Biometrika (1980), **67**, 1, *pp.* 133–43
Printed in Great Britain

Confidence bands for a survival curve from censored data

BY W. J. HALL AND JON A. WELLNER
Department of Statistics, University of Rochester, New York

SUMMARY

For arbitrarily right-censored data, the Kaplan–Meier product-limit estimator $\hat{S}_N^0$ provides a nonparametric estimate of the survival function $S^0 = 1-F^0$. We provide large-sample simultaneous confidence bands for S^0, centred at $\hat{S}_N^0$. The derivation uses the weak convergence of $N^{\frac{1}{2}}\{\hat{S}_N^0(t)-S^0(t)\}$, on a finite interval, to a Gaussian process, a theorem of Breslow & Crowley (1974), and transforms both the time and space axes of the limiting process to achieve a Brownian bridge limit. Parameters in the transformation are replaced by uniformly consistent estimates to form the bands. The new bands reduce to the well-known Kolmogorov bands in the absence of censoring. Comparisons are made with recent bands of Gillespie & Fisher (1979) and V. N. Nair. The bands are illustrated by imposing some different kinds of random censorship on a set of uncensored data.

Some key words: Brownian bridge; Censored data; Goodness-of-fit test; Kaplan–Meier estimator; Survival function.

1. INTRODUCTION

Consider the problem of estimating a survival function $S^0 = 1-F^0$ and of providing simultaneous confidence bands for S^0. In the case of complete, or uncensored, data, the usual empirical survival function estimates S^0 and confidence bands are provided by the well-known bands based on the Kolmogorov statistic (Birnbaum, 1952). Also see Doksum (1977) for other bands and related work in the case of complete data. The Kolmogorov bands have been extended to completely truncated or censored samples, in which detailed data are available only up to a fixed time T or up to an order statistic $X_{(r)}$ for a fixed r, by Barr & Davidson (1973); see also Koziol & Byar (1975) and Dufour & Maag (1978).

For arbitrarily right-censored data, which often occur in medical applications, the Kaplan & Meier (1958) 'product-limit' estimator provides a nonparametric estimate of the survival function S^0. But confidence bands to accompany the Kaplan–Meier estimator have been unavailable.

Our object here is to provide such bands, at least for moderate or large sample sizes. We confine attention to the 'random censorship' model (Breslow & Crowley, 1974), but see Meier (1975) for interesting results concerning 'fixed censorship'. Section 2 contains preliminary results and notation. In § 3 we present confidence bands for S^0 and their asymptotic justification, based on transformation of the Breslow–Crowley weak convergence result to a Brownian bridge form. These bands become, essentially, the usual Kolmogorov bands in the absence of censoring. In § 4, we give width comparisons in several cases of interest and speculate on size of samples necessary for adequate approximation. In § 5 we illustrate the bands by imposing random censorship on some complete survival data given by Bjerkedal (1960). Appropriate tables appear in § 6, and an Appendix contains some proofs and comments on derivations.

Recently, other work, one paper (Gillespie & Fisher, 1979) and an unpublished report of V. N. Nair, have come to our attention; both contain alternative large-sample confidence bands based on the Kaplan–Meier estimate, the Breslow–Crowley weak convergence theorem for it, and Efron's (1967) observation that the limiting process can be transformed to Brownian motion. Nair presents two alternative bands: one is based on Efron's transformation to Brownian motion, instead of our transformation to Brownian bridge, and it does not reduce to anything recognizable in the uncensored case. The other uses a weighted metric and a resulting transformation to an Ornstein–Uhlenbeck limit process. It can be modified so as to reduce to the corresponding bands in the uncensored case. The bands of Gillespie & Fisher are based on the same transformation to Brownian motion used by Nair, but employing linear boundaries; they also fail to reduce to the Kolmogorov bands in the uncensored case. In contrast to ours, none of these bands as given is symmetric around the Kaplan–Meier estimate, although they could be so modified. Aalen (1976) has suggested Kolmogorov-type tests for cumulative hazard functions in a more general multiple decrement model; although Aalen's Corollary 1, p. 24, and the discussion following it is related to the present work, our perspective is quite different.

Our statistic, and its asymptotic distribution, can, of course, also be used for goodness-of-fit testing when testing a simple null hypothesis as in the usual Kolmogorov test; see also § 6. The natural analogue of the two-sample Kolmogorov–Smirnov test and other extensions will be treated elsewhere.

For the user, the confidence bands are described in § 3 and illustrated in § 5.

2. Preliminaries

Let $X_1^0, \ldots, X_N^0$ be independent positive random variables with common continuous distribution function F^0 and survival function $S^0 = 1-F^0$. Let $Y_1, \ldots, Y_N$ be independent positive random variables having common left-continuous distribution function H, and suppose that the X^0's and Y's are independent. For any distribution function G, let

$$T_G = \inf\{t \geqslant 0: G(t) = 1\} \leqslant \infty.$$

Suppose that we observe the N pairs (X_i, δ_i), with $X_i = \min(X_i^0, Y_i)$ and δ_i the indicator function of $X_i^0 \leqslant Y_i$ for $i = 1, \ldots, N$. Thus the X's are a random sample from the distribution function F with $1-F = (1-F^0)(1-H)$, and the subdistribution functions of the uncensored and censored observations are given by

$$\tilde{F}(x) = \mathrm{pr}(X \leqslant x, \delta = 1) = \int_0^x (1-H)\,dF^0, \quad \tilde{H}(x) = \mathrm{pr}(X \leqslant x, \delta = 0) = \int_0^x (1-F^0)\,dH,$$

respectively, so that $F = \tilde{F} + \tilde{H}$; note that $T_F = \min(T_{F^0}, T_H)$. This is the random censorship model for arbitrary right censoring.

