\sum_j a_j^2 Q_t(I_j)\right)^{1/2}, \tag{2.9}$$

assuming that the series in (2.8) and (2.9) are both finite. Fix $0<\delta\leqslant 2-1/\alpha$; we take $V=2-\delta$ and $a_j=|j|^{1/(2-2\delta)}$. Then

$$\sum_j \frac{1}{a_j^V}<\infty$$

and

$$\begin{aligned}\sum_j a_j^2 Q_t(I_j) &= \sum_j |j|^{1/(1-\delta)}P(Y+B(t)\in I_j)\\ &\leqslant E|Y+B(t)|^{1/(1-\delta)}\\ &\leqslant 2^{\delta/(1-\delta)}(E|B(1)|^{1/(1-\delta)}+E|Y|^{1/(1-\delta)})<\infty\end{aligned}$$

if $E|Y|^{1/(1-\delta)}<\infty$. We label the brackets in (2.7) as $[\phi_{mL},\phi_{mU}]$, $m\in\{1,\dots,J_\varepsilon\}$.

Suppose now we partition $[0,1]$ into K_ε intervals $[t_k,t_{k+1}]$ to be chosen later. For $\phi\in[\phi_{mL},\phi_{mU}]$ and $t\in[t_k,t_{k+1}]$ we have

$$\begin{aligned}\phi(B(t)+Y) &= \phi(B(t_k)+Y)+\phi(B(t)+Y)-\phi(B(t_k)+Y)\\ &\leqslant \phi_{mU}(B(t_k)+Y)+|B(t)-B(t_k)|^\alpha\\ &\leqslant \phi_{mU}(B(t_k)+Y)+\Delta_k\equiv U_{mk},\end{aligned} \tag{2.10}$$

where

$$\Delta_k\equiv \sup_{t\in[t_k,t_{k+1}]}|B(t)-B(t_k)|^\alpha. \tag{2.11}$$

Similarly, a lower bound is given by

$$\phi(B(t)+Y)\geqslant \phi_{mL}(B(t_k)+Y)-\Delta_k\equiv L_{mk}. \tag{2.12}$$

Note that by Lévy's inequality

$$
\begin{aligned}
E\Delta_k^2 &= E \sup_{t\in[t_k,t_{k+1}]} |B(t)-B(t_k)|^{2\alpha} \\
&= E \sup_{s\in[0,t_{k+1}-t_k]} |B(s)|^{2\alpha} \leqslant 2E|B(t_{k+1}-t_k)|^{2\alpha} \\
&= 2M_\alpha(t_{k+1}-t_k)^\alpha = \varepsilon^2
\end{aligned}
$$

if the partition $[t_k, t_{k+1}]$ of $[0,1]$ is chosen with $t_k = k(\varepsilon^2/2M_\alpha)^{1/\alpha}$. Thus, we have at most

$$
K = K_\varepsilon \leqslant (3M_\alpha)^{1/\alpha}\left(\frac{1}{\varepsilon}\right)^{2/\alpha}
$$

such intervals for every $0<\varepsilon\leqslant 1$. Furthermore, we have

$$
U_{mk} - L_{mk} = 2\varepsilon \sum_j a_j 1_{I_j}(B(t_k)+Y) + 2\Delta_k,
$$

so that

$$
\begin{aligned}
\{E(U_{mk}-L_{mk})^2\}^{1/2} &\leqslant \left\{E\Big(2\varepsilon\sum_j a_j 1_{I_j}(B(t_k)+Y)\Big)^2\right\}^{1/2} + 2\{E\Delta_k^2\}^{1/2} \\
&= \left\{4\varepsilon^2 \sum_j |j|^{1/(1-\delta)} \Pr\{Y_1 + B_1(t_k)\in I_j\}\right\}^{1/2} + 2\{E\Delta_k^2\}^{1/2} \\
&\leqslant 2\varepsilon\{2^{\delta/(1-\delta)}(E|B(1)|^{1/(1-\delta)} + E|Y|^{1/(1-\delta)})\}^{1/2} + 2\varepsilon \\
&= K\varepsilon
\end{aligned}
$$

for a fixed constant $K<\infty$. Hence, the $L_2(P)$ size of the brackets $[L_{mk}, U_{mk}]$ does not exceed $K\varepsilon$, whereas their number is at most

$$
J_\varepsilon K_\varepsilon \leqslant \exp(K(1/\varepsilon)^V)K'\left(\frac{1}{\varepsilon}\right)^{2/\alpha},
$$

that is

$$
\log N_{[\,]}(\varepsilon, \mathscr{F}_\alpha, L_2(P)) \leqslant \widetilde{K}\left(\frac{1}{\varepsilon}\right)^V. \tag{2.13}
$$

Finite-dimensional convergence follows from the fact that our bracketing argument implies that the collection $\mathscr{F}_\alpha$ has an envelope F which is square integrable, or, alternatively from Ito (1983, Theorem 6.1, part (i), p. 27). Since the bound (2.13) together with finite-dimensional convergence imply the hypotheses of Ossiander's (1987) bracketing CLT for the empirical processes $\mathbb{G}_n$ (also see Van der Vaart and Wellner 1996, Section 2.5.2, pp. 129–133), we have proved the following theorem.

Theorem 1. *Suppose that $E|Y|^r<\infty$ for some $r>1$, and $\alpha>\frac{1}{2}$. Then $\mathbb{Z}_n \rightsquigarrow \mathbb{Z}$ in $l^\infty(\mathscr{F}_\alpha)$ where $\mathbb{Z}(t,\phi) = \mathbb{G}_P(f_{t,\phi})$ is a P-Brownian bridge process, uniformly contin-*

uous with respect to ρ_P, indexed by the collection $\mathscr{F}_\alpha$: i.e. a mean zero Gaussian process with covariance function

$$\mathrm{Cov}(\mathbb{Z}(s,\phi),\mathbb{Z}(t,\psi)) = E(\phi(B(s)+Y)\psi(B(t)+Y)) \\ -E(\phi(B(s)+Y))E(\psi(B(t)+Y)).$$

Remark. (1) $[0,1]$ and $C_1^\alpha(R)$ can be replaced, in the definition of $\mathscr{F}_\alpha$, by $[0,M]$ and $C_L^\alpha(R)$ for any M and L finite.

(2) It is easy to see that $\rho_P^2(f_{s,\phi},f_{t,\psi}) \leqslant 2|t-s| + 2\|\phi-\psi\|_\infty^2$. Hence, the limit process $\mathbb{Z}(t,\phi)$, defined for all $t \in R^+ \cup \{0\}$ and $\phi \in C^\alpha(R)$, has a version whose sample paths are d-uniformly continuous on every set $[0,M] \times C_L^\alpha(R)$, $M,L<\infty$, for the distance $d((s,\phi),(t,\psi)) = |t-s| \vee \|\phi-\psi\|_\infty$.

(3) The integrability condition in the previous theorem is not far from best possible: as shown in Arcones (1994) – see also Giné and Zinn (1986) for $\alpha=1$ – the condition $\sum_j P(Y\in[j,j+1))^{1/2}<\infty$ is necessary and sufficient for $C_1^\alpha(R)$ to be $\mathscr{L}(Y)$-Donsker, $1/2<\alpha\leqslant 1$; this condition is implied by $E|Y|^\tau<\infty$ if $\tau>1$, and implies $E|Y|<\infty$ if moreover the sequence $P(|Y|\in[j,j+1))$ is eventually non-increasing.

3. A limit theorem for branching Brownian motion particles

Suppose a particle starts at a random position at time $t=0$, performs a Brownian motion path during an exponentially distributed time, and then either dies or splits into two (with probability $\frac{1}{2}$ for each possibility); if it splits into two, these two particles in turn perform Brownian paths again during exponentially distributed times after which they either split into two or die, and so on. We assume no interaction, i.e. the initial positions, Brownian motion paths, lifetimes, are all independent. We are interested in the asymptotic distribution (as $n\to\infty$) of the total number of descendents of n independent such particles that are alive and in $A\subset R$ at time t, $\eta_t^n(A)$. Typically the literature (Walsh, 1986, Ch. 8 and references therein) considers the limit in distribution of the action of the random measures η_t^n on smooth rapidly decreasing functions,

$$\frac{\eta_t^n(\phi)-E\eta_t^n(\phi)}{\sqrt{n}} \tag{3.1}$$

(where $\eta(\phi)=\int\phi\,\mathrm{d}\eta$) as processes in $t\in[0,1]$ for a fixed function ϕ. The model we consider differs from Walsh's in that we start with n particles at i.i.d. positions Y_i, $i=1,\dots,n$, instead of with infinitely many according to a Poisson point process, and that the exponential times for our processes do not vary with n. We show that, for this model, the processes (3.1) converge in law uniformly in t and ϕ (as opposed to just uniformly in t), for ϕ in the unit ball of $C_1^\gamma(R)$ for some $\gamma>\frac{1}{2}$ (as opposed to C^∞, rapidly decreasing ϕ) if the common initial distribution of the particles satisfies the moment condition $E|Y|^\tau<\infty$ for some $\tau>1$.

