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On the isoperimetric constant, covariance inequalities and L_p-Poincaré inequalities in dimension one

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First, we derive in dimension one a new covariance inequality of $L_1 - L_\infty$ type that characterizes the isoperimetric constant as the best constant achieving the inequality. Second, we generalize our result to $L_p - L_q$ bounds for the covariance. Consequently, we recover Cheeger's inequality without using the co-area formula. We also prove a generalized weighted Hardy type inequality that is needed to derive our covariance inequalities and that is of independent interest. Finally, we explore some consequences of our covariance inequalities for L_p-Poincaré inequalities and moment bounds. In particular, we obtain optimal constants in general L_p-Poincaré inequalities for measures with finite isoperimetric constant, thus generalizing in dimension one Cheeger's inequality, which is a L_p-Poincaré inequality for $p = 2$, to any real $p \geq 1$.

Keywords: Cheeger's inequality; covariance formula; covariance inequality; isoperimetric constant; moment bounds; Poincaré inequality

1. The isoperimetric constant

For a measure μ on $\mathbb{R}^d$ (we will focus on $d = 1$), an isoperimetric inequality is an inequality of the form,

$$\mu^+(A) \geq c \min\{\mu(A), 1 - \mu(A)\}, \tag{1.1}$$

where $c > 0$, A is an arbitrary measurable set in $\mathbb{R}^d$ and $\mu^+(A)$ stands for the $\mu-$perimeter of A, defined to be

$$\mu^+(A) = \lim \inf_{h \to 0^+} \frac{\mu(A^h) - \mu(A)}{h},$$

where $A^h = \{x \in \mathbb{R}^d : \exists a \in A, |x - a| < h\}$ is an h-neighborhood of A. The optimal value of $c = \mathrm{Is}(\mu)$ in (1.1) is referred to as the *Cheeger isoperimetric constant of* μ. It turns out that the isoperimetric constant is linked to the best constant in Poincaré's inequality, this is the celebrated Cheeger's inequality: it says that if $\lambda > 0$ satisfies, for every smooth (i.e., locally Lipschitz) function g on $\mathbb{R}^d$,

$$\lambda \operatorname{Var}(g) \leq \int |\nabla g|^2 \, d\mu, \tag{1.2}$$

with $\mathrm{Var}(g) = \mathbb{E}[(g - \mathbb{E}[g])^2] = \int (g - \int g\, d\mu)^2\, d\mu$ then we can take

$$\lambda \geq \big(\mathrm{Is}(\mu)\big)^2/4.$$

On $\mathbb{R}$, the isoperimetric constant achieves the following identity (Bobkov and Houdré [8], Theorem 1.3),

$$\mathrm{Is}(\mu) = \operatorname*{essinf}_{a<x<b} \frac{f(x)}{\min\{F(x), 1 - F(x)\}}, \tag{1.3}$$

where f is the density of the absolutely continuous part of μ, F is the distribution function of μ, $a = \inf\{x : F(x) > 0\}$ and $b = \sup\{x : F(x) < 1\}$.

In addition to being defined via relations on sets (1.1), the isoperimetric constant can be stated through the use of a functional inequality. Indeed, the isoperimetric constant is also the optimal constant satisfying the following analytic (Cheeger-type) inequality (see, for instance, p. 192, Bobkov and Houdré [8]),

$$c \int \big|g - \mathrm{med}_\mu(g)\big|\, d\mu \leq \int |\nabla g|\, d\mu, \tag{1.4}$$

where g is an integrable, (locally) Lipschitz function on $\mathbb{R}^d$ and $\mathrm{med}_\mu(g)$ is the median of g with respect to μ.

Inequality (1.4) is also termed an L_1-Poincaré inequality. Instances of L_1-Poincaré inequalities could also be considered for a centering of g by its mean rather than its median, since

$$\int \big|g - \mathrm{med}_\mu(g)\big|\, d\mu \leq \int \big|g - \mathbb{E}_\mu(g)\big|\, d\mu \leq 2 \int \big|g - \mathrm{med}_\mu(g)\big|\, d\mu. \tag{1.5}$$

Bobkov and Houdré [9], Chapter 14, studied related optimal constants in Sobolev-type inequalities defined from Orlicz-norm.