The Kaplan–Meier product-limit estimate $\hat{F}_N^0 = \hat{S}_N^0$ of F^0 is

$$\hat{S}_N^0(t) = \begin{cases} \prod_{\{k:\, X_k \leqslant t\}} \{(N-R_k)/(N-R_k+1)\}^{\delta_k} & (t < X_{(N)}), \\ 0 & (t \geqslant X_{(N)}), \end{cases} \tag{2·1}$$

where $X_{(N)} = \max(X_1, \ldots, X_N)$ and R_k is the rank of $(X_k, 1-\delta_k)$ in the lexicographic ordering of $(X_1, 1-\delta_1), \ldots, (X_N, 1-\delta_N)$.

THEOREM 1 (Breslow & Crowley, 1974). *If F^0 and H are continuous and $T < T_F$ with $F(T) < 1$, then the process $Z_N^*(t) = N^{\frac{1}{2}}\{\hat{S}_N^0(t) - S^0(t)\}$, $0 \leqslant t \leqslant T$, converges weakly to a zero mean Gaussian process Z^* with covariance function*

$$\mathrm{cov}\{Z^*(s), Z^*(t)\} = C(s)\,S^0(s)\,S^0(t) \quad (s \leqslant t), \tag{2·2}$$

where

$$C(s) = \int_0^s (1-F)^{-2}\, d\tilde{F} = \int_0^s (1-F^0)^{-2}(1-H)^{-1}\, dF^0 \quad (s < T_F). \tag{2·3}$$

The theorem remains true when H is degenerate, the 'truncation case'. Meier (1975) gives a similar theorem for the fixed censorship model; his theorem suggests that our bands given in § 3 remain valid for this case as well.

The function C, which reflects the 'heaviness' of censoring through the factor $(1-H)^{-1}$ in its integrand, plays a basic role in the formation and interpretation of our bands. Associated with C is the distribution function K defined by

$$K(t) = C(t)\{1+C(t)\}^{-1} \quad (0 \leqslant t < T_F),$$

and $K(t) = 1$ for $t \geqslant T_F$. By writing $(1-H)^{-1} = 1 + H(1-H)^{-1}$,

$$C(t) = F^0(t)\{1-F^0(t)\}^{-1} + \int_0^t H(1-H)^{-1}(1-F^0)^{-2}\, dF^0 \geqslant F^0(t)\{1-F^0(t)\}^{-1},$$

and, using $\tilde{F} + \tilde{H} = F$, we obtain

$$C(t) = \int_0^t (1-F)^{-2}\, dF - \int_0^t (1-F)^{-2}\, d\tilde{H} \leqslant F(t)\{1-F(t)\}^{-1}.$$

Hence for $0 \leqslant t < \infty$,

$$F^0(t) \leqslant K(t) \leqslant F(t). \tag{2·4}$$

Now let $\bar{K} = 1 - K = (1+C)^{-1}$ and let B^0 denote a tied Brownian motion, or Brownian bridge process, on [0,1]. Note that the process Z^* is well defined on $[0, T_H)$ and that in law

$$\{Z^*(t)\}_{0 \leqslant t < T_F} = \{B^0\{K(t)\}\, S^0(t)/\bar{K}(t)\}_{0 \leqslant t < T_F}$$

since they are both zero mean Gaussian processes with covariance (2·2). Equivalently, in law on $[0, T_F)$ (Doob, 1949)

$$Z^* \bar{K}/S^0 = B^0 \circ K, \tag{2·5}$$

where $\circ$ denotes functional composition. Thus the limiting process Z^* is related to a Brownian bridge process B^0 by a rescaling of the state space and a monotone transformation of the time axis, and K enters as a 'natural time scale'. Notice that (2·5) reduces, in the absence of censoring, to the standard fact that in law $\{Z^*(t)\} = \{B^0\{F^0(t)\}\}$. Subject to such a reduction, the transformation in (2·5) can be shown to be unique.

Consideration of the convergence in Theorem 1 together with the distributional identity (2·5) leads naturally to large-sample bands for the survival function S^0 in terms of the unknown function $\bar{K}$. To implement our bands we will estimate C, and hence $\bar{K}$, as follows. Let F_N and $\tilde{F}_N$ denote (Breslow & Crowley, 1974, p. 445) the left-continuous empirical distribution function of all the observations and the left-continuous empirical distribution function of those observations that are uncensored respectively, and let F_N^+ be the right-continuous version of F_N; $NF_N^+(x)$ is the number of $X_i \leqslant x$. Define

$$C_N(t) = \int_{[0,t)} (1-F_N^+)^{-1}(1-F_N)^{-1}\, d\tilde{F}_N \quad (0 \leqslant t < X_{(N)}), \tag{2·6}$$

and $C_N(t) = \infty$ for $t \geqslant X_{(N)}$, and $K_N = C_N(1+C_N)^{-1}$. This estimator agrees with standard practice: $\hat{S}_N^0(t)\{C_N(t)/N\}^{\frac{1}{2}}$ is the estimated standard error of $\hat{S}_N^0(t)$ as an estimate of $S^0(t)$ for fixed t (Greenwood, 1926; Dixon & Brown, 1977, p. 752). To calculate C_N and $\bar{K}_N$ we use the formulae

$$C_N(t) = N \Sigma_{\{i:\, X_i<t\}} (N-i)^{-1}(N-i+1)^{-1}\,\delta_i, \quad \bar{K}_N(t) = \{1+C_N(t)\}^{-1}. \tag{2·7}$$

The following proposition gives two additional properties of C_N which will be of use in forming confidence bands for S^0.