We essentially follow Walsh for the description of the Branching Brownian motion process. Let $\mathscr{A}$ be the set of multiindices

$$\mathscr{A}=\{\alpha=(\alpha_1,\dots,\alpha_p): p\in\mathbb{N}, \alpha_1\in\mathbb{N}, \alpha_r\in\{1,2\} \text{ for } r\geqslant 2\}.$$

We define $|\alpha|=p$ if $\alpha=(\alpha_1,\ldots,\alpha_p)$, and $\alpha\leqslant\alpha'$ if $|\alpha|\leqslant|\alpha'|$ and $\alpha_1=\alpha'_1,\ldots,\alpha_{|\alpha|}=\alpha'_{|\alpha|}$. For $\alpha\in\mathscr{A}$, define the "predecessors" of α by $\alpha-1=(\alpha_1,\ldots,\alpha_{p-1}),\alpha-2=(\alpha_1,\ldots,\alpha_{p-2})$, $\ldots,\alpha-(|\alpha|-1)=\alpha_1$ (e.g. if $\alpha=(4,1,2)$, then $\alpha-1=(4,1)$, and $\alpha-2=4$). (Walsh (1986) seems to contain a slight error in the description of the set $\mathscr{A}$, allowing $\alpha_j\in\mathbb{N}$ for $2\leqslant j\leqslant p$). We also set

$$\mathscr{A}_{k,p}=\{\alpha\in\mathscr{A}:\alpha\geqslant k,\ |\alpha|=p\},\qquad \mathscr{A}_k=\{\alpha\in\mathscr{A}:\alpha\geqslant k\}=\bigcup_{p=1}^{\infty}\mathscr{A}_{k,p},$$

and note that $\mathrm{Card}(\mathscr{A}_{k,p})=2^{p-1}$, $p\in\mathbb{N}$. Define now the following independent collections of random variables and processes:

$\{Y_k:k\in\mathbb{N}\}$, i.i.d. real random variables with law μ,

$\{B^\alpha_t:t\in[0,1],\alpha\in\mathscr{A}\}$, i.i.d. standard one-dimensional Brownian motions starting at 0,

$\{S^\alpha:\alpha\in\mathscr{A}\}$, i.i.d. exponential τ random variables, and

$\{N^\alpha:\alpha\in\mathscr{A}\}$, i.i.d. with $P(N^\alpha=0)=P(N^\alpha=2)=\frac{1}{2}$.

These variables, all independent, are the building blocks of our branching Brownian motions. The *birth time* of the α-th particle is defined as $\beta(\alpha)=0$ if $|\alpha|=1$, and, for $|\alpha|>1$,

$$\beta(\alpha)=\begin{cases}\sum_{j=1}^{|\alpha|-1}S^{\alpha-j} & \text{if } N^{\alpha-j}=2 \quad \text{for } j=1,\ldots,|\alpha|-1,\\ \infty & \text{otherwise,}\end{cases}$$

and its death time by

$$\zeta(\alpha)=\beta(\alpha)+S^\alpha.$$

We also set $h^\alpha(t)=1_{[\beta(\alpha),\zeta(\alpha))}(t)$. Then, letting ∂ be the cemetery (or, more accurately, limbo) we set

$$X^\alpha(t)\equiv X^\alpha_t=\begin{cases}\partial & \text{if } t\notin[\beta(\alpha),\zeta(\alpha))\\ X^\alpha(\beta(\alpha))+\int_0^t h^\alpha(s)\,\mathrm{d}B^\alpha_s & \text{if } t\in[\beta(\alpha),\zeta(\alpha)),\end{cases}\tag{3.2}$$

where

$$X^\alpha(\beta(\alpha))=Y_{\alpha-(|\alpha|-1)}+\sum_{i=1}^{|\alpha|-1}(B^{\alpha-i}_{\zeta(\alpha-i)}-B^{\alpha-i}_{\beta(\alpha-i)}).\tag{3.3}$$

With the convention that for all functions ϕ and sets $A\subset R$, $\phi(\partial)=0=1_A(\partial)$, the random measure η^n_t is defined as

$$\eta^n_t=\sum_{k=1}^{n}\sum_{\alpha\in\mathscr{A}_k}\delta_{X^\alpha_t},\quad n\in\mathbb{N},\ t\in[0,1].$$

That is,

$$\eta_t^n(\phi) = \sum_{k=1}^{n} \sum_{\alpha \in \mathscr{A}_k} \phi(X_t^\alpha).$$

In particular, $\eta_t^n(A)$ is the number of particles (starting from n particles at the positions $Y_1, \dots, Y_n$ at time $t=0$) that are alive and in A at time t.

Next, we translate this setup into the language of empirical process theory. We define (Ω, Σ, Q) as

$$\Omega := \prod_{k=1}^{\infty} \Omega_k := \prod_{k=1}^{\infty} (R \times C[0,1]^{\mathscr{A}_k} \times R^{\mathscr{A}_k} \times \{0,2\}^{\mathscr{A}_k}),$$

with σ-algebra Σ equal to the product of the corresponding Borel σ-algebras and with Q the product measure induced by the laws of Y_k, B^α, S^α, N^α, $k \in \mathbb{N}, \alpha \in \mathscr{A}$. We let T be the set of functions $f : [0,1] \to R \cup \{\partial\}$ for which there exist $0 \leqslant \beta \leqslant \zeta \leqslant 1$ such that $f(x) = \partial$ if $0 \leqslant x < \beta$ or $\zeta \leqslant x \leqslant 1$, and $f(x)$ is a continuous real function on $[\beta, \zeta)$. We set $\mathscr{A}' = \{1,2\} \cup \{1,2\}^2 \cup \cdots \cup \{1,2\}^p \cup \cdots$ and $S = T \times T^{\mathscr{A}'}$, and equip S with the σ-algebra that makes the maps (one sufices)

$$\mathbb{X}_k : \Omega_k \to S = T \times T^{\mathscr{A}'},$$
$$\mathbb{X}_k(y_k, b_t^\alpha, s^\alpha, n^\alpha) = (x_t^\alpha : \alpha \in k \cup \{(k, \alpha') : \alpha' \in \mathscr{A}'\},\ t \in [0,1])$$

measurable, where x_t^α is defined as in (3.2) and (3.3) with Y_k, $B_t^\alpha, \dots$ replaced by y_k, $b_t^\alpha, \dots$. We also equip S with the law of $\mathbb{X}_k$, $P = Q \circ \mathbb{X}_1^{-1}$. Then, obviously, the S-valued random variables $\mathbb{X}_k$ may be considered as the coordinate functions on a product probability space, each with law P. Next we define the following class of functions $\mathscr{F}$ on S:

$$\mathscr{F} = \{f_{t,\phi} : t \in [0,1],\ \phi \in C_1^\gamma\}$$

for some $\gamma > \frac{1}{2}$, where

$$f_{t,\phi}(x) = \sum_{\alpha} \phi(x_t^\alpha), \quad x \in S.$$

For instance,

$$f_{t,\phi}(\mathbb{X}_j) = \sum_{\alpha \geqslant j} \phi(X_t^\alpha),$$

and therefore,

$$\mathbb{Z}_n(t,\phi) \equiv \frac{\eta_t^n(\phi) - E\eta_t^n(\phi)}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (f_{t,\phi}(\mathbb{X}_j) - Pf_{t,\phi})$$

is the *empirical process* for the i.i.d. sequence $\{\mathbb{X}_i\}$ indexed by the class of functions $\mathscr{F}$. For ease of notation we will write

$$\kappa_t^j(\phi) = f_{t,\phi}(\mathbb{X}_j), \qquad j \in \mathbb{N},\ t \in [0,1],\ \phi \in C_1^\gamma(R),$$

and, if $\mathscr{L}(\mathbb{X}) = \mathscr{L}(\mathbb{X}_j)$, $\kappa_t(\phi) = f_{t,\phi}(\mathbb{X})$.

Theorem 2. *Suppose that* $E|Y|^{\lambda}<\infty$ *for some* $\lambda>1$, *and* $\gamma>\frac{1}{2}$. *Then* $\mathbb{Z}_n \rightsquigarrow \mathbb{Z}$ *in* $l^{\infty}(\mathscr{F})$ *where* $\mathbb{Z}(t,\phi)=\mathbb{G}_P(f_{t,\phi})$ *is a P-Brownian bridge process, uniformly continuous with respect to* ρ_P, *indexed by the collection* $\mathscr{F}$, *i.e. a mean zero Gaussian process with covariance function*

$$\mathrm{Cov}(\mathbb{Z}(s,\phi),\mathbb{Z}(t,\psi))=E(\kappa_s(\phi)\kappa_t(\psi))-E(\phi(B(s)+Y))E(\psi(B(t)+Y))$$

for $\phi,\psi\in C_1^{\gamma}(R)$.

An upper bound for the L_2-distance between $\mathbb{Z}(t,\phi)$ and $\mathbb{Z}(s,\psi)$ is given at the end of this section. Computing its exact value requires cumbersome computations which we omit.