In a different direction, inequality (1.4) may also be seen as a special instance of a covariance inequality. Indeed, let us denote $\mathrm{sign}(x) = 2\mathbf{1}_{\{x \geq 0\}} - 1$ the sign of a real number x. Then by definition of a median, it follows that

$$\mathbb{E}_\mu\big[\mathrm{sign}\big(g - \mathrm{med}_\mu(g)\big)\big] = \int \mathrm{sign}\big(g - \mathrm{med}_\mu(g)\big)\, d\mu = 0$$

and so,

$$\int \big|g - \mathrm{med}_\mu(g)\big|\, d\mu = \mathrm{Cov}_\mu\big(g - \mathrm{med}_\mu(g), \mathrm{sign}\big(g - \mathrm{med}_\mu(g)\big)\big), \tag{1.6}$$

where $\mathrm{Cov}_\mu(g, h) = \int g(h - \mathbb{E}_\mu[h])\, d\mu$.

A natural question then arises: can the isoperimetric constant be defined as the optimal constant in a covariance inequality bounding, for suitable functions g and h, their covariance $\mathrm{Cov}_\mu(g, h)$ by the L_1 moment of ∇g, that is $\int |\nabla g|\, d\mu$? Surely, the upper-bound on the covariance will also depend on some function of the magnitude of ∇h.

Recently, Menz and Otto [25] have established in dimension one what they call an *asymmetric Brascamp–Lieb inequality*: if μ is a strictly log-concave measure on $\mathbb{R}$, then for smooth and square integrable functions g and h on $\mathbb{R}$,

$$\left|\mathrm{Cov}_\mu(g,h)\right| \le \|g'\|_1 \left\|\frac{h'}{\varphi''}\right\|_\infty, \tag{1.7}$$

where φ is the potential of the measure μ, defined by the relation $d\mu = \exp(-\varphi)\,dx$, and the norms are taken with respect to μ. This result has been generalized by Carlen, Cordero-Erausquin and Lieb [13] to higher dimension and to $L_p - L_q$ versions (rather than $L_1 - L_\infty$). For a strictly log-concave measure μ on $\mathbb{R}^d$, $d\mu = \exp(-\varphi)\,dx$, any square integrable locally Lipschitz functions g and h and $p \in [2, +\infty)$, $p^{-1} + q^{-1} = 1$, it holds that

$$\left|\mathrm{Cov}_\mu(g,h)\right| \le \left\|\lambda_{\min}^{(2-p)/p} \mathrm{Hess}_\varphi^{-1/p} \nabla g\right\|_q \left\|\mathrm{Hess}_\varphi^{-1/p} \nabla h\right\|_p,$$

where $\lambda_{\min}(x)$ is the least eigenvalue of $\mathrm{Hess}_{\varphi(x)}$.

Such results seem close in their form to what we would want to have to generalize (1.4). However, it is well known (see, for instance, Ledoux [24]) that for a log-concave measure μ, if $\lambda_{\min}(x) \ge \rho > 0$ – which means that μ is strongly log-concave, see, for instance, Saumard and Wellner [29] – then

$$\frac{\mathrm{Is}(\mu)^2}{4} \le \lambda_P \le 36\,\mathrm{Is}(\mu)^2 \quad \text{and} \quad \rho \le \lambda_P,$$

where λ_P is the (optimal) Poincaré constant of μ. In particular, Inequality (1.7) implies in this case,

$$\left|\mathrm{Cov}_\mu(g,h)\right| \le \rho^{-1} \|g'\|_1 \|h'\|_\infty, \tag{1.8}$$

for smooth and square integrable functions g and h on $\mathbb{R}$. But $\rho^{-1} \ge (36\,\mathrm{Is}(\mu)^2)^{-1}$ and we a priori don't know if the right-hand side of (1.8) could be changed to $u\,\mathrm{Is}(\mu)^{-2}\|g'\|_1\|h'\|_\infty$, where u would be a universal constant – note that we will give a positive answer to this question in the following. Hence, the connection between inequalities of the form of (1.8) and the isoperimetric constant is not straightforward.

One of the reasons for this difficulty is that if we try to approximate the function $\mathrm{sign}(g - \mathrm{med}_\mu(g))$ appearing in (1.6) by a sequence of smooth functions g_n in order to use inequality (1.7) or (1.8) and take the limit, then the sequence of sup-norms $\|g_n'\|_\infty$ will diverge to infinity and it is thus hopeless to recover the L_1-Poincaré inequality (1.4) at the limit. Another limitation of asymmetric Brascamp–Lieb inequalities is that they hold for strictly log-concave measures while our expected covariance inequality would ideally be valid for any measure.