PROPOSITION.
(a) (*Consistency*). *If* $F(T) < 1$,

$$\sup_{0\leqslant t\leqslant T} |C_N(t) - C(t)| = o_p(1).$$

(b) (*Reduction*). *If all the observations are uncensored then*

$$\hat{S}^0_N(t)/\bar{K}_N(t) = \{1+C_N(t)\}\hat{S}^0_N(t) = 1 \quad (0 \leqslant t < \infty)$$

with the convention that $\infty \cdot 0 = 1$.

See the Appendix for proofs. While (a) of the Proposition remains true for other obvious estimators of C such as $\int_{[0,t)} (1-F_N)^{-2} d\tilde{F}_N$ or $\int_{[0,t)} (1-\tilde{F}^+_N)^{-2} dF_N$, the 'reduction property' (b) fails for these estimators. Although the convergence in Theorem 1 is equivalent to convergence on $[0, T_F)$, it may fail at T_F as may (a) above.

We conclude this section by introducing notation for some limiting distributions. Let B^0 denote a Brownian bridge process on $[0, 1]$, and for $0 < a \leqslant 1$, $0 \leqslant \lambda < \infty$ set

$$G^+_a(\lambda) = \text{pr}\,\{\sup_{0\leqslant t\leqslant a} B^0(t) \leqslant \lambda\} = 1 - \bar{\Phi}[\lambda\{a(1-a)\}^{-\frac{1}{2}}] - e^{-2\lambda^2}\bar{\Phi}[\lambda(1-2a)\{a(1-a)\}^{-\frac{1}{2}}], \tag{2·8}$$

$$G_a(\lambda) = \text{pr}\,\{\sup_{0\leqslant t\leqslant a} |B^0(t)| \leqslant \lambda\}$$

$$= 1 - 2\bar{\Phi}[\lambda\{a(1-a)\}^{-\frac{1}{2}}] + 2\sum_{k=1}^{\infty} (-1)^k e^{-2k^2\lambda^2}[\bar{\Phi}\{r(2k-d)\} - \bar{\Phi}\{r(2k+d)\}], \tag{2·9}$$

where Φ is the standard normal distribution function, $\bar{\Phi} = 1 - \Phi$, $r = \lambda\{(1-a)/a\}^{\frac{1}{2}}$ and $d = (1-a)^{-1}$.

Thus

$$G^+(\lambda) = G^+_1(\lambda) = 1 - e^{-2\lambda^2} < G^+_a(\lambda) \quad (a < 1),$$

$$G(\lambda) = G_1(\lambda) = 1 + 2\sum_{k=1}^{\infty} (-1)^k e^{-2k^2\lambda^2} < G_a(\lambda) \quad (a < 1).$$

The distribution functions G^+ and G are simply the well-known and extensively tabulated one- and two-sided Kolmogorov–Smirnov limiting distributions respectively. The distributions G^+_a and G_a are easily obtained from results concerning linear barriers for Brownian motion with drift (Cox & Miller, 1965, p. 211; Anderson, 1960); see the Appendix for further details. Selected percentiles of the distributions G^+_a and G_a are given in § 6, and for more extensive tables for G_a see Koziol & Byar (1975).

3. CONFIDENCE BANDS FOR S^0

We now give the confidence bands.

THEOREM 2. *If $T < T_F$ so that $F(T) < 1$, and F^0 and H are continuous, then as $N \to \infty$*

$$\text{pr}\{S^0(t) \leqslant \hat{S}_N^0(t) + \lambda D_N(t) \text{ for all } 0 \leqslant t \leqslant T\} \to G_a^+(\lambda) > G^+(\lambda), \tag{3·1}$$

$$\text{pr}\{\hat{S}_N^0(t) - \lambda D_N(t) \leqslant S^0(t) \leqslant \hat{S}_N^0(t) + \lambda D_N(t) \text{ for all } 0 \leqslant t \leqslant T\} \to G_a(\lambda) > G(\lambda), \tag{3·2}$$

where

$$D_N(t) = N^{-\frac{1}{2}} \hat{S}_N^0(t) / \bar{K}_N(t), \tag{3·3}$$

$a = K(T) = C(T)\{1 + C(T)\}^{-1}$ and $\bar{K}_N$ is given by (2·7).

Proof. It suffices to prove (3·2); the proof of (3·1) is similar. The left-hand side of (3·2) equals

$$\begin{aligned} \text{pr}\{N^{\frac{1}{2}} |\hat{S}_N^0(t) - S^0(t)| \leqslant \lambda \hat{S}_N^0(t)/\bar{K}_N(t) \text{ for all } 0 \leqslant t \leqslant T\} &\to \text{pr}\{|Z^*(t)| \leqslant \lambda S^0(t)/\bar{K}(t) \text{ for all } 0 \leqslant t \leqslant T\} \\ &= \text{pr}\{|Z^*(t)|\, \bar{K}(t)/S^0(t) \leqslant \lambda \text{ for all } 0 \leqslant t \leqslant T\} \\ &= \text{pr}[|B^0\{K(t)\}| \leqslant \lambda \text{ for all } 0 \leqslant t \leqslant T] \\ &= G_a(\lambda), \end{aligned}$$

with $a = K(T)$, where the convergence holds by virtue of Theorem 1 and (a) of the Proposition, and the next to last equality holds by (2·5).

