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## OSCILLATING BROWNIAN MOTION

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### Abstract

An 'oscillating' version of Brownian motion is defined and studied. 'Ordinary' Brownian motion and 'reflecting' Brownian motion are shown to arise as special cases. Transition densities, first-passage time distributions, and occupation time distributions for the process are obtained explicitly. Convergence of a simple oscillating random walk to an oscillating Brownian motion process is established by using results of Stone (1963).

OSCILLATING BROWNIAN MOTION; DIFFUSION PROCESS; TRANSITION PROBABILITIES;  
ARCSIN LAW

### 1. Introduction

Let  $(B(t))_{t \geq 0}$  be a standard Brownian motion process and let  $\sigma_+, \sigma_- > 0$ , be two positive numbers. (We allow one of  $\sigma_+$  or  $\sigma_-$  to be  $+\infty$ , but not both.) Consider the process  $(Y(t))_{t \geq 0}$  obtained from  $B$  as follows: let  $\sigma^2(y) = \sigma_+^2, y \geq 0$ ,  $\sigma_-^2, y < 0$ ; let

$$A_x(t) = \int_0^t \sigma^{-2}(B(s) + x) ds = \int_{\mathcal{R}} L(t, y - x) m(dy) \quad t \geq 0, x \in \mathcal{R}$$

where  $m(dy) = 2\sigma^{-2}(y)dy$  and  $L$  is local time for  $B$ ; and define

$$Y(t) = B(A_x^{-1}(t)) + x \quad t \geq 0.$$

We call  $Y$  *oscillating Brownian motion*. Our definition of  $Y$  is just the recipe used by Ito and McKean (1965) to construct a diffusion process with speed measure  $m$  from Brownian motion; they prove (Chapter 5) that for an arbitrary speed measure  $m$  with support an interval  $Q \subset \mathcal{R}$ , the process  $Y$  defined above is a strong Markov, conservative diffusion process on  $Q$  with the desired speed measure  $m$ .

For our particular choice of the function  $\sigma^2$  (and speed measure  $m$ ) the additive functional  $A_x$  may be written as

$$A_x(t) = \sigma_-^2 t + (\sigma_+^2 - \sigma_-^2) \lambda \{s \leq t: B(s) + x \geq 0\} \quad t \geq 0$$

where  $\lambda$  denotes Lebesgue measure. Suppose temporarily that  $0 < \sigma_{\pm} < \infty$ . For a fixed sample path,  $B(t)(\omega)$ ,  $A_x(t)(\omega)$  is a continuous piecewise linear function with slope  $\sigma_-^2$  or  $\sigma_+^2$  depending on whether  $B + x$  is  $< 0$  or  $\geq 0$ . Similarly,  $A_x^{-1}(t)(\omega)$  is continuous and piecewise linear with alternating slopes  $\sigma_-^2$  and  $\sigma_+^2$ . In fact, it is seen that  $A_x^{-1}$  scales time by  $\sigma_+^2$  or  $\sigma_-^2$  depending on whether  $B + x$  is  $\geq 0$  or  $< 0$ . Since Brownian motion starting from zero changes sign very frequently, the process  $Y$  ‘oscillates’ in that it behaves like a Brownian motion which changes variance parameter each time it crosses zero. This will be made more precise in the sequel.

In the well-known special case  $\sigma_+ = \sigma_- = \sigma$ ,  $Y$  is clearly ordinary Brownian motion with variance parameter  $\sigma^2$ . Another case of interest is  $\sigma_+ = \sigma$ ,  $\sigma_- = +\infty$ ; then  $Y$  is reflecting Brownian motion (in this case  $A_x(t) = \sigma^2 \lambda \{s \leq t: B(s) + x \geq 0\}$ ); now see Ito and McKean (1965), pp. 81–82).

Our object here is to calculate the distributions of several functionals of the process  $Y$  and to identify these as the limiting distributions of the corresponding functionals of a simple oscillating random walk,  $Y_n$ . We calculate the transition densities, first-passage time distributions, and some occupation time distributions for the process  $Y$ ; these calculations are standard using the techniques of Ito and McKean (1965). The occupation time distributions obtained are related to results of Lamperti (1958) and Dwass and Karlin (1963).