Proof. As in the simpler case considered in the previous section, we will deduce the central limit theorem for the processes $\eta_t^n(\phi)$ from the bracketing CLT for empirical processes. To verify the hypotheses of the bracketing CLT for $\mathscr{F}$ and P, we proceed by analogy with Ito's case. First we must show that $E(\kappa_t(\phi))^2<\infty$ for all $t\in[0,1]$, $\phi\in C_1^{\gamma}(R)$. Note that, since $\mathbb{X}^{\alpha}(t)\neq\partial$ only if $h^{\alpha}(t)\neq 0$,

$$\begin{aligned}
E\left(\sum_{\alpha\geqslant 1}\phi(X^{\alpha}(t))\right)^2 &= \sum_{\alpha\geqslant 1}E\phi^2(X^{\alpha}(t))+\sum_{\alpha\neq\alpha'\geqslant 1}E\phi(X^{\alpha}(t))\phi(X^{\alpha'}(t))\\
&\leqslant \sum_{\alpha\geqslant 1}Eh^{\alpha}(t)+\sum_{\alpha\neq\alpha'\geqslant 1}Eh^{\alpha}(t)h^{\alpha'}(t)\\
&\leqslant \sum_{\alpha\geqslant 1}Eh^{\alpha}(t)+\sum_{\alpha\neq\alpha'\geqslant 1}\{Eh^{\alpha}(t)Eh^{\alpha'}(t)\}^{1/2}\\
&= \left\{\sum_{\alpha\geqslant 1}\{Eh^{\alpha}(t)\}^{1/2}\right\}^2.
\end{aligned}$$

and we show below (see also Walsh, 1986), that $\sum_{\alpha\geqslant 1}Eh^{\alpha}(t)=1$ and $\sum_{\alpha\geqslant 1}\{Eh^{\alpha}(t)\}^{1/2}<\infty$. With the same notation as in the last section, we have the following analogues of (1.10) and (1.12):

$$\begin{aligned}
\kappa_t(\phi) &= \kappa_{t_l}(\phi)+(\kappa_t(\phi)-\kappa_{t_l}(\phi))\\
&\leqslant \kappa_{t_l}(\phi_{mU})+\sup_{t\in[t_l,t_{l+1}]}\sup_{\phi\in C_1^{\gamma}(R)}|\kappa_t(\phi)-\kappa_{t_l}(\phi)|\\
&\equiv \kappa_{t_l}(\phi_{mU})+\Delta_l
\end{aligned}$$

(for the first inequality, note that κ_{t_l} is monotone in ϕ) and

$$\kappa_t(\phi)\geqslant\kappa_{t_l}(\phi_{mL})-\Delta_l.$$

Hence, as in Section 2, in order to prove that

$$\log N_{[\,]}(\varepsilon,\mathscr{F},L_2(P))\leqslant\tilde{K}\left(\frac{1}{\varepsilon}\right)^V$$

with $V=2-\delta$, $0<\delta\leqslant 2-1/\gamma$, it suffices to show

$$E\Delta_l^2\leqslant K|t_{l+1}-t_l|^\gamma \tag{3.4}$$

and

$$E\{\kappa_{t_l}(\phi_{mU})-\kappa_{t_l}(\phi_{mL})\}^2\leqslant K\varepsilon^2 \tag{3.5}$$

for fixed constants K. To ease notation, set $t_l:=t_0$, $t_{l+1}:=t_0+h$. We first observe that, since $\phi(\partial)=0$,

$$\begin{aligned}
&\sup_{\phi\in C_1^\gamma(R)}|\kappa_t(\phi)-\kappa_{t_0}(\phi)|\\
&=\sup_{\phi\in C_1^\gamma(R)}\left|\sum_{\alpha\geqslant k}[\phi(X_t^\alpha)-\phi(X_{t_0}^\alpha)]\right|\\
&=\sup_{\phi\in C_1^\gamma(R)}\left|\sum_{\alpha\geqslant k}[\phi(X_t^\alpha)-\phi(X_{t_0}^\alpha)]h^\alpha(t)h^\alpha(t_0)+\phi(X_t^\alpha)h^\alpha(t)(1-h^\alpha(t_0))\right.\\
&\qquad\left.-\phi(X_{t_0}^\alpha)(1-h^\alpha(t))h^\alpha(t_0)\right|\\
&\leqslant\sum_{\alpha\geqslant k}|X_t^\alpha-X_{t_0}^\alpha|^\gamma h^\alpha(t)h^\alpha(t_0)+\sum_{\alpha\geqslant k}h^\alpha(t)(1-h^\alpha(t_0))+\sum_{\alpha\geqslant k}(1-h^\alpha(t))h^\alpha(t_0)\\
&\equiv A_t^l+C_t^l+D_t^l.
\end{aligned}$$

Hence,

$$E\Delta_l^2\leqslant 4\left\{E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h}A_t^l\right)^2\right]+E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h}C_t^l\right)^2\right]+E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h}D_t^l\right)^2\right]\right\}.$$

Note that

$$\begin{aligned}
&E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h}A_t^l\right)^2\right]\\
&\leqslant E\sup_{t_0\leqslant t\leqslant t_0+h}\left\{\sum_{\alpha\geqslant k}|X_t^\alpha-X_{t_0}^\alpha|^{2\gamma}h^\alpha(t)h^\alpha(t_0)\right\}\\
&\quad+E\sup_{t_0\leqslant t\leqslant t_0+h}\left\{\sum_{\alpha\neq\alpha'\geqslant k}|X_t^\alpha-X_{t_0}^\alpha|^\gamma|X_t^{\alpha'}-X_{t_0}^{\alpha'}|^\gamma h^\alpha(t)h^\alpha(t_0)h^{\alpha'}(t)h^{\alpha'}(t_0)\right\}\\
&\equiv I+II.
\end{aligned}$$

Now we note that

$$\begin{aligned}
|X_t^\alpha-X_{t_0}^\alpha|^{2\gamma}h^\alpha(t)h^\alpha(t_0)&=\left|\int_{t_0}^t h^\alpha(s)\,\mathrm{d}B_s^\alpha\right|^{2\gamma}h^\alpha(t)h^\alpha(t_0)\\
&=|B^\alpha(\zeta(\alpha)\wedge t)-B^\alpha(\beta(\alpha)\vee t_0)|^{2\gamma}h^\alpha(t)h^\alpha(t_0)\\
&=|B^\alpha(t)-B^\alpha(t_0)|^{2\gamma}h^\alpha(t)h^\alpha(t_0)
\end{aligned}$$

and therefore, by independence of B^α and h^α, and by the properties of Brownian motion,

$$\begin{aligned} I &\leqslant \sum_{\alpha\geqslant k} E \sup_{t_0\leqslant t\leqslant t_0+h} |B^\alpha(t)-B^\alpha(t_0)|^{2\gamma} Eh^\alpha(t_0) \\ &= c_\gamma h^\gamma \sum_{\alpha\geqslant k} Eh^\alpha(t_0) \end{aligned}$$

for some constant $c_\gamma < \infty$. Now

$$\sum_{\alpha\geqslant k} Eh^\alpha(t_0) = \sum_{p=1}^{\infty} \sum_{\alpha\in\mathscr{A}_{k,p}} Eh^\alpha(t_0)$$

and, if N_t is a Poisson process with intensity τ, then for $\alpha\in\mathscr{A}_{k,p}$,

$$\begin{aligned} Eh^\alpha(t_0) &= \frac{1}{2^{p-1}} P(S^{\alpha-(p-1)}+\cdots+S^{\alpha-1}\leqslant t_0 < S^{\alpha-(p-1)}+\cdots+S^{\alpha-1}+S^\alpha) \\ &= \frac{1}{2^{p-1}} P(N_{t_0}=p-1) = \frac{1}{2^{p-1}} \mathrm{e}^{-\tau t_0}\frac{(\tau t_0)^{p-1}}{(p-1)!}, \end{aligned} \tag{3.6}$$

which, since Card $(\mathscr{A}_{k,p})=2^{p-1}$, gives

$$\sum_{\alpha\geqslant k} Eh^\alpha(t_0) = \sum_{p=1}^{\infty} \mathrm{e}^{-\tau t_0}\frac{(\tau t_0)^{p-1}}{(p-1)!} = 1.$$

We thus conclude that $I\leqslant c_\gamma h^\gamma$. To bound II, note that independence of $\{B^\alpha\}$ and $\{h^\alpha\}$ yields

$$\begin{aligned} II &\leqslant \sum_{\alpha\neq\alpha'\geqslant k} E \sup_{t_0\leqslant t\leqslant t_0+h} |B_t^\alpha - B_{t_0}^\alpha|^\gamma E \sup_{t_0\leqslant t\leqslant t_0+h} |B_t^{\alpha'} - B_{t_0}^{\alpha'}|^\gamma Eh^\alpha(t_0)h^{\alpha'}(t_0) \\ &\leqslant C_\gamma h^\gamma \sum_{\alpha\neq\alpha'\geqslant k} Eh^\alpha(t_0)h^{\alpha'}(t_0) \\ &\leqslant C_\gamma h^\gamma \sum_{\alpha\neq\alpha'\geqslant k} \sqrt{Eh^\alpha(t_0)\,Eh^{\alpha'}(t_0)} \quad \text{by Cauchy–Schwarz} \\ &\leqslant C_\gamma h^\gamma \left[\sum_{\alpha\geqslant k}\sqrt{Eh^\alpha(t_0)}\right]^2. \end{aligned}$$

But for $\alpha\in\mathscr{A}_{k,p}$, $Eh^\alpha(t_0)$ is given by (3.6). Hence

$$\begin{aligned} \sum_{\alpha\geqslant k}\sqrt{Eh^\alpha(t_0)} &= \sum_{p=1}^{\infty}\sum_{\alpha\in\mathscr{A}_{k,p}} \left\{\frac{1}{2^{p-1}}\mathrm{e}^{-\tau t_0}\frac{(\tau t_0)^{p-1}}{(p-1)!}\right\}^{1/2} \\ &= \sum_{p=1}^{\infty} \mathrm{e}^{-(\tau t_0)/2}\frac{(\sqrt{2\tau t_0})^{p-1}}{\sqrt{(p-1)!}} \\ &\leqslant \sum_{p=1}^{\infty} \frac{(\sqrt{2\tau})^{p-1}}{\sqrt{(p-1)!}} \\ &\equiv C_\tau < \infty. \end{aligned}$$

This yields $II \leqslant ch^{\gamma}$, and hence we have shown that

$$E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h} A_t^l\right)^2\right] \leqslant ch^{\gamma} \tag{3.7}$$

for some $c<\infty$.