For fixed T, smaller than the largest uncensored observation to ensure $T < T_F$, and large N, we obtain conservative one- or two-sided β confidence bands for S^0 in the interval $[0, T]$ from (3·1) or (3·2) by choosing $\lambda = \lambda_\beta$ so that $G^+(\lambda) = \beta$ or $G(\lambda) = \beta$ respectively; see Table 1. From (b) of the Proposition, the reduction property, it follows that these bands are precisely the usual Kolmogorov bands restricted to $[0, T]$ when all the observations are uncensored.

A more complicated, but less conservative, way to proceed is as follows: fix T smaller than the largest uncensored observation, estimate a by $\hat{a} = K_N(T)$, use this $\hat{a}$ to find an estimated $\lambda = \lambda_\beta$ in Table 1, call it $\hat{\lambda}$, and then assert, in the case of two-sided bands, that

$$\text{pr}\{\hat{S}_N^0(t) - \hat{\lambda} D_N(t) \leqslant S^0(t) \leqslant \hat{S}_N^0(t) + \hat{\lambda} D_N(t) \text{ for all } 0 \leqslant t \leqslant T\} \simeq \beta \tag{3·4}$$

for large N with $\bar{K}_N$ and D_N given by (2·7) and (3·3) and $\hat{S}_N^0$ by (2·1). It is not difficult to formalize this as a limit theorem, but we will not do so here. It is clear from Table 1 that this more complicated procedure will be of interest only when $\hat{a}$ differs substantially from 1, e.g. less than $\frac{3}{4}$.

We conjecture that the present bands remain valid for the interval $0 \leqslant t \leqslant \max\{X_i : \delta_i = 1\}$ with $a = K(T_H)$; a proof would require an extension of Theorem 1.

By comparison, a pointwise confidence interval for $S^0(t)$ at any given fixed t with asymptotic confidence coefficient β is

$$\hat{S}_N^0(t) \pm z_\beta \hat{S}_N^0(t) \{C_N(t)/N\}^{\frac{1}{2}}, \tag{3·5}$$

where $\bar{\Phi}(z_\beta) = \frac{1}{2}(1 - \beta)$. See Thomas & Grunkemeier (1975) for information about these and other intervals.

4. Properties of the bands

To get some feeling for the 'price of censoring' it is natural to compare the width of the bands provided by Theorem 2 with the width of similar bands in the uncensored case. In the presence of censoring, however, we are frequently only able to make confidence statements about S^0 for a bounded subset of its support, and hence it seems unfair to compare the bands of Theorem 2, valid on $[0, T]$, with the usual Kolmogorov bands, valid on $[0, \infty)$. Thus we will compare, for a fixed confidence coefficient β, the bands of Theorem 2, with $a = K(T)$ and λ_a chosen so that $G_a(\lambda) = \beta$, with the Kolmogorov bands for the interval $[0, T]$ using

$$b = F^0(T) \leqslant K(T) = a$$

and the corresponding critical constant $\lambda_b \leqslant \lambda_a$.

It is easily seen that the asymptotic width of the bands in Theorem 2, for the less conservative procedure suggested in the discussion following the theorem, is $2\lambda_a N^{-\frac{1}{2}} S^0(t)/\bar{K}(t)$, while the asymptotic width of the Kolmogorov bands is $2\lambda_b N^{-\frac{1}{2}}$. Hence the ratio of the asymptotic widths is

$$r(t) = r(\text{censored: uncensored}) = \frac{\lambda_a S^0(t)}{\lambda_b \bar{K}(t)}. \tag{4·1}$$

Since $\lambda_b \leqslant \lambda_a$ and $\bar{K}(t) \leqslant S^0(t)$ by (2·4), both factors are $\geqslant 1$, so that $r \geqslant 1$ always. Also, it is easily shown, using $C \leqslant F(1-F)^{-1}$, that r is nondecreasing in t.

To examine the effect of censoring on r, and on functions C and K, consider the following four cases, arranged in increasing severity of censorship. Cases 3 and 4 are probably of greatest interest for typical medical settings. All four cases will be illustrated in § 5.

Case 1: uncensored. If $H \equiv 0$, then $C = F^0(1-F^0)^{-1} = F^0/S^0$, so that $K = F^0$, and (2·2) reduces to $F^0(s)\{1-F^0(t)\}$ for $s \leqslant t$ the familiar covariance function of the empirical process for complete data. Of course $r(t) = 1$ identically in this case.

Case 2: censoring with unbounded support. If $T_F = T_H = \infty$ and H is continuous, then $C(s) \geqslant F^0(s)/S^0(s)$ with strict inequality for some s and $K(t) \geqslant F^0(t)$ increases continuously to 1 as $t \to \infty$. In this case $r(t) \to \infty$ as $t \to \infty$, by L'Hôpital's rule, and hence the bands will be quite wide in the tail, but will still be valid for most of the support of S^0.

For example, if $1 - H(t) = \{S^0(t)\}^\theta$ with $0 < \theta < \infty$, then as $t \to \infty$

$$S^0(t)/\bar{K}(t) = (1+\theta)^{-1}[\theta S^0(t) + \{S^0(t)\}^{-\theta}] \to \infty.$$

Case 3: 'light' censoring with bounded support. If $T_F = T_H < \infty$ and $C(T_H-) < \infty$, then $K(T_H-) = C(T_H-)\{1 + C(T_H-)\}^{-1} < 1$ while $K(T_H) = 1$, so that K has a jump of height $\{1 + C(T_H-)\}^{-1}$ at T_H. Hence $\bar{K}(t) \geqslant \bar{K}(T_H-) > 0$ and $1 \leqslant S^0(t)/\bar{K}(t) \leqslant S^0(T_H)/\bar{K}(T_H-) < \infty$ for all $0 \leqslant t < T_H$. Therefore

$$\sup_{t<T_H} r(t) = \frac{\lambda_a}{\lambda_b} \frac{S^0(T_H-)}{\bar{K}(T_H-)} < \infty$$

in this case, but the bands are valid only for intervals $[0, T]$ with $T < T_H$.