Far less standard is the convergence of an ‘oscillating random walk’ process  $Y_n$  to the process  $Y$  which is discussed in Section 4. We use results of Stone (1963) to establish the convergence in the case of a simple oscillating random walk,  $Y_n$ , but have been unable to prove the convergence in the case of more general increment distributions. This problem has connections with the work of Kemperman (1974) concerning the recurrence of oscillating random walks, and the convergence to diffusion processes with discontinuous diffusion coefficients.

## 2. Distributions of some functionals of $Y$

Denote the transition densities of the process  $Y$  with respect to the speed measure  $m$  by  $p(t, x, y)$ ; thus

$$P_x(Y(t) \in dy) = p(t, x, y)m(dy), \quad (t, x, y) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}.$$

As is shown in Section 4.11 of Ito and McKean (1965), these densities exist, are continuous, and are symmetric in  $x$  and  $y$ :  $p(t, x, y) = p(t, y, x)$  for all  $x, y \in \mathbb{R}$ . The following theorem gives these densities for the oscillating Brownian motion process  $Y$  defined in Section 1. Here, and in the following, we denote the transition density of ordinary Brownian motion by  $p^*(t, x, y)$ :

$$p^*(t, x, y) = (2\pi t)^{-1/2} \exp(-(y - x)^2/2t).$$

*Theorem 1* (transition densities for  $Y$ ). Let  $\theta \equiv \sigma_- / (\sigma_+ + \sigma_-) \equiv (1+r)^{-1}$ ,  $r \equiv \sigma_+ / \sigma_-$ . Then

$$p(t, x, y) = \begin{cases} \frac{1}{2}\theta\sigma_+ \left\{ (1+r)p^* \left( t, \frac{x}{\sigma_+}, \frac{y}{\sigma_+} \right) + (1-r)p^* \left( t, \frac{-x}{\sigma_+}, \frac{y}{\sigma_+} \right) \right\} & 0 \leq y \leq x \\ \theta\sigma_+ p^* \left( t, \frac{x}{\sigma_+}, \frac{y}{\sigma_-} \right) & y \leq 0 \leq x \\ \frac{1}{2}\theta\sigma_+ \left\{ \left(1 + \frac{1}{r}\right) p^* \left( t, \frac{x}{\sigma_-}, \frac{y}{\sigma_-} \right) + \left(1 - \frac{1}{r}\right) p^* \left( t, \frac{-x}{\sigma_-}, \frac{y}{\sigma_-} \right) \right\} & y \leq x \leq 0. \end{cases}$$

Since  $p(t, x, y)$  is a symmetric function of  $x$  and  $y$ , Theorem 1 determines  $p(t, x, y)$  for all  $x, y \in R$ . Observe that in the special case  $\sigma_+ = \sigma_- = \sigma$ ,  $r = 1$ , and we recover the transition density of ordinary Brownian motion with variance parameter  $\sigma^2$ . In the case  $\sigma_+ = \sigma$ ,  $\sigma_- = +\infty$ ,  $r = 0$ , and the densities given in Theorem 1 reduce to those of reflecting Brownian motion (cf. Ito and McKean, p. 41). Taking  $x = 0$ , we get the one-dimensional distributions of  $Y$ ; these are ‘pieced half-normal’ with variance parameters  $\sigma_-^2$  and  $\sigma_+^2$  and masses  $1 - \theta$  and  $\theta$  on the negative and positive half-axes respectively.

*Corollary 1.*

$$p(t, 0, y) = \begin{cases} \theta\sigma_+ p^*(t, 0, (y/\sigma_-)) & y < 0 \\ \theta\sigma_+ p^*(t, 0, (y/\sigma_+)) & y \geq 0. \end{cases}$$

This is the density with respect to the speed measure  $m$ ; converting this to a density with respect to Lebesgue measure  $\lambda$  ( $m$  is absolutely continuous with respect to  $\lambda$  in our case) one obtains

$$p_\lambda(t, 0, y) = \begin{cases} \frac{2(1-\theta)}{(2\pi\sigma_-^2 t)^{1/2}} \exp(-y^2/2\sigma_-^2 t) & y < 0 \\ \frac{2\theta}{(2\pi\sigma_+^2 t)^{1/2}} \exp(-y^2/2\sigma_+^2 t) & y \geq 0. \end{cases}$$

Using this, the mean and variance of  $Y$  may be computed easily: for  $t \geq 0$ ,  $E_0(Y(t)) = 0$  and  $E_0(Y(t)^2) = \sigma_+ \sigma_- t$ .