Next we bound the C_t^l component of Δ_l:

$$\begin{aligned}
E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h} C_t^l\right)^2\right] &= E\sup_{t_0\leqslant t\leqslant t_0+h}\left(\sum_{\alpha\geqslant k} h^{\alpha}(t)(1-h^{\alpha}(t_0))\right)^2\\
&= E\sup_{t_0\leqslant t\leqslant t_0+h}\left(\sum_{\alpha\geqslant k} 1_{[0,\beta(\alpha))}(t_0)1_{[\beta(\alpha),\zeta(\alpha))}(t)\right)^2\\
&\leqslant E\left(\sum_{\alpha\geqslant k} 1_{\{t_0<\beta(\alpha)\leqslant t_0+h\}}\right)^2\\
&\leqslant \sum_{\alpha\geqslant k}\Pr(t_0<\beta(\alpha)\leqslant t_0+h)\\
&\quad+\sum_{\alpha\neq\alpha'\geqslant k} E(1_{[t_0<\beta(\alpha)\leqslant t_0+h]}1_{[t_0<\beta(\alpha')\leqslant t_0+h]})\\
&\leqslant \left\{\sum_{\alpha\geqslant k}[\Pr(t_0<\beta(\alpha)\leqslant t_0+h)]^{1/2}\right\}^2
\end{aligned}$$

by Cauchy–Schwarz and rearranging. Although it is possible to simply bound this last sum, it will probably be easier for the reader to follow the arguments if we proceed by first bounding the sum of the diagonal terms and then the sum of the off-diagonal terms. Now

$$\begin{aligned}
\sum_{\alpha\geqslant k}\Pr(t_0<\beta(\alpha)\leqslant t_0+h) &= \sum_{p=1}^{\infty}\sum_{\alpha\in\mathscr{A}_{k,p}}\Pr(t_0<\beta(\alpha)\leqslant t_0+h)\\
&= \sum_{p=2}^{\infty}\Pr(t_0<S_1+\cdots+S_{p-1}\leqslant t_0+h)
\end{aligned}$$

where S_i are i.i.d. exponential(τ). Setting $r=p-1$, and letting N_t be the Poisson process with intensity τ, we can write these probabilities as follows:

$$\begin{aligned}
\sum_{\alpha\geqslant k}\Pr(t_0<\beta(\alpha)\leqslant t_0+h) &= \sum_{r=1}^{\infty}\Pr(N_{t_0}<r,\ N_{t_0+h}-N_{t_0}\geqslant 1,\ N_{t_0+h}\geqslant r)\\
&= \sum_{r=1}^{\infty}\left(\sum_{s=1}^{r}\Pr(N_{t_0+h}-N_{t_0}=s,\ r-s\leqslant N_{t_0}<r)\right.\\
&\quad\left.+\sum_{s=r+1}^{\infty}\Pr(N_{t_0+h}-N_{t_0}=s, N_{t_0}<r-s)\right)
\end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{s=1}^{\infty}\sum_{r=s}^{\infty} \Pr(N_h = s)\Pr(r-s \leqslant N_{t_0} < r) \\
&\quad + \sum_{s=1}^{\infty}\sum_{r=1}^{s-1} \Pr(N_h = s) \\
&\leqslant \sum_{s=1}^{\infty} \Pr(N_h = s)\sum_{r=s}^{\infty} \Pr(r-s \leqslant N_{t_0}) + \sum_{s=1}^{\infty} s\Pr(N_h = s) \\
&= \Pr(N_h \geqslant 1)EN_{t_0} + EN_h = (1-\mathrm{e}^{-\tau h})t_0\tau + \tau h \\
&\leqslant (\tau^2 t_0 + \tau)h.
\end{aligned}$$

Along these same lines,

$$\begin{aligned}
&\sum_{\alpha \geqslant k} [\Pr(t_0 < \beta(\alpha) \leqslant t_0 + h)]^{1/2} \\
&= \sum_{p=2}^{\infty} \sum_{\alpha \in \mathscr{A}_{k,p}} [\Pr(t_0 < \beta(\alpha) \leqslant t_0 + h)]^{1/2} \\
&= \sum_{p=2}^{\infty} 2^{p-1} \left[\frac{1}{2^{p-1}} \Pr(N_{t_0} < p-1, N_{t_0+h} - N_{t_0} \geqslant 1,\ N_{t_0+h} \geqslant p-1)\right]^{1/2} \\
&= \sum_{p=2}^{\infty} 2^{(p-1)/2} \left(\sum_{s=1}^{p-1} \Pr(N_{t_0+h} - N_{t_0} = s, p-1-s \leqslant N_{t_0} < p-1)\right. \\
&\quad \left. + \sum_{s=p}^{\infty} \Pr(N_{t_0+h} - N_{t_0} = s, N_{t_0} < p-1-s)\right)^{1/2} \\
&\leqslant \sum_{p=2}^{\infty} 2^{(p-1)/2} \sum_{s=1}^{p-1} \Pr(N_h = s)^{1/2} \Pr(p-1-s \leqslant N_{t_0} < p-1)^{1/2} \\
&\quad + \sum_{p=2}^{\infty} 2^{(p-1)/2} \sum_{s=p}^{\infty} \Pr(N_h = s)^{1/2} \Pr(N_{t_0} < p-1-s)^{1/2} \\
&\leqslant \sum_{s=1}^{\infty}\sum_{p=2}^{\infty} 1_{[s \leqslant p-1]} 2^{(p-1)/2} \left\{\mathrm{e}^{-\tau h}\frac{(\tau h)^s}{s!}\right\}^{1/2} \Pr(N_{t_0} > p-s-1)^{1/2} \\
&\quad + \sum_{s=1}^{\infty}\sum_{p=2}^{\infty} 1_{[s \geqslant p]} 2^{(p-1)/2} \left\{\mathrm{e}^{-\tau h}\frac{(\tau h)^s}{s!}\right\}^{1/2} \\
&\leqslant C\mathrm{e}^{-\tau h/2} \sum_{s=1}^{\infty} \frac{(\sqrt{2\tau h})^s}{\sqrt{s!}} \\
&\leqslant Ch^{1/2}
\end{aligned}$$

for some constant C depending on τ. Hence, for some constant C_τ

$$E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h} C_t^l\right)^2\right] \leqslant C_\tau h. \tag{3.8}$$

Finally, the D_t^l component of Δ_l can be similarly bounded as follows:

$$\begin{aligned} E\left[\left(\sup_{t_0\leqslant t\leqslant t_0+h} D_t^l\right)^2\right] &= E\sup_{t_0\leqslant t\leqslant t_0+h}\left(\sum_{\alpha\geqslant k} h^\alpha(t_0)(1-h^\alpha(t))\right)^2 \\ &= E\sup_{t_0\leqslant t\leqslant t_0+h}\sum_{\alpha\geqslant k} h^\alpha(t_0)(1-h^\alpha(t)) \\ &\quad + E\sup_{t_0\leqslant t\leqslant t_0+h}\sum_{\alpha\neq\alpha'\geqslant k} h^\alpha(t_0)(1-h^\alpha(t))h^{\alpha'}(t_0)(1-h^{\alpha'}(t)). \end{aligned}$$

Since $t_0\leqslant t\leqslant t_0+h$, we have

$$\begin{aligned} h^\alpha(t_0)(1-h^\alpha(t)) &= 1_{[\beta(\alpha),\zeta(\alpha))}(t_0)1_{[\zeta(\alpha),\infty)}(t) \\ &\leqslant 1_{[\beta(\alpha),\zeta(\alpha))}(t_0)1_{[\zeta(\alpha),\infty)}(t_0+h), \end{aligned}$$

and hence the diagonal terms above are bounded by

$$\begin{aligned} E\sup_{t_0\leqslant t\leqslant t_0+h}\sum_{\alpha\geqslant k} h^\alpha(t_0)(1-h^\alpha(t)) &= \sum_{\alpha\geqslant k}\Pr(\beta(\alpha)\leqslant t_0<\zeta(\alpha)\leqslant t_0+h) \\ &= \sum_{p=2}^{\infty}\Pr(N_{t_0}=p-1,\ N_{t_0+h}\geqslant p) \\ &\leqslant \sum_{p=2}^{\infty}\Pr(N_{t_0+h}-N_{t_0}\geqslant 1)\Pr(N_{t_0}=p-1) \\ &= 1-\mathrm{e}^{-\tau h}\leqslant \tau h. \end{aligned}$$