For example, if H puts mass 1 at T_H, then $C(t) = F^0(t)/S^0(t)$ for $0 \leqslant t < T_H$, and is infinite for $t \geqslant T_H$, so that $K(t) = F^0(t)$ and 1, respectively. This is truncation-type censoring, corresponding to all patients entering a study simultaneously in a medical setting (Barr & Davidson, 1973).

Note that Case 3 also includes 'lighter than uniform' censoring $1 - H(t) = (1 - t/T_H)^\gamma$ with $0 < \gamma < 1$, assuming $F^0(T_H) < 1$. To illustrate this concretely, if somewhat unrealistically, suppose that $1 - H(t) = (1-t)^{\frac{1}{2}}$ $(0 \leqslant t \leqslant 1)$ and $1 - F^0(t) = 1 - \frac{1}{2}t$ $(0 \leqslant t \leqslant 2)$. Then

$$C(t) = 1 + \pi - (1-t)^{\frac{1}{2}}(1 - \tfrac{1}{2}t)^{-1} - 4 \arctan\{(1-t)^{\frac{1}{2}}\} \quad (0 \leqslant t \leqslant 1),$$

so that $C(1-) = 1+\pi$ and $K(1-) = (1+\pi)(2+\pi)^{-1} \simeq 0{\cdot}806 < 1$. Thus

$$S^0(t)/\bar{K}(t) \leqslant S^0(1)/\bar{K}(1-) = \tfrac{1}{2}(2+\pi) \simeq 2{\cdot}57$$

for all $0 \leqslant t < 1$. Since $b = F^0(1) = \frac{1}{2}$, $a = K(1-) = 0{\cdot}806$, for $\beta = 0{\cdot}90$ we find $\lambda_b = 1{\cdot}133$ and $\lambda_a = 1{\cdot}222$. Therefore

$$\sup_{0\leqslant t<1} r(t) = (\lambda_a/\lambda_b)\{S^0(1)/\bar{K}(1-)\} \simeq 2{\cdot}77.$$

Case 4: 'heavy' censoring with bounded support. If $T_F = T_H < \infty$, H is continuous, and $C(T_H-) = \infty$ then, as $t \uparrow T_H$, $\lim K(t) = 1$ and K is continuous. Therefore, if $S^0(T_H) > 0$, $S^0(t)/\bar{K}(t) \rightarrow \infty$ as $t \rightarrow T_H$ and

$$\sup_{t<T_H} r(t) = \infty$$

in this case.

For example, if $S^0(T_H) > 0$ and $1 - H(t) = 1 - t/T_H$ $(0 \leqslant t \leqslant T_H)$, then $K(T_H-) = K(T_H) = 1$ and $r(t)$ increases logarithmically as $t \rightarrow T_H$.

For extremely heavy censorship the upper band of Theorem 2 may fail to be nonincreasing, since $N^{-\frac{1}{2}}\hat{S}^0_N(t)/\bar{K}_N(t) \simeq N^{-\frac{1}{2}}S^0(t)/\bar{K}(t)$ may increase more rapidly than $\hat{S}^0_N(t) \simeq S^0(t)$ decreases. In such cases the upper band may obviously be replaced by its greatest nonincreasing minorant without affecting any of the probability statements. This should not be a serious problem in practice.

We now comment on the size of samples needed for adequate approximation using these asymptotic methods. Since our bands are not distribution-free except asymptotically, due to censoring, no general conclusions are possible. But some guidance can be obtained from the uncensored and truncation cases.

For the uncensored case with $a = 1$, our bands reduce to the Kolmogorov bands on an unrestricted interval. If we use the critical constants from Table 1 to construct bands, then the true finite-sample coverage probability β_N may be found by interpolation from Table 42 of Odeh *et al.* (1977). At $N = 15$, the true coverage probability is $0{\cdot}994$ corresponding to nominal $\beta = 0{\cdot}99$ and $0{\cdot}791$ corresponding to nominal $\beta = 0{\cdot}75$; at $N = 50$, the corresponding coverage probabilities are $0{\cdot}992$ and $0{\cdot}772$. From another perspective, use of the asymptotics has led to bands which are unnecessarily wide: 4% too wide when $N = 15$, 3% when $N = 25$ and 2% when $N = 50$, for β between $0{\cdot}75$ and $0{\cdot}99$. Further comparisons are given by Birnbaum (1952, Table 2).

Using computations of Dufour & Maag (1978), we can do a similar evaluation for the case of truncated samples, corresponding to H degenerate at T_H, for $N = 10\,(5)\,25$ and β ranging from $0{\cdot}85$ to $0{\cdot}99$. Comparison of their critical constants with ours in Table 1, for a ranging from $0{\cdot}5$ to 1, shows bands based on asymptotic theory to be 4–5% too wide when $N = 15$ and 3% too wide when $N = 25$. The conservative nature of all of these comparisons is noteworthy.

It seems likely that our two-sided bands can be used for sample sizes in this range without serious difficulty.

By contrast, the computations of Gillespie & Fisher (1979) suggest that sample sizes well in excess of 200 may be needed to validate asymptotic theory for their bands, even in the uncensored case. Moreover, asymptotic theory apparently overestimates true coverage probability for their bands, and their bands are highly asymmetric in contrast to the symmetry of ours; the latter could be corrected by using asymptotically equivalent symmetric bands.