Other variables of interest are the passage times

$$T_y = \inf\{s \geq 0: Y(s) = y\}.$$

Recall that  $Y(0) = x \in R$  with  $(P_x)$  probability one (w.p. 1). Since  $Y$  ‘acts like’ ordinary Brownian motion with variance parameter  $\sigma_+^2$  or  $\sigma_-^2$  depending on

whether it is  $\geq 0$  or  $< 0$ , one expects that in the cases  $0 \leq y < x$  and  $x < y \leq 0$ ,  $T_y$  will have the same distribution as for an ordinary Brownian motion with variance  $\sigma_+^2$  or  $\sigma_-^2$  respectively. However, in the cases  $0 \leq x < y$ ,  $x \leq 0 < y$ ,  $y < x \leq 0$ , or  $y < 0 \leq x$ , the passage time  $T_y$  depends on the behavior of the process on both sides of zero and hence may be expected to have somewhat different distributions. That this is indeed the case is shown by the following theorem.

*Theorem 2* (passage time distributions for  $Y$ ). For  $0 = y < x$ ,

$$P_x(T_0 \in dt) = \frac{(x/\sigma_+)}{(2\pi t^3)^{1/2}} \exp(-x^2/2\sigma_+^2 t) dt;$$

a similar formula holds for  $x < y = 0$  by replacing  $(x/\sigma_+)$  by  $(-x/\sigma_-)$ . When  $0 = x < y$ ,

$$P_0(T_y \in dt) = \frac{2}{r+1} \cdot \frac{(y/\sigma_+)}{(2\pi t^3)^{1/2}} \sum_{k=0}^{\infty} q^k (2k+1) \exp(-(2k+1)^2 y^2 / 2\sigma_+^2 t) dt$$

where  $q \equiv (r-1)/(r+1)$ ; again a similar formula holds for  $y < x = 0$  by replacing  $(y/\sigma_+)$  by  $(-y/\sigma_-)$  and  $r$  by  $r^{-1}$ .

In the case  $\sigma_+ = \sigma_-$ ,  $r = 1$ , only the first term of the series contributes, and the formula above becomes the well-known passage time distribution for ordinary Brownian motion (as in the first part of the theorem). Using the strong Markov property and standard techniques, the Laplace transform of the first-passage time in the two-barrier case,  $T_{ab} = \min\{T_a, T_b\}$  with  $a < 0 < b$ , can be obtained from Theorem 2 (and some of the results of Section 3). The resulting expression is quite unwieldy and we will not pursue it here. We do, however, have the following obvious corollary of the second part of Theorem 2.

*Corollary 2* (distribution of  $\sup_{0 \leq s \leq t} Y(s)$ ).

$$\begin{aligned} P_0(\sup_{0 \leq s \leq t} Y(s) < y) &= P_0(T_y > t) \\ &= \frac{4}{r+1} \sum_{k=0}^{\infty} q^k \left\{ \Phi \left( \frac{(2k+1)y}{\sigma_+ t^{1/2}} \right) - \frac{1}{2} \right\} \end{aligned}$$

where  $\Phi$  denotes the standard normal distribution function and  $q \equiv (r-1)/(r+1)$ .

Another functional of interest (when  $Y(0) = 0$  w.p. 1) is the occupation time of the positive half-axis,

$$M(t) = \lambda \{s \leq t: Y(s) \geq 0\}.$$

In the case  $r = 1$ , the distribution of  $M(1) \stackrel{d}{=} M(t)/t$  is well known to be the

classical arcsin law originally obtained by Lévy (1939) (see also Ito and McKean (1965), p. 57 and Kac (1951), p. 192). This is extended to general  $r \geq 0$  by the following theorem, which is related to a theorem of Lamperti (1958) (take  $\delta = 1/2$  in (1.4) there).

*Theorem 3* (occupation time of  $R^+$ ). For  $t > 0$   $M(1) \stackrel{d}{=} M(t)/t$  and, for  $0 < r < \infty$ ,  $M(1)$  has the distribution

$$P_0(M(1) \in du) = \frac{1}{\pi} \cdot \frac{1}{\{u(1-u)\}^{1/2}} \cdot \frac{r}{1-(1-r^2)u} du, \quad 0 < u < 1.$$

When  $r = 0$ ,  $M(1) = 1$  with probability 1; when  $r = \infty$ ,  $M(1) = 0$  with probability 1.