Similarly,

$$\begin{aligned} &E\sup_{t_0\leqslant t\leqslant t_0+h}\sum_{\alpha\neq\alpha'\geqslant k} h^\alpha(t_0)(1-h^\alpha(t))h^{\alpha'}(t_0)(1-h^{\alpha'}(t)) \\ &\leqslant \sum_{\alpha\neq\alpha'\geqslant k} E1_{[\beta(\alpha)\leqslant t_0<\zeta(\alpha)\leqslant t_0+h]}1_{[\beta(\alpha')\leqslant t_0<\zeta(\alpha')\leqslant t_0+h]} \\ &\leqslant \sum_{\alpha\neq\alpha'\geqslant k}[\Pr(\beta(\alpha)\leqslant t_0<\zeta(\alpha)\leqslant t_0+h)\Pr(\beta(\alpha')\leqslant t_0<\zeta(\alpha')\leqslant t_0+h)]^{1/2} \\ &\leqslant \left\{\sum_{\alpha\geqslant k}[\Pr(\beta(\alpha)\leqslant t_0<\zeta(\alpha)\leqslant t_0+h)]^{1/2}\right\}^2 \end{aligned}$$

where

$$\begin{aligned} &\sum_{\alpha\geqslant k}[\Pr(\beta(\alpha)\leqslant t_0<\zeta(\alpha)\leqslant t_0+h)]^{1/2} \\ &\quad = \sum_{p=1}^{\infty}\sum_{\alpha\in\mathscr{A}_{k,p}}[\Pr(\beta(\alpha)\leqslant t_0<\zeta(\alpha)\leqslant t_0+h)]^{1/2} \end{aligned}$$

$$= \sum_{p=1}^{\infty} 2^{p-1} \left(\frac{1}{2^{p-1}} \Pr(N_{t_0} = p-1, N_{t_0+h} \geqslant p) \right)^{1/2}$$

$$= \sum_{p=1}^{\infty} 2^{(p-1)/2} \Pr(N_{t_0+h} - N_{t_0} \geqslant 1)^{1/2} \Pr(N_{t_0} = p-1)^{1/2}$$

$$\leqslant \sqrt{\tau h} \sum_{p=1}^{\infty} \mathrm{e}^{-\tau t_0/2} \frac{(\sqrt{2\tau t_0})^{p-1}}{\sqrt{(p-1)!}}$$

$$\leqslant C_\tau \sqrt{h}.$$

Therefore

$$E\left[\left(\sup_{t_0 \leqslant t \leqslant t_0+h} D_t^l\right)^2\right] \leqslant C_\tau h \tag{3.9}$$

for some $C_\tau < \infty$. The bounds (3.7)–(3.9), in combination with (3.6), prove the inequality (3.4).

Finally, we prove (3.5). We replace t_l by t_1 and k by 1. First note that from the definition of X_t^α the independence of the variables N, S, Y, B, and our previous calculations

$$E1_{[j,j+1)}(X_t^\alpha) = \frac{1}{2^{|\alpha|-1}} E1_{[j,j+1)}(B_t + Y) \Pr(S_1 + \cdots + S_{|\alpha|-1} \leqslant t < S_1 + \cdots + S_{|\alpha|})$$

$$= \mathrm{e}^{-\tau t} \frac{(\tau t/2)^{|\alpha|-1}}{(|\alpha|-1)!} \Pr(B_t + Y \in I_j).$$

Let $\phi(x) \equiv \sum_{-\infty}^{\infty} a_j 1_{[j,j+1)}(x)$. Then

$$E(\kappa_t(\phi_{mU}) - \kappa_t(\phi_{mL}))^2$$

$$= E\left\{ \sum_{\alpha \geqslant 1} 2\varepsilon \phi(X_t^\alpha) \right\}^2$$

$$\leqslant 4\varepsilon^2 \sum_{\alpha, \alpha' \geqslant 1} \{E\phi^2(X_t^\alpha)\}^{1/2} \{E\phi^2(X_t^{\alpha'})\}^{1/2}$$

$$= 4\varepsilon^2 \left\{ \sum_{\alpha \geqslant 1} \sqrt{E\phi^2(X_t^\alpha)} \right\}^2$$

$$= 4\varepsilon^2 \left\{ \sum_{\alpha \geqslant 1} \left[E\left(\sum_{j=-\infty}^{\infty} a_j 1_{[j,j+1)}(X_t^\alpha) \right)^2 \right]^{1/2} \right\}^2$$

$$= 4\varepsilon^2 \left\{ \sum_{\alpha \geqslant 1} \left[E \sum_{j=-\infty}^{\infty} a_j^2 1_{[j,j+1)}(X_t^\alpha) \right]^{1/2} \right\}^2$$

$$= 4\varepsilon^2 \left\{ \sum_{\alpha \geqslant 1} \left[\mathrm{e}^{-\tau t} \frac{(\tau t/2)^{|\alpha|-1}}{(|\alpha|-1)!} \right]^{1/2} \left[\sum_{j=-\infty}^{\infty} a_j^2 \Pr(B_t + Y \in I_j) \right]^{1/2} \right\}^2$$

$$= 4\varepsilon^2 \left(2e^{-\tau t/2} \sum_{p=1}^{\infty} \frac{(2\tau t)^{(p-1)/2}}{((p-1)!)^{1/2}} \right)^2 \sum_{j=-\infty}^{\infty} a_j^2 \Pr(B_t+Y \in I_j)$$

$$\leqslant C_\tau 4\varepsilon^2 \sum_{j=-\infty}^{\infty} a_j^2 \Pr(B_t + Y \in I_j)$$

which, under the same assumption on Y as in Section 1, and for $a_j = |j|^{1/(2-2\delta)}$, is dominated by a constant times ε^2, proving (3.5).

Finite-dimensional convergence follows from the fact that $E(\kappa_t(\phi))^2 < \infty$ for all $t \in [0,1]$, $\phi \in C_1^\gamma(R)$. □

The limiting Gaussian process $\{\mathbb{Z}(t,\phi) : t \in [0,1], \phi \in C_1^\gamma(R)\}$ is sample continuous with respect to its covariance structure

$$\rho_P^2((t,\phi),(s,\psi)) = E(\mathbb{Z}(t,\phi) - \mathbb{Z}(s,\psi))^2 = Var(\kappa_t^1(\phi) - \kappa_s^1(\psi)).$$

In order to determine the meaning of "continuity of $\mathbb{Z}$ with respect to ρ_P", and of the "tightness" condition on (the centered version of) the process $\eta_t^n(\phi)$ associated to the convergence in law given in Theorem 2 (see the introduction), we should have a sensible upper estimate on ρ_P. This will allow us to compare with the type of convergence and with the limit continuity properties implied by weak convergence in $D([0,1], \mathscr{S}')$. For $0 \leqslant s \leqslant t \leqslant 1$ and $\phi, \psi \in C_1^\gamma(R)$, we have

$$\rho_P(f_{t,\phi}, f_{s,\psi}) \leqslant \rho_P(f_{t,\phi}, f_{s,\phi}) + \rho_P(f_{s,\phi}, f_{s,\psi}).$$

The first term is bounded by $C|t-s|^{\gamma/2}$ by A, C, and D in the previous proof. For the second term, we proceed as in the proof of (3.5):

$$E\left(\sum_\alpha (\phi - \psi)(X_s^\alpha)\right)^2 \leqslant \left\{ \sum_\alpha (E(\phi-\psi)^2(X_s^\alpha))^{1/2} \right\}^2$$

$$= \left\{ \sum_\alpha \left(e^{-\tau s} \frac{(\tau s/2)^{|\alpha|-1}}{(|\alpha|-1)!} E[(\phi-\psi)^2(Y+B_t)] \right)^{1/2} \right\}^2$$

$$= e^{-\tau s/2} \left(\sum_{r=0}^{\infty} \frac{(2\tau s)^{r/2}}{\sqrt{r!}} \right)^2 E[(\phi-\psi)^2(Y+B_t)]$$

$$\leqslant C_\tau \|\phi - \psi\|_\infty^2.$$

The conclusion is

$$\rho_P((t,\phi),(s,\psi)) \leqslant K\{|t-s|^{\gamma/2} + \|\phi-\psi\|_\infty\}$$

with a considerable error made in the replacement of $\sup_{s\in[0,1]}\{E(\phi-\psi)^2(B_s+Y)\}^{1/2}$ by $\|\phi-\psi\|_\infty$. Hence, $\mathbb{Z}(t,\phi)$ is sample continuous with respect to the distance

$$d((t,\phi),(s,\psi)) = |t-s| \vee \|\phi-\psi\|_\infty,$$

and therefore, $[0,1] \times C_1^\gamma(R)$ being compact for d, there is a version of $\mathbb{Z}$ all of whose sample paths are uniformly continuous for the distance d on $[0,1] \times C_1^\gamma(R)$ (this can be extended to $[0,T] \times C_M^{\gamma}(R)$ for all finite T and M). The condition that a version of $\mathbb{Z}$ be in $C([0,1],\mathscr{S}')$ is much weaker. Likewise, the asymptotic equicontinuity condition for the process η_t^n, even with ρ_P replaced by d, is much stronger than the tightness given by Mitoma's (1983) theorem.

4. A limit theorem for interacting diffusion processes

Our goal in this section is to provide a strengthening of a central limit theorem of Tanaka and Hitsuda (1981) similar to the way Theorems 1 and 2 in the previous sections strengthen the theorems of Ito (1983) and Walsh (1986). The main difference in this section is that the particles interact.