5. ILLUSTRATION OF THE CONFIDENCE BANDS

Here we illustrate the confidence bands by imposing random censorship on data given by Bjerkedal (1960). Bjerkedal gave various doses of tubercle bacilli to groups of 72 guinea pigs and recorded their survival times. We concentrate on his Regimen 4.3, which contains no censored observations, so that we have a complete data set initially.

The usual empirical survival function and 90% two-sided Kolmogorov confidence bands for S^0 for the complete or uncensored set of 72 observations are shown by dotted lines in Fig. 1. We used the asymptotic critical constant $\lambda = 1{\cdot}224$ from Table 1 for all the bands.

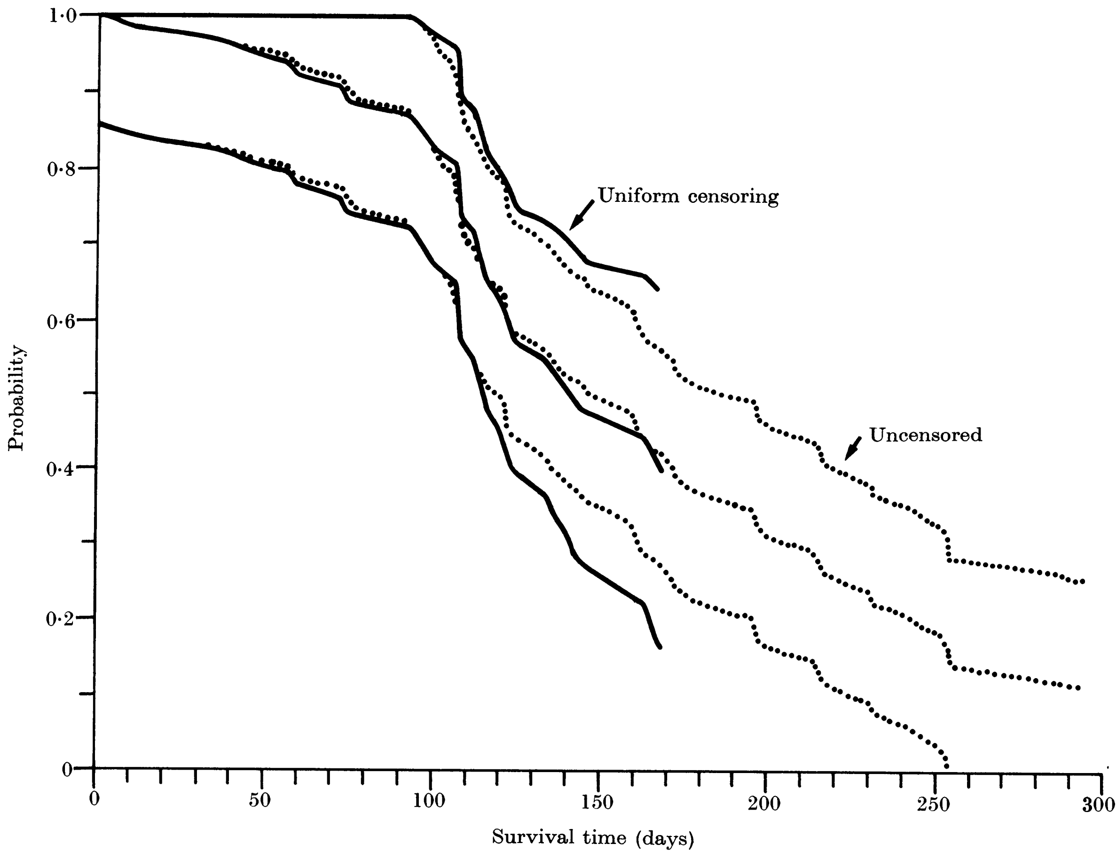

Fig. 1. Survival function estimates and 90% confidence bands: 'uniform' censoring and uncensored.

The plot has been truncated at 300 days; only eight survival times, 327, 342, 347, 361, 402, 432, 458, 555, exceed 300 days. The width of the band is $2\lambda N^{-\frac{1}{2}} \simeq 0{\cdot}289$ away from the tails. All the graphs should be step functions, but we have interpolated linearly between observations for convenience in plotting.

The solid lines in Fig. 1 represent the Kaplan–Meier estimator and 90% two-sided confidence bands for Bjerkedal's data after applying uniform censoring: random variables Y_i were drawn from the uniform distribution $H(t) = t/220$ $(0 \leqslant t \leqslant 220)$ as if 'patients', or guinea pigs, arrived at random during a 220-day study period, corresponding to Case 4 of § 4. In other words, we treated Bjerkedal's data as the X_i^0's of the beginning of § 2, generated 72 independent Y_i's as noted, and obtained X_i's and δ_i's as defined in § 2.

This censoring resulted in 23 uncensored and 49 censored observations or 68% censoring, with the last uncensored observation at 168 days. Note that the estimator has changed very little from the uncensored case over the interval for which data are available for estimation, but that the confidence bands have widened substantially: for example the width at 160 days is approximately 0·406 compared with 0·289 in the uncensored case. This is in keeping with the discussion in § 4.

We also applied 'exponential' censoring and light censoring with bounded support corresponding to Cases 2 and 3 of § 4. We have not displayed the results here. The confidence bands were very similar to those for the uncensored case, over the period for which data were available, but slightly wider.