In any case it follows that  $E_0(M(1)) = \theta$  and  $\text{Var}_0(M(1)) = \frac{1}{2}\theta(1-\theta)$ .

By calculations similar to those of p. 58 of Ito and McKean (1965), we also obtain the following conditional occupation time distribution, which is related to the computations on p. 1162 of Dwass and Karlin (1963).

*Theorem 4* (occupation time of  $R^+$ , given  $Y(1) = 0$ ). For  $0 < r < \infty$ ,  $0 < u < 1$ ,

$$P_0(M(1) \leq u \mid Y(1) = 0) = (r(r+1)/r(u)(r(u)+1))u,$$

where  $r(u) \equiv [1-u(1-r^2)]^{1/2}$ .

### 3. Proofs; computation of the transforms

As is well known, the continuous, positive, increasing and decreasing solutions of the equation

$$(1) \quad \frac{d}{dm} \frac{d}{dx} g = sg \quad s > 0,$$

play a basic role in diffusion theory; this is explained in considerable detail in Section 4.6 of Ito and McKean (1965). In our case the infinitesimal generator  $(d/dm)(d/dx)$  is simply

$$(2) \quad \frac{d}{dm} \frac{d}{dx} = \frac{1}{2}\sigma^2(x) \frac{d^2}{dx^2} = \begin{cases} \frac{1}{2}\sigma_+^2 \frac{d^2}{dx^2} & x \geq 0 \\ \frac{1}{2}\sigma_-^2 \frac{d^2}{dx^2} & x < 0. \end{cases}$$

Let  $g_1$  and  $g_2$  denote positive increasing and decreasing solutions of (1) respectively with Wronskian  $B \equiv g_1'g_2 - g_1g_2' = 1$  for all  $x \in R$ .

*Lemma 1.* For the infinitesimal generator given by (2), the functions  $g_1$  and  $g_2$ , for  $s > 0$ , are

$$g_1(x, s) = \begin{cases} \exp(-\sqrt{2s}|x|/\sigma_-) & x \leq 0 \\ \frac{1}{2}(1+r)\exp(\sqrt{2s}x/\sigma_+) + \frac{1}{2}(1-r)\exp(-\sqrt{2s}x/\sigma_+) & x \geq 0 \end{cases}$$

$$g_2(x, s) = \begin{cases} \frac{\theta\sigma_+}{2\sqrt{2s}} \left\{ \left(1 + \frac{1}{r}\right) \exp(\sqrt{2s}|x|/\sigma_-) + \left(1 - \frac{1}{r}\right) \exp(-\sqrt{2s}|x|/\sigma_-) \right\} & x \leq 0 \\ \frac{\theta\sigma_+}{2\sqrt{2s}} \exp(-\sqrt{2s}x/\sigma_+) & x \geq 0. \end{cases}$$

*Proof.* Straightforward differentiation and examination of derivatives. These functions were computed in series form for a more general class of diffusions by Stone (1963); take  $\beta = 0$  on pp. 657 and 658.

Now the theorems of Section 2 are easily proved using the functions  $g_1$  and  $g_2$ . For example, it is known (cf. Ito and McKean, pp. 149 ff.) that if  $r$  is defined by

$$r(s, x, y) = \int_0^\infty e^{-st} p(t, x, y) dt$$

where  $p(t, x, y)$  is the transition density introduced in Section 2, then

$$r(s, x, y) = \begin{cases} g_1(x, s)g_2(y, s) & x \leq y \\ g_2(x, s)g_1(y, s) & x \geq y. \end{cases}$$

Hence,  $r$  is easily obtained from Lemma 1:

$$r(s, x, y) = \begin{cases} \frac{\theta\sigma_+}{2\sqrt{2s}} \left\{ (1+r)\exp(-\sqrt{2s}|y-x|/\sigma_+) \right. \\ \qquad \qquad \qquad \left. + (1-r)\exp(-\sqrt{2s}|y+x|/\sigma_+) \right\} & 0 \leq y \leq x \\ \frac{\theta\sigma_+}{\sqrt{2s}} \exp\left(-\sqrt{2s}\left(\frac{x}{\sigma_+} - \frac{y}{\sigma_-}\right)\right) & y \leq 0 \leq x \\ \frac{\theta\sigma_+}{2\sqrt{2s}} \left\{ \left(1 + \frac{1}{r}\right) \exp(-\sqrt{2s}|y-x|/\sigma_-) \right. \\ \qquad \qquad \qquad \left. + \left(1 - \frac{1}{r}\right) \exp(-\sqrt{2s}|y+x|/\sigma_-) \right\} & y \leq x \leq 0. \end{cases}$$

The statement of Theorem 1 now follows by (easy) inversion of the Laplace transform. Corollaries 1 and 2 are immediate consequences of Theorem 1.