Here is a description of the particle system considered by Tanaka and Hitsuda (1981). Consider the diffusion process $\{X^{(n)}(t): t \geqslant 0\} = \{(X_1^{(n)}(t),\dots,X_n^{(n)}(t)): t \geqslant 0\}$ in R^n with generator

$$K^{(n)}\phi = \frac{1}{2}\sum_{i=1}^{n}\frac{\partial^2\phi}{\partial x_i^2} + \sum_{i=1}^{n}\left(\frac{1}{n-1}\sum_{i\neq j} b(x_i,x_j)\right)\frac{\partial\phi}{\partial x_i}$$

and initial positions $X^{(n)}(0) = (Y_1,\dots,Y_n)$ where $Y_1, Y_2,\dots$ are i.i.d. with distribution μ on R. As in Tanaka and Hitsuda (1981), we will focus on the case in which the interaction function b is given by $b(x,y) = -\lambda(x-y)$ with $\lambda > 0$. Then $X^{(n)}(t)$ can be obtained as the solution of the stochastic integral equations

$$X_k^{(n)}(t) = Y_k + B_k(t) - \frac{n}{n-1}\lambda\int_0^t \left(X_k^{(n)}(s) - \frac{1}{n}\sum_{j=1}^{n} X_j^{(n)}(s)\right)\mathrm{d}s, \tag{4.1}$$

for $t \geqslant 0$, $k \in \{1,\dots,n\}$, where $Y_1, Y_2,\dots$ are i.i.d. μ and $B_1, B_2,\dots$ are i.i.d. standard Brownian motions starting at 0 independent of the Y's. Intuitively, as $n \to \infty$, the averages $n^{-1}\sum_{j=1}^{n} X_j^{(n)}(s)$ should converge to the mean v of the initial distribution, $v = \int x\mu(\mathrm{d}x)$, and the system of equations (4.1) should converge to the (non-interacting) system of equations

$$X_k(t) = Y_k + B_k(t) - \lambda\int_0^t (X_k(s) - v)\,\mathrm{d}s, \quad t \geqslant 0,\ k \in \{1,\dots,n\} \tag{4.2}$$

for the same initial positions and Brownian motions. As in Tanaka and Hitsuda (1981), we will assume, without loss of generality, that $v = 0$. Then Eqs. (4.2) are those governing a system of independent Ornstein–Uhlenbeck processes $X_k(t)$ with initial positions $X_k(0) = Y_k$. It is well-known that Eqs. (4.2) (with $v = 0$) have the solutions

$$X_k(t) = \mathrm{e}^{-\lambda t}Y_k + \int_0^t \mathrm{e}^{-\lambda(t-s)} dB_k(s), \quad t \geqslant 0; \tag{4.3}$$

see Tanaka and Hitsuda (1981, (2.8) p. 418) or Breiman (1968, Section 16.1, pp. 347–350), and especially (16.6) on p. 349 with Breiman's $(\gamma, \alpha, V_1(0))$ taken to be our $(1, \lambda, Y_k)$. Note that $X_k(t) = \mathrm{e}^{-\lambda t}Y_k + V_k(t)$ where $V_k(t) \equiv \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}B_k(s) \sim \mathrm{N}(0, \sigma_\lambda^2(t))$

and $\sigma_\lambda^2(t) = (1 - e^{-2\lambda t})/2\lambda$. Hence, the marginal distribution of $X_k(t)$ is $\mathscr{L}(X_t) = \mathscr{L}(e^{-\lambda t}Y) * N(0, \sigma_\lambda^2(t))$. Note that as $\lambda \to 0$ it follows that

$$\mathscr{L}(X_t) \to_d \mathscr{L}(Y) * N(0,t) = \mu * g_t \equiv \mu_t$$

in agreement with Section 2. On the other hand, $\mathscr{L}(X_t) \to N(0, 1/(2\lambda))$ as $t \to \infty$.

Now consider the processes

$$\mathbb{Z}_n(t,\phi) = \frac{1}{\sqrt{n}} \sum_{k=1}^n \{\phi(X_k^{(n)}(t)) - E\phi(X_k(t))\}$$

for $t \in [0,1]$, $\phi \in C_1^\gamma(R)$, $\gamma > 1$. Motivated by (4.1) and (4.2), our strategy will be to decompose $\mathbb{Z}_n$ as

$$\begin{aligned}\mathbb{Z}_n(t,\phi) &= \frac{1}{\sqrt{n}} \sum_{k=1}^n \{\phi(X_k^{(n)}(t)) - \phi(X_k(t))\} + \frac{1}{\sqrt{n}} \sum_{k=1}^n \{\phi(X_k(t)) - E\phi(X_k(t))\} \\ &\equiv \mathbb{Z}_n^{(1)}(t,\phi) + \mathbb{Z}_n^{(2)}(t,\phi).\end{aligned}$$

Note that $\mathbb{Z}_n^{(2)}$ is a process with i.i.d. summands

$$f_{t,\phi}^{(2)}(\mathbb{X}_k) - Pf_{t,\phi}^{(2)} = \phi(X_k(t)) - E\phi(X_k(t))$$

where $\mathbb{X}_k \equiv \{X_k(t) : 0 \leqslant t \leqslant 1\}$ are the i.i.d. Ornstein–Uhlenbeck processes given by (4.3), and P denotes the law of $\mathbb{X}_1$ on $C[0,1]$. Thus

$$\mathbb{Z}_n^{(2)}(t,\phi) = \mathbb{G}_n(f_{t,\phi}^{(2)})$$

where $\mathbb{P}_n = n^{-1} \sum_{k=1}^n \delta_{\mathbb{X}_k}$, $\mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - P)$ and

$$\mathscr{F}^{(2)} \equiv \{f_{t,\phi}(x) = \phi(x(t)) : \phi \in C_1^\gamma(R), t \in [0,1]\}.$$

On the other hand, the process $\mathbb{Z}_n^{(1)}$ involves the interactions between the particles in $X^{(n)}(t)$. It will turn out that the interaction term $\mathbb{Z}_n^{(1)}$ is asymptotically equivalent to $\mathbb{Z}_n^{(3)}$ where

$$\mathbb{Z}_n^{(3)}(t,\phi) = \mathbb{G}_n(f_{t,\phi}^{(3)}), \qquad f_{t,\phi}^{(3)} \in \mathscr{F}^{(3)}$$

and

$$\mathscr{F}^{(3)} \equiv \left\{ f_{t,\phi}^{(3)}(x) = \lambda(E\phi'(X_1(t))) \int_0^t x(s)\,ds : \phi \in C_1^\gamma(R), t \in [0,1] \right\}.$$

We therefore set

$$f_{t,\phi}(x) = \phi(x(t)) + \lambda(E\phi'(X_1(t))) \int_0^t x(s)\,ds$$

and consider the collection of functions

$$\mathscr{F} = \{f_{t,\phi} : \phi \in C_1^\gamma(R),\ t \in [0,1]\},$$

where $C_L^\gamma(R)$, $1<\gamma<2$, is defined as follows: $\phi \in C_L^\gamma(R)$ if $|\phi'(x)-\phi'(y)| \leqslant L|x-y|^{\gamma-1}$ for all $x, y \in R$, $\|\phi\|_\infty \leqslant L$, $\|\phi'\|_\infty \leqslant L$. Thus our goal is to prove the following theorem:

Theorem 3. *Suppose that* $\gamma > 1$, $EY_1 = 0$, *and* $EY_1^2 < \infty$. *Then* $\mathbb{Z}_n \rightsquigarrow \mathbb{Z}$ *in* $l^\infty(\mathscr{F})$ *where* $\mathbb{Z} \equiv \{\mathbb{Z}(t,\phi) : t \in [0,1], \phi \in C_1^\gamma(R)\} = \{\mathbb{G}_P(f_{t,\phi}) : f_{t,\phi} \in \mathscr{F}\}$ *is a mean 0 Gaussian process, uniformly continuous with respect to* ρ_P, *indexed by the collection* $[0,1] \times C_1^\gamma(R)$, *and with covariance function*

$$\begin{aligned}\operatorname{Cov}(\mathbb{Z}(s,\phi), \mathbb{Z}(t,\psi)) = \operatorname{Cov}\Bigg(&\phi(X_1(s)) + \lambda(E\phi'(X_1(s))) \int_0^s X_1(u)\,\mathrm{d}u,\\ &\psi(X_1(t)) + \lambda(E\psi'(X_1(t))) \int_0^t X_1(u)\,\mathrm{d}u\Bigg),\end{aligned}$$

X_1 *as defined by* (4.3) *for* $s,t \in [0,1]$, *and* $\phi, \psi \in C_1^\gamma(R)$.

Proof. Our proof will rely heavily on Lemma 2.1 of Tanaka and Hitsuda (1981, p. 417), which shows that

$$X_k^{(n)}(t) = X_k(t) + n^{-1/2} Y_k^{(n)}(t),$$

where

$$\begin{aligned}Y_k^{(n)}(t) = {}& \lambda \int_0^t Z^{(n)}(s)\,\mathrm{d}s + \frac{\lambda}{n-1} \int_0^t \exp\left\{-\frac{n\lambda(t-s)}{n-1}\right\} Z^{(n)}(s)\,\mathrm{d}s\\ &- \frac{n^{1/2}}{n-1} \int_0^t \exp\left\{-\frac{n\lambda(t-s)}{n-1}\right\} X_k(s)\,\mathrm{d}s\end{aligned}$$

and

$$Z^{(n)}(t) = \frac{1}{\sqrt{n}} \sum_{j=1}^n X_j(t).$$

Define a new sequence of processes $\mathbb{Z}_n^{(3)}$ by

$$\mathbb{Z}_n^{(3)}(t,\phi) = \lambda[E\phi'(X_1(t))] \int_0^t \sqrt{n}\,\overline{X}_n(s)\,\mathrm{d}s,$$

where $\overline{X}_n(t) \equiv n^{-1} \sum_{j=1}^n X_j(t)$. We first show that

$$R_n(t,\phi) \equiv \mathbb{Z}_n^{(1)}(t,\phi) - \mathbb{Z}_n^{(3)}(t,\phi)$$

satisfies

$$\sup_{t\in[0,1],\,\phi\in C_1^\gamma(R)} |R_n(t,\phi)| = \mathrm{o}_\mathrm{p}^*(1). \tag{4.4}$$