We conclude by illustrating the less conservative bands (3·4) for the uniformly censored data of Fig. 1 and the pointwise confidence intervals (3·5). We chose $T = 160$ since the last uncensored observation was 168, and found $\hat{a} = 0{\cdot}66$ and thence $\lambda = 1{\cdot}200$. The corresponding 90% confidence band on the interval [0, 160] is virtually identical, only 2% narrower, to that in Fig. 1 where $\lambda = 1{\cdot}224$ was used instead. If a band was desired over a shorter interval, or if the censoring were of truncation type, it would be slightly narrower yet. By contrast, an appropriate 90% confidence interval for $S^0(160)$, given in (3·5), is $0{\cdot}481 \pm 0{\cdot}129$ with width 0·258, whereas the simultaneous band has width 0·406 at $t = 160$. For the uncensored data, a 90% interval for $S^0(160)$ has width 0·193 and the band has width 0·289.

Programs for calculation of the bands are available from the authors.

6. Critical points of G_a^+ and G_a

Table 1 gives selected critical points of the distributions G_a^+ and G_a for a few values of a; more extensive tables are available from the authors. For comments regarding computation, see the Appendix. Note that for a larger than about 0·75 and probabilities larger than

Table 1. *Selected percentiles of G_a^+, with value of λ such that $G_a^+(\lambda) = \beta$, and selected percentiles of G_a, with value of λ such that $G_a(\lambda) = \beta$*

β		$a = 0{\cdot}10$	$a = 0{\cdot}25$	$a = 0{\cdot}40$	$a = 0{\cdot}50$	$a = 0{\cdot}60$	$a = 0{\cdot}75$	$a = 0{\cdot}90$	$a = 1{\cdot}00$
0·99	G_a^+	0·782	1·157	1·358	1·438	1·486	1·514	1·517	1·517
	G_a	0·851	1·256	1·470	1·552	1·600	1·626	1·628	1·628
0·95	G_a^+	0·599	0·894	1·062	1·134	1·181	1·217	1·224	1·224
	G_a	0·682	1·014	1·198	1·273	1·321	1·354	1·358	1·358
0·90	G_a^+	0·504	0·759	0·909	0·976	1·023	1·063	1·073	1·073
	G_a	0·599	0·894	1·062	1·133	1·181	1·217	1·224	1·224
0·75	G_a^+	0·357	0·546	0·665	0·723	0·768	0·814	0·832	0·833
	G_a	0·471	0·711	0·854	0·920	0·967	1·008	1·019	1·019
0·50	G_a^+	0·213	0·334	0·420	0·466	0·506	0·555	0·585	0·589
	G_a	0·356	0·544	0·663	0·720	0·765	0·809	0·827	0·828
0·25	G_a^+	0·102	0·167	0·218	0·250	0·280	0·324	0·366	0·379
	G_a	0·272	0·420	0·518	0·567	0·608	0·652	0·675	0·676

0·50 or 0·75, in the upper right-hand corner of the table, the critical points of $G^+ = G_1^+$ and $G = G_1$ will suffice for most practical purposes; this is to be expected since $B^0(1) = 0$ and hence the probability that

$$\sup_{0 \leqslant t \leqslant 1} B^0(t) > \sup_{0 \leqslant t \leqslant a} B^0(t)$$

is very small for values of a near 1. Small values of a will be of interest in situations involving heavy or truncation-type censoring however.

More extensive tabulations of percentiles of G_a are given by Koziol & Byar (1975); our values, computed independently, agree with theirs. They discuss the use of such tables for goodness-of-fit testing and other related testing problems; of course our results can likewise be used for hypothesis testing.

We acknowledge the assistance of Kartik Patel in preparing the figure. The research of the second author was supported in part by the U.S. National Science Foundation.

APPENDIX

Proofs and derivations

To prove (a) of the Proposition, let

$$C_N^*(t) = \int_{[0,t)} (1-F_N)^{-2}\, d\tilde{F}_N$$

and write

$$C_N^*(t) - C(t) = \int_0^t \{(1-F_N)^{-2} - (1-F)^{-2}\}\, d\tilde{F}_N + \int_0^t (1-F)^{-2}\, d(\tilde{F}_N - \tilde{F})$$

$$= \int_0^t \{(1-F_N) + (1-F)\}(1-F)^{-2}(1-F_N)^{-2}(F_N - F)\, d\tilde{F}_N + \{\tilde{F}_N(t) - \tilde{F}(t)\}\{1-F(t)\}^{-2}$$

$$-2\int_0^t (\tilde{F}_N - \tilde{F})(1-F)^{-3}\, dF$$

after integration by parts. Hence, using the Glivenko–Cantelli theorem and $F(T) < 1$,

$$\sup_{0\leqslant t\leqslant T} |C_N^*(t) - C(t)| \leqslant \sup_{0\leqslant t\leqslant T} |F_N(t) - F(t)| \int_0^T \{(1-F) + (1-F_N)\}(1-F)^{-2}(1-F_N)^{-2}\, d\tilde{F}_N$$

$$+ \sup_{0\leqslant t\leqslant T} |F_N(t) - F(t)| \left[2\int_0^T (1-F)^{-3}\, dF + \{1-F(T)\}^{-2}\right]$$

$$= o_p(1)\, O_p(1) + o_p(1)\, O(1) = o_p(1).$$

The conclusion now follows easily from

$$\sup_{0\leqslant t\leqslant T} |C_N(t) - C_N^*(t)| = o_p(1).$$

To prove (b) let $X_{(1)} \leqslant \ldots \leqslant X_{(N)}$ denote the ordered sample. Since C_N and S_N^0 are left-continuous step functions, it suffices to prove the equality for $t = X_{(j)}+$ $(j = 1, \ldots, N)$. But when there is no censoring $\delta_j = 1$ $(j = 1, \ldots, N)$, and hence

$$\{1 + C_N(X_{(j)}+)\}\hat{S}_N^0(X_{(j)}+) = \left\{1 + N \sum_{i=1}^{j} (N-i)^{-1}(N+1-i)^{-1}\right\}(1 - j/N)$$

$$= 1$$

for $j = 1, \ldots, N-1$ by an easy calculation.