The proof of Theorem 2 is equally straightforward. From Section 4.6 of Ito and McKean,

$$E_x(\exp(-sT_0)) = g_2(x)/g_2(0) = \exp(-\sqrt{2s}x/\sigma_+) \quad x > 0$$

which inverts to give the first statement of Theorem 2. Similarly, for  $y > 0$

$$\begin{aligned} E_0(\exp(-sT_y)) &= g_1(0)/g_1(y) \\ &= \frac{2}{1+r} \cdot \exp(-\sqrt{2s}y/\sigma_+) \{1 - q \exp(-2\sqrt{2s}y/\sigma_+)\}^{-1} \\ &= \frac{2}{1+r} \sum_{k=0}^{\infty} q^k \exp(-(2k+1)\sqrt{2s}y/\sigma_+) \end{aligned}$$

with  $q \equiv (r-1)/(r+1)$ . Another easy inversion yields the second half of the statement of Theorem 2.

Theorem 3 requires a little more effort. We first use the method of Kac (1951) as outlined in Ito and McKean, pp. 54–57 to compute

$$\alpha(s, v) = E_0 \left\{ \int_0^{\infty} e^{-st} e^{-vM(t)} dt \right\} \quad s, v > 0.$$

Kac's formula says that

$$u(x) = E_x \left\{ \int_0^{\infty} e^{-st} e^{-vM(t)} dt \right\} \quad s, v > 0, x \in R$$

is the bounded solution of

$$(3) \quad \frac{d}{dm} \frac{d}{dx} u - (s + v 1_{[0, \infty)}) u = 1.$$

(Here  $1_A$  is the function which is 1 for  $x \in A$  and 0 otherwise.) Let  $h_1$  and  $h_2$  denote the increasing and decreasing solutions of (3). Then, from Lemma 1,

$$\begin{aligned} h_1(x, s) = g_1(x, s) &= \exp(-\sqrt{2s}|x|/\sigma_-) & x \leq 0 \\ h_2(x, s) = g_2(x, s+v) &= \frac{\theta\sigma_+}{\sqrt{2(s+v)}} \exp(-\sqrt{2(s+v)}x/\sigma_+) & x \geq 0, \end{aligned}$$

and the Wronskian is

$$B = h_1'(0)h_2(0) - h_1(0)h_2'(0) = \theta \left\{ r \left( \frac{s}{s+v} \right)^{1/2} + 1 \right\}.$$

Hence

$$\begin{aligned} \alpha(s, v) = u(0) &= \int_0^{\infty} e^{-st} dt \int_0^{\infty} e^{-vw} P_0(M(t) \in dw) \\ &= B^{-1} \left\{ h_2(0) \int_{-\infty}^0 h_1(x) m(dx) + h_1(0) \int_0^{\infty} h_2(x) m(dx) \right\} \\ &= \frac{r\sqrt{s+v} + \sqrt{s}}{\sqrt{s+v} + r\sqrt{s}} \frac{1}{\sqrt{s(s+v)}}, \end{aligned}$$

and it remains only to invert this transform. We do this by way of the following lemmas.

*Lemma 2.* For all  $t > 0$ ,  $P_0(M(t)/t \leq u) = P_0(M(1) \leq u)$ .

*Proof.* This follows upon noting that  $A_0^{-1}(tu) \stackrel{d}{=} tA_0^{-1}(u)$  and hence, using the scaling properties of Brownian motion  $B$ ,

$$\begin{aligned} \frac{M(t)}{t} &= \int_0^1 1_{[0, \infty)}(Y(tu))du \\ &= \int_0^1 1_{[0, \infty)}(B(A_0^{-1}(tu)))du \\ &\stackrel{d}{=} \int_0^1 1_{[0, \infty)}(B(tA_0^{-1}(u)))du \\ &\stackrel{d}{=} \int_0^1 1_{[0, \infty)}(\sqrt{t}B(A_0^{-1}(u)))du \\ &= \int_0^1 1_{[0, \infty)}(Y(u))du = M(1). \end{aligned}$$

The lemma may also be proved by noting that  $\alpha$  has the form  $\alpha(s, v) = s^{-1}\beta(s/v)$  and examining the inverse transform.