To prove (4.4), we further decompose R_n. Let $W_n(t) \equiv \lambda \int_0^t \sqrt{n}\,\overline{X}_n(s)\,\mathrm{d}s$. Then we can rewrite R_n as

$$\begin{aligned}
R_n(t,\phi) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \{\phi(X_k(t) + n^{-1/2} Y_k^{(n)}(t)) - \phi(X_k(t) + n^{-1/2} W_n(t))\} \\
&\quad + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \{\phi(X_k(t) + n^{-1/2} W_n(t)) - \phi(X_k(t)) - \phi'(X_k(t)) n^{-1/2} W_n(t)\} \\
&\quad + \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \{\phi'(X_k(t)) n^{-1/2} W_n(t) - [E\phi'(X_1(t))] n^{-1/2} W_n(t)\} \\
&\equiv A_n(t,\phi) + C_n(t,\phi) + D_n(t,\phi).
\end{aligned}$$

Now it follows from (4.4), since $\phi \in C_1^{\gamma}(R)$ implies that $|\phi(y) - \phi(x)| \leqslant |y - x|$, that we have

$$\begin{aligned}
|A_n(t,\phi)| &\leqslant \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \frac{1}{\sqrt{n}} |Y_k^{(n)}(t) - W_n(t)| \\
&\leqslant \frac{1}{n} \sum_{k=1}^{n} \left\{ \left| \frac{\lambda}{n-1} \int_0^t \exp\left\{ -\frac{n\lambda(t-s)}{n-1} \right\} \sqrt{n}\,\overline{X}_n(s)\,\mathrm{d}s \right| \right. \\
&\quad \left. + \left| \frac{\sqrt{n}}{n-1} \int_0^t \exp\left\{ -\frac{n\lambda(t-s)}{n-1} \right\} X_k(s)\,\mathrm{d}s \right| \right\} \\
&\leqslant \frac{\lambda}{n-1} \left| \int_0^t \exp\left\{ -\frac{n\lambda(t-s)}{n-1} \right\} \sqrt{n}\,\overline{X}_n(s)\,\mathrm{d}s \right| \\
&\quad + \frac{\sqrt{n}}{n-1} \int_0^t \exp\left\{ -\frac{n\lambda(t-s)}{n-1} \right\} \frac{1}{n} \sum_{k=1}^{n} |X_k(s)|\,\mathrm{d}s \\
&\leqslant \frac{\lambda+1}{(n-1)\sqrt{n}} \sum_{k=1}^{n} \int_0^t |X_k(s)|\,\mathrm{d}s,
\end{aligned}$$

where, by Doob's inequality,

$$\begin{aligned}
E \sup_{0 \leqslant s \leqslant 1} |X_k(s)| &\leqslant E|Y_1| + E \sup_{0 \leqslant s \leqslant 1} \left| \int_0^s \mathrm{e}^{\lambda u}\,\mathrm{d}B_1(u) \right| \\
&\leqslant E|Y_1| + \left\{ E\left(\sup_{0 \leqslant s \leqslant 1} \left| \int_0^s \mathrm{e}^{\lambda u}\,\mathrm{d}B_1(u) \right| \right)^2 \right\}^{1/2} \\
&\leqslant E|Y_1| + 2\left\{ E\left(\left| \int_0^1 \mathrm{e}^{\lambda u}\,\mathrm{d}B_1(u) \right| \right)^2 \right\}^{1/2} \\
&= E|Y_1| + 2\left\{ \int_0^1 \mathrm{e}^{2\lambda u}\,\mathrm{d}u \right\}^{1/2} < \infty.
\end{aligned}$$

Hence, it follows that

$$\sup_{t\in[0,1],\phi\in C_1^{\gamma}(R)} |A_n(t,\phi)| \leqslant \frac{\lambda+1}{(n-1)\sqrt{n}} \sum_{k=1}^{n} \sup_{0\leqslant s\leqslant 1} |X_k(s)| \to 0 \qquad \text{a.s.} \tag{4.5}$$

To handle C_n, we first use $\phi(y)-\phi(x)=\phi'(\tilde{x})(y-x)$ for some point $\tilde{x}$ with $|\tilde{x}-x|\leqslant|y-x|$ to rewrite C_n as

$$\begin{aligned} C_n(t,\phi) &= \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \{\phi'(\widetilde{X}_k(t)) - \phi'(X_k(t))\} n^{-1/2} W_n(t) \\ &= W_n(t) \frac{1}{n} \sum_{k=1}^{n} \{\phi'(\widetilde{X}_k(t)) - \phi'(X_k(t))\}. \end{aligned}$$

Hence, using $|\phi'(\tilde{x})-\phi'(x)|\leqslant|\tilde{x}-x|^{\gamma-1}\leqslant|y-x|^{\gamma-1}$, it follows that

$$\begin{aligned} |C_n(t,\phi)| &\leqslant |W_n(t)| \frac{1}{n} \sum_{k=1}^{n} |n^{-1/2} W_n(t)|^{\gamma-1} \\ &= n^{-(\gamma-1)/2} |W_n(t)|^{\gamma}. \end{aligned}$$

The process $W_n(t)$ has the same law as the process

$$(1-\mathrm{e}^{-\lambda t}) \frac{1}{\sqrt{n}} \sum_{k=1}^{n} Y_k + \lambda \int_0^t \mathrm{e}^{-\lambda(t-s)}\, \mathrm{d}B(s),$$

which is sample bounded on $[0,1]$, and therefore its absolute value has an a.s. finite supremum over $[0,1]$. Hence

$$\sup_{t\in[0,1],\phi\in C_1^{\gamma}(R)} |C_n(t,\phi)| = \mathrm{o}_{\mathrm{p}}^{*}(1). \tag{4.6}$$

Finally, to show that D_n is $\mathrm{o}_{\mathrm{p}}(1)$ uniformly in t and $\phi\in C_1^{\gamma}(R)$, note that

$$D_n(t,\phi) = W_n(t) \left\{ \frac{1}{n} \sum_{k=1}^{n} \phi'(X_k(t)) - E\phi'(X_1(t)) \right\}$$

and hence

$$\sup_{t\in[0,1],\phi\in C_1^{\gamma}(R)} |D_n(t,\phi)| \leqslant \sup_{0\leqslant t\leqslant 1} |W_n(t)| \|\mathbb{P}_n - P\|_{\mathscr{G}} = \mathrm{o}_{\mathrm{p}}(1) \tag{4.7}$$

since $\sup_{0\leqslant t\leqslant 1} |W_n(t)| = \mathrm{O}_{\mathrm{p}}(1)$ as argued immediately above, and, as we will argue below, the class of functions $\mathscr{G}$ defined by

$$\mathscr{G} \equiv \{\phi'(x(t)) : \phi\in C_1^{\gamma}(R),\ 0\leqslant t\leqslant 1\}$$

is a P-Glivenko–Cantelli class of functions. Once we have shown that $\mathscr{G}$ is Glivenko–Cantelli, then it follows from (4.4) that

$$\begin{aligned} \mathbb{Z}_n(t,\phi) &= \mathbb{Z}_n^{(3)}(t,\phi) + \mathbb{Z}_n^{(2)}(t,\phi) + R_n(t,\phi) \\ &= \mathbb{G}_n(f_{t,\phi}) + \mathrm{o}_{\mathrm{p}}^{*}(1) \end{aligned}$$

where

$$f_{t,\phi}(x) = \phi(x(t)) + \lambda\{E\phi'(X(t))\} \int_0^t x(s)\,\mathrm{d}s \equiv f_{t,\phi}^{(2)}(x) + f_{t,\phi}^{(3)}(x). \tag{4.8}$$

Thus, it remains to show that $\mathscr{G}$ is Glivenko–Cantelli, and that the class of functions

$$\mathscr{F} = \{f_{t,\phi} : \phi \in C_1^\gamma(R), 0 \leqslant t \leqslant 1\},$$

with $f_{t,\phi}$ as defined by (4.8), is P-Donsker.

To show that $\mathscr{F}$ is P-Donsker it suffices to separately show that $\mathscr{F}^{(2)}$ and $\mathscr{F}^{(3)}$ are P-Donsker (see Van der Vaart and Wellner, 1996, Example 2.10.7, p. 192). For the first, let $\mathbb{X} =_d \mathbb{X}_1$ and write $\mathbb{X} = \{X(t) = \mathrm{e}^{-\lambda t}Y + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}B(s) : t \in [0,1]\}$ where $Y \sim \mu$ and B is a standard Brownian motion independent of Y. As in Sections 2 and 3, let the brackets for $C_1^\gamma(R)$ be denoted by $[\phi_{m,L}, \phi_{m,U}]$, $m \in \{1,\dots,J_\varepsilon\}$. Then, with $0 \leqslant t_1 \leqslant \cdots \leqslant t_{K(\varepsilon)} \leqslant 1$, we have, for $\phi \in [\phi_{m,L}, \phi_{m,U}]$ and $t_k \leqslant t \leqslant t_{k+1}$,

$$\begin{aligned} \phi(X(t)) &= \phi(X(t_k)) + (\phi(X(t)) - \phi(X(t_k))) \\ &\leqslant \phi_{m,U}(X(t_k)) + \Delta_k \end{aligned}$$

where

$$\Delta_k \equiv \sup_{t \in [t_k, t_{k+1}]} \sup_{\phi \in C_1^\gamma(R)} |\phi(X(t)) - \phi(X(t_k))|,$$