We now provide some details concerning the derivation of (2·8) and (2·9). A Brownian bridge process B^0 on $[0, 1]$ may be obtained from a standard Brownian motion B on $[0, \infty)$ since in law

$$\{B^0(t)\}_{0\leqslant t<1} = \left\{(1-t)\, B\left(\frac{t}{1-t}\right)\right\}_{0\leqslant t<1};$$

this is simply a restatement of the well-known transformation of Doob (1949) that in law

$$\left\{(1+t)\, B^0\left(\frac{t}{1+t}\right)\right\}_{0\leqslant t<\infty} = \{B(t)\}_{0\leqslant t<\infty}.$$

Hence

$$\{\sup_{0\leqslant t\leqslant a} B^0(t)\leqslant\lambda\} = \left\{(1-t)\,B\left(\frac{t}{1-t}\right)\leqslant\lambda \text{ for all } t\leqslant a\right\}$$
$$= \{B(s)\leqslant\lambda(1+s) \text{ for all } s\leqslant b\}$$
$$= \{B_{-\lambda}(s)\leqslant\lambda \text{ for all } s\leqslant b\},$$

where B_μ is Brownian motion with drift μ and $b = a/(1-a)$.

Hence our $G_a^+(\lambda)$ is precisely $P(x_0, x;\, t)$ of Cox & Miller (1965, p. 211), with their μ, σ, x_0, a, x, t replaced by $-\lambda$, 1, 0, λ, λ, b respectively. Their (28) may be evaluated by integrating their formula (71) and thus we obtain (2·8); for an alternative derivation see Schey (1977).

To obtain (2·9), we proceed as above and note that $G_a(\lambda) = \text{pr}(|B(s)|\leqslant\lambda(1+s)$ for all $s\leqslant b)$ (Anderson, 1960, equation (4·59)). Replacement of his c, d, μ, T by our λ, λ, 0, b yields (2·9). Formula (2·9) is also a special case of (5·9) of Anderson & Darling (1952).

The two distributions are of course related. It is easily verified that $G_a(\lambda) > 2G_a^+(\lambda)-1$, with near equality for large λ or small a, and G_a^+ is readily computed; see also Schey (1977).

References

Aalen, O. (1976). Nonparametric inference in connection with multiple decrement models. *Scand. J. Statist.* **3**, 15–27.

Anderson, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* **31**, 165–97.

Anderson, T. W. & Darling, D. A. (1952). Asymptotic theory of certain 'goodness of fit' criteria based on stochastic processes. *Ann. Math. Statist.* **23**, 193–213.

Barr, D. R. & Davidson, T. (1973). A Kolmogorov–Smirnov test for censored samples. *Technometrics* **15**, 739–57.

Birnbaum, Z. W. (1952). Numerical tabulation of the distribution of Kolmogorov's statistic for finite sample size. *J. Am. Statist. Assoc.* **47**, 425–41.

Bjerkedal, T. (1960). Acquisition of resistance in guinea pigs infected with different doses of virulent tubercle bacilli. *Am. J. Hyg.* **72**, 130–48.

Breslow, N. & Crowley, J. (1974). A large sample study of the life table and product limit estimate under random censorship. *Ann. Statist.* **2**, 437–53.

Cox, D. R. & Miller, H. D. (1965). *The Theory of Stochastic Processes.* London: Chapman and Hall.

Dixon, W. J. & Brown, M. B. (1977). BMDP-77, *Biomedical Computer Programs, P-Series.* Berkeley: University of California Press.

Doksum, K. (1977). Some graphical methods in statistics; a review and some extensions. *Statist. Neerl.* **31**, 53–68.

Doob, J. L. (1949). A heuristic approach to the Kolmogorov–Smirnov theorems. *Ann. Math. Statist.* **20**, 393–403.

Dufour, R. & Maag, U. R. (1978). Distribution results for modified Kolmogorov–Smirnov statistics for truncated or censored samples. *Technometrics* **20**, 29–32.

Efron, B. (1967). The two-sample problem with censored data. *Proc. 5th Berkeley Symp.* **4**, 831–53.

Gillespie, M. J. & Fisher, L. (1979). Confidence bands for the Kaplan–Meier survival curve estimate. *Ann. Statist.* **7**, 920–4.

Greenwood, M. (1926). The natural duration of cancer. *Reports on Public Health and Medical Subjects* 33. London: H.M. Stationery Office.

Kaplan, E. L. & Meier, P. (1958). Nonparametric estimation from incomplete observations. *J. Am. Statist. Assoc.* **53**, 457–81.

Koziol, J. A. & Byar, D. P. (1975). Percentage points of the asymptotic distributions of one and two sample K–S statistics for truncated or censored data. *Technometrics* **17**, 507–10.

Meier, P. (1975). Estimation of a distribution function from incomplete observations. In *Perspectives in Probability and Statistics*, Ed. J. Gani, pp. 67–87. London: Academic Press.

Odeh, R. E., Owen, D. B., Birnbaum, Z. W. & Fisher, L. (1977). *Pocket Book of Statistical Tables.* New York: Dekker.

Schey, H. M. (1977). The asymptotic distribution of the one-sided Kolmogorov–Smirnov statistic for truncated data. *Comm. Statist.* A **6**, 1361–6.

Thomas, D. R. & Grunkemeier, G. L. (1975). Confidence interval estimation of survival probabilities for censored data. *J. Am. Statist. Assoc.* **70**, 865–71.

[*Received May* 1979. *Revised October* 1979]