Now let  $F(u) \equiv P_0(M(1) \leq u)$ ,  $0 \leq u \leq 1$ ;  $F$  is clearly 0 for  $u < 0$  and 1 for  $u \geq 1$ .

*Lemma 3.*  $\alpha(s, 1) = \int_0^1 (s + u)^{-1} dF(u)$ .

*Proof.* By Lemma 2 and a change of variables

$$\begin{aligned} \alpha(s, v) &= \int_0^\infty e^{-st} dt \int_0^t e^{-v\omega} P_0(tM(1) \in d\omega) \\ &= \int_0^\infty e^{-st} dt \int_0^1 e^{-vut} dF(u). \end{aligned}$$

The statement of the lemma follows by setting  $v = 1$ , applying Fubini's theorem, and computing the inner integral.

Lemma 3 identifies  $\alpha(s, 1)$  as the Stieltjes transform of  $F$  (see Widder (1941), pp. 325 ff.). To complete the proof of Theorem 3 we must invert to find  $F$  or its density function  $f$ . The cases  $r = 0$  and  $+\infty$  are easy: when  $r = 0$ ,  $\alpha(s, 1) = (s + 1)^{-1}$ , and  $F(u) = 1_{[1, \infty)}(u)$ ; when  $r = \infty$ ,  $\alpha(s, 1) = s^{-1}$ , and  $F(u) = 1_{[0, \infty)}(u)$ .

For  $0 < r < \infty$ , first rewrite  $\alpha(s, 1)$  as

$$\alpha(s, 1) = \frac{r}{\sqrt{s(s+1)\{1+(1-r^2)s\}}} + \frac{(1-r^2)}{\{1+(1-r^2)s\}};$$

then apply Theorem 7b of Widder, p. 340 to obtain, with  $f(u) = dF(u)/du$ ,

$$\begin{aligned} f(u) &= \lim_{\theta \rightarrow \pi} \{\alpha(ue^{-i\theta}, 1) - \alpha(ue^{i\theta}, 1)\}/2\pi i \\ &= \frac{1}{\pi} \cdot \frac{r}{\sqrt{u(1-u)\{1-(1-r^2)u\}}} \cdot \lim_{\theta \rightarrow \pi} \left\{ \frac{e^{i\theta/2} - e^{-i\theta/2}}{2i} \right\} \\ &= \frac{1}{\pi} \cdot \frac{r}{\sqrt{u(1-u)\{1-(1-r^2)u\}}} \quad 0 < u < 1, \end{aligned}$$

since the second term of  $\alpha(s, 1)$  is continuous at  $ue^{i\pi} = -u$ . This completes the proof of Theorem 3.

To prove Corollary 3, we proceed by way of a moment generating function: let  $\mu_n = E_0(M(1)^n)$  and set

$$e(w) = \sum_{n=0}^{\infty} \mu_n w^n \quad 0 \leq w < 1.$$

*Lemma 4.*  $e(w) = -w^{-1}\alpha(-w^{-1}, 1) = z \cdot (r+z)/(1+rz)$  where  $z \equiv (1-w)^{-1/2}$ .

*Proof.* The first equality follows easily from Lemma 3 and an expansion of  $(1-uw)^{-1}$  as a geometric series. The second equality is obtained straightforwardly from the expression for  $\alpha$ .

Corollary 3 follows from Lemma 4 by easy differentiation.

#### 4. Convergence to $Y$

Consider the ‘oscillating random walk’  $\{Y_n^*\}_{n \geq 0}$  defined as follows: let  $\{U_n\}_{n \geq 1}$  be i.i.d. with  $P(U_n = 1) = \frac{1}{2} = P(U_n = -1)$ ; let  $\{V_n\}_{n \geq 1}$  be i.i.d. with  $P(V_n = 1) = p/2 = P(V_n = -1)$  and  $P(V_n = 0) = 1-p$  with  $0 \leq p < 1$ . Now set  $Y_0^* = 0$  and, for  $n \geq 0$ , set

$$Y_{n+1}^* = \begin{cases} Y_n^* + U_{n+1} & \text{if } Y_n^* \geq 0, \\ Y_n^* + V_{n+1} & \text{if } Y_n^* < 0. \end{cases}$$

Define the normalized processes  $\{Y_n(t): 0 \leq t < \infty\}_{n \geq 0}$  by  $Y_n(t) = n^{-1/2}Y_{[nt]}^*$ . The process  $\{Y_n^*\}_{n \geq 0}$  is a very special case of an ‘oscillating random walk’ as defined by Kemperman (1974). Kemperman obtained Wiener–Hopf type factorizations for such processes and studied recurrence questions.