and similarly

$$\phi(X(t)) \geqslant \phi_{m,L}(X(t_k)) - \Delta_k.$$

Much as in Sections 2 and 3, in order to show that

$$\log N_{[\,]}(\varepsilon, \mathscr{F}^{(2)}, L_2(P)) \leqslant \widetilde{K}\left(\frac{1}{\varepsilon}\right)^V \tag{4.9}$$

with $V = 2 - \delta$, $0 < \delta \leqslant \frac{1}{2}$, it suffices to show

$$E\Delta_k^2 \leqslant K|t_{k+1} - t_k| \tag{4.10}$$

and

$$E\{\phi_{mU}(X(t_k)) - \phi_{mL}(X(t_k))\}^2 \leqslant K\varepsilon^2 \tag{4.11}$$

for fixed constants K. Again to ease notation, set $t_k := t_0$, $t_{k+1} := t_0 + h$. To prove (4.10) we will bound $\Delta_{t_0,h} \equiv \sup_{t_0 \leqslant t \leqslant t_0+h} \sup_{\phi \in C_1^\gamma(R)} |\phi(X_t) - \phi(X_{t_0})|$. Now, since $\phi \in C_1^\gamma(R)$,

$$\begin{aligned} \Delta_{t_0,h} &\leqslant \sup_{t_0 \leqslant t \leqslant t_0+h} |X_t - X_{t_0}| \\ &= \sup_{t_0 \leqslant t \leqslant t_0+h} \left| (\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda t_0})Y + \int_0^t \mathrm{e}^{-\lambda(t-s)}\,\mathrm{d}B(s) - \int_0^{t_0} \mathrm{e}^{-\lambda(t_0-s)}\,\mathrm{d}B(s) \right| \\ &\leqslant \sup_{t_0 \leqslant t \leqslant t_0+h} \left| (\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\lambda t_0})(Y + \int_0^t \mathrm{e}^{\lambda s}\,\mathrm{d}B(s)) \right| \\ &\quad + \mathrm{e}^{-\lambda t_0} \sup_{t_0 \leqslant t \leqslant t_0+h} \left| \int_{t_0}^t \mathrm{e}^{\lambda s}\,\mathrm{d}B(s)) \right| \end{aligned}$$

$$\begin{aligned}
&\leqslant \lambda e^{-\lambda t_0} h \left\{ |Y| + \sup_{t_0 \leqslant t \leqslant t_0+h} \left| \int_0^t e^{\lambda s}\, dB(s)) \right| \right\} \\
&\quad + e^{-\lambda t_0} \sup_{t_0 \leqslant t \leqslant t_0+h} \left| \int_{t_0}^t e^{\lambda s}\, dB(s)) \right| \\
&\equiv \lambda e^{-\lambda t_0} h\{|Y| + S_1\} + e^{-\lambda t_0} S_2.
\end{aligned}$$

Since $\{\int_0^t e^{\lambda s}\, dB(s) : t \in [0,1]\}$ and $\{\int_{t_0}^t e^{\lambda s}\, dB(s) : t \in [t_0, t_0+h]\}$ are martingales, it follows by Doob's inequality that

$$\begin{aligned}
E(S_1^2) &= E\left(\sup_{t_0 \leqslant t \leqslant t_0+h} \left| \int_0^t e^{\lambda s}\, dB(s)) \right| \right)^2 \\
&\leqslant 4E\left(\int_0^{t_0+h} e^{\lambda s}\, dB(s)) \right)^2 \\
&= 4\int_0^{t_0+h} e^{2\lambda s}\, ds = \frac{2}{\lambda}\{e^{2\lambda(t_0+h)} - 1\}
\end{aligned}$$

and

$$\begin{aligned}
E(S_2^2) &= E\left(\sup_{t_0 \leqslant t \leqslant t_0+h} \left| \int_{t_0}^t e^{\lambda s}\, dB(s)) \right| \right)^2 \\
&\leqslant 4E\left(\int_{t_0}^{t_0+h} e^{\lambda s}\, dB(s)) \right)^2 \\
&= 4\int_{t_0}^{t_0+h} e^{2\lambda s}\, ds = \frac{2}{\lambda}\{e^{2\lambda(t_0+h)} - e^{2\lambda t_0}\}.
\end{aligned}$$

Hence, we find that

$$\begin{aligned}
E(\Delta_{t_0,h})^2 &\leqslant 2\lambda^2 e^{-2\lambda t_0} h^2 E\{|Y| + S_1\}^2 + 2e^{-2\lambda t_0} E\{S_2\}^2 \\
&\leqslant 4\lambda^2 e^{-2\lambda t_0} h^2 \{EY^2 + \frac{2}{\lambda}(e^{2\lambda(t_0+h)} - 1)\} \\
&\quad + 2e^{-2\lambda t_0} \frac{2}{\lambda}\{e^{2\lambda(t_0+h)} - e^{2\lambda t_0}\} \\
&\leqslant K_\lambda h,
\end{aligned}$$

proving that (4.10) holds.

To show that (4.11) holds, we compute, using $\{EW^r\}^{1/r} \leqslant \{EW^s\}^{1/s}$ for $0 < r \leqslant s < \infty$ in the second inequality,

$$E\{\phi_{mU}(X(t_k)) - \phi_{mL}(X(t_k))\}^2 = E\left\{ 2\varepsilon \sum_{j=-\infty}^{\infty} a_j 1_{[j,j+1]}(X(t_k)) \right\}^2$$

$$
\begin{aligned}
&= 4\varepsilon^2 \sum_{j=-\infty}^{\infty} a_j^2 P(X(t_k) \in I_j) \\
&\leqslant 4\varepsilon^2 E\,|X(t_k)|^{1/(1-\delta)} \\
&\leqslant 4\varepsilon^2 \{E\,|X(t_k)|^2\}^{1/2(1-\delta)} \\
&\leqslant 4\{E(Y^2) + (1 - \mathrm{e}^{-2\lambda t_k})/(2\lambda))\}^{1/2(1-\delta)}\varepsilon^2 \\
&\leqslant 4\{E(Y^2) + (1 - \mathrm{e}^{-2\lambda})/(2\lambda))\}^{1/2(1-\delta)}\varepsilon^2 \\
&= K\varepsilon^2,
\end{aligned}
$$

completing the proof of (4.11). Hence $\mathscr{F}^{(2)}$ is P-Donsker.

To show that $\mathscr{F}^{(3)}$ is P-Donsker, we again invoke the bracketing theorem. For $0<h<1$, let $t_k = kh$, $k=0,1,\ldots,[1/h]+1$, and let $l \in \{-[1/h]-1,\ldots,-1,0,1,\ldots,[1/h]\}$. Given $t\in[0,1]$, let k be such that $t_k \leqslant t < t_{k+1}$, and let l be such that $lh \leqslant E\phi'(X_1(t)) < (l+1)h$. Then

$$
(E\phi'(X(t)))\int_0^t X_1(s)\,\mathrm{d}s \leqslant (l+1)h\left|\int_0^{t_k} X_1(s)\,\mathrm{d}s\right| + (l+1)h \sup_{t\in[t_k,t_{k+1}]}\left|\int_{t_k}^t X_1(s)\,\mathrm{d}s\right|
$$

and

$$
(E\phi'(X(t)))\int_0^t X_1(s)\,\mathrm{d}s \geqslant lh\left|\int_0^{t_k} X_1(s)\,\mathrm{d}s\right| - (l+1)h \sup_{t\in[t_k,t_{k+1}]}\left|\int_{t_k}^t X_1(s)\,\mathrm{d}s\right|,
$$

so that we can take the upper and lower bounding functions as defining the brackets. Thus, we have at most $((2/h)+1)^2$ brackets whose L_2-size is dominated by the square root of

$$
2h^2 \sup_{t\in[0,1]} E\left(\int_0^t X_1(s)\,\mathrm{d}s\right)^2 + 4\max_k E \sup_{t\in[t_k,t_{k+1}]}\left(\int_{t_k}^t X_1(s)\,\mathrm{d}s\right)^2 \leqslant Kh
$$

by previous computations. This shows that the class $\mathscr{F}^{(3)}$ satisfies the hypotheses of the bracketing CLT for P and is therefore P-Donsker.

Finally, to prove that the class $\mathscr{G}$ is P-Glivenko–Cantelli, we use the Blum–Dehardt law of large numbers (e.g. Van der Vaart and Wellner, 1996, Theorem 2.4.1, p. 122). We must show that

$$
\log N_{[\,]}(\varepsilon,\mathscr{G},L_1(P)) < \infty \tag{4.12}
$$

for all $\varepsilon>0$. If $\gamma = 1+\delta$, then $\mathscr{G} \subset C_1^{\delta}(R)$, and the arguments and estimates necessary to prove (4.12) are not too different from those in the proof of (4.9) for $\mathscr{F}^{(2)}$. Thus we omit them. $\square$

Remark. It is easy to see, from estimates in the previous proof, that the distance ρ_P associated to the limiting Gaussian process of Theorem 3 satisfies

$$
\rho_P^2(f_{s,\phi}, f_{t,\psi}) \leqslant C\{|t-s| + \|\phi-\psi\|_\infty^2 + \|\phi'-\psi'\|_\infty^2\},
$$

and therefore, the process $\mathbb{Z}(t,\phi)$ is sample continuous for the distance $d((s,\phi),(t,\psi)) = |s-t| \vee \|\phi-\psi\|_\infty \vee \|\phi'-\psi'\|_\infty$.

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