Stone (1963) used probabilistic methods (local time for Brownian motion) to construct birth and death processes and random walks from Brownian motion and thereby prove the convergence of these processes to limiting diffusion processes. The crucial requirement in Stone’s construction is that the processes be ‘skip-free’ in both directions. The processes  $Y_n$  defined above satisfy this requirement and hence Stone’s Theorem 2, p. 650, may be used to establish convergence of  $Y_n$  to the oscillating Brownian motion process  $Y$ .

More formally, for our process  $Y_n$  we have, (in Stone’s notation except that his  $W_n$  is replaced by our  $Y_n$ ) for  $n \geq 1$ ,  $E_n = \{i/\sqrt{n} : -\infty < i < \infty\}$ ,  $\theta_n = 1/n$ ,  $x_n = 0$ ,  $q_i^{(n)} = 0$ ,  $\alpha_i^{(n)} = i/\sqrt{n}$ ,

$$P\left(Y_n\left(\frac{k+1}{n}\right) = \frac{i \pm 1}{\sqrt{n}} \mid Y_n\left(\frac{k}{n}\right) = \frac{i}{\sqrt{n}}\right) = \frac{1}{2}, \quad i \geq 0$$

$$P\left(Y_n\left(\frac{k+1}{n}\right) = \frac{i \pm 1}{\sqrt{n}} \mid Y_n\left(\frac{k}{n}\right) = \frac{i}{\sqrt{n}}\right) = \frac{p}{2}, \quad i < 0$$

$$P\left(Y_n\left(\frac{k+1}{n}\right) = \frac{i}{\sqrt{n}} \mid Y_n\left(\frac{k}{n}\right) = \frac{i}{\sqrt{n}}\right) = 1 - p, \quad i < 0,$$

and

$$m_n\left\{\frac{i}{\sqrt{n}}\right\} = \begin{cases} 2n^{-1/2} & i \geq 0 \\ 2p^{-1}n^{-1/2} & i < 0, \end{cases}$$

so

$$\lim_{n \rightarrow \infty} m_n(x) \equiv m(x) = \begin{cases} 2x & x \geq 0 \\ 2p^{-1}x & x < 0. \end{cases}$$

This is just the speed measure for our process  $Y$  with  $\sigma_+^2 = 1$ ,  $\sigma_-^2 = p$ . Now Stone’s Theorem 2 yields the convergence of the processes  $Y_n$ : since  $Y$  is continuous, Stone’s  $J$ -convergence becomes uniform convergence on bounded time intervals, and hence

$$\rho_K(Y_n, Y) \equiv \sup_{0 \leq t \leq K} |Y_n(t) - Y(t)| \xrightarrow{p} 0, \quad n \rightarrow \infty,$$

for all  $K > 0$  where  $Y_n$  denotes the process constructed as in Stone (1963) (and of course  $Y(0) = 0$  w.p. 1). This implies that any version of the oscillating random walk process  $Y_n$  converges weakly to the diffusion process  $Y$ . In particular,

$$Y_n(1) = n^{-1/2} Y_n^* \xrightarrow{d} Y(1), \quad n \rightarrow \infty,$$

where  $Y(1)$  has the pieced half-normal law given by Corollary 1 with  $t = 1$ ,  $\sigma_+^2 = 1$ ,  $\sigma_-^2 = p$ ; and, if  $M_n \equiv \#\{k \leq n: Y_k \geq 0\}/n$ , then

$$M_n \xrightarrow{d} M(1), \quad n \rightarrow \infty,$$

where  $M(1)$  has the distribution given by Theorem 3.

It would be interesting to have a proof of this convergence for more general increments than our  $U_n, V_n$  above; do zero means and finite second moments suffice for this convergence? Does the modified arcsin law of Theorem 3 hold under some sort of symmetry assumption (as in the case of ordinary random walk)? Here Stone's methods seem to break down and we have been able to establish the convergence only in special cases.

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