In Section 2, taking the dimension to be one, we establish a covariance inequality that is valid for any measure on $\mathbb{R}$ and that indeed generalizes the L_1-Poincaré inequality (1.4). Then we will consider in Section 3 extensions of our covariance inequalities that are related to L_p-Poincaré inequalities, for $p \ge 1$. In particular, we will prove and make use of some generalized (weighted) Hardy-type inequalities, that are of independent interest. We will explore further consequences in terms of moment estimates of our new covariance inequalities in Section 4.

2. A $L_1 - L_\infty$ covariance inequality

All norms, expectations and covariances will be taken with respect to a probability measure μ on $\mathbb{R}$ so that we will skip related indices in the notations.

Notice that if a L_1-Poincaré inequality (with a centering by the median) holds for a measure μ, that is there exists $c_1 > 0$ such that for every smooth integrable g,

$$c_1 \| g - \mathrm{med}(g) \|_1 \leq \| g' \|_1, \tag{2.1}$$

then if h is in L_∞, we get

$$\begin{aligned} \left|\mathrm{Cov}(g,h)\right| &= \left|\mathbb{E}\left[\left(g-\mathrm{med}(g)\right)\left(h-\mathbb{E}[h]\right)\right]\right| \leq \| g - \mathrm{med}(g)\|_1 \| h - \mathbb{E}[h] \|_\infty \\ &\leq c_1^{-1} \| g' \|_1 \| h_0 \|_\infty, \end{aligned} \tag{2.2}$$

where $h_0 := h - \mathbb{E}[h]$. Moreover, the optimal constant in (2.1) is $c_1 = \mathrm{Is}(\mu)$.

The following theorem states an inequality in dimension one that is sharper in terms of the control of h.

Theorem 2.1. *Let μ be a probability measure with a positive density f on $\mathbb{R}$, cumulative distribution F and median $m \in \mathbb{R}$. Let $g \in L_\infty(F)$ and $h \in L_1(F)$. Assume also that g and h are absolutely continuous. Then we have,*

$$\begin{aligned} &\left|\mathrm{Cov}(g,h)\right| \\ &\quad \leq \mathrm{Is}(\mu)^{-1} \max\left\{ \sup_{a<x\leq m} \left| \frac{\int_a^x h\,dF}{F(x)} - \mathbb{E}[h] \right|, \sup_{b>x>m} \left| \frac{\int_x^b h\,dF}{1-F(x)} - \mathbb{E}[h] \right| \right\} \int_{\mathbb{R}} \left|g'\right| dF, \end{aligned} \tag{2.3}$$

where $a = \inf\{x : F(x) > 0\}$ and $b = \sup\{x : F(x) < 1\}$.

Before proving Theorem 2.1, let us recall a representation formula for the covariance of two functions which first appeared in Menz and Otto [25] and was further studied in Saumard and Wellner [31]. We define a non-negative and symmetric kernel K_μ on $\mathbb{R}^2$ by

$$K_\mu(x,y) = F(x \wedge y) - F(x)F(y), \qquad \text{for all } (x,y) \in \mathbb{R}^2, \tag{2.4}$$

where $F(x) = F_\mu(x) = \mu((-\infty, x])$ is the distribution function associated with the probability measure μ on $(\mathbb{R}, \mathcal{B})$.

Lemma 2.1 (Corollary 2.2, Saumard and Wellner [31]). *If g and h are absolutely continuous and $g \in L_p(F)$, $h \in L_q(F)$ for some $p \in [1,\infty]$ and $p^{-1} + q^{-1} = 1$, then*

$$\mathrm{Cov}(g,h) = \iint_{\mathbb{R}^2} g'(x) K_\mu(x,y) h'(y)\,dx\,dy. \tag{2.5}$$

In fact, Lemma 2.1 can be seen as a special instance of a covariance representation lemma due to (Höffding [17]) (see also (Hoeffding [16]) for a translation of the German original paper into English). Hoeffding's covariance representation is as follows: Let X and Y be two real random variables with finite second moments. Then

$$\mathrm{Cov}(X, Y) = \iint K_{(X,Y)}(x, y)\, dx\, dy,$$

with $K_{(X,Y)}(x, y) = F_{(X,Y)}(x, y) - F_X(x)F_Y(y) = \mathbb{P}(X \le x, Y \le y) - \mathbb{P}(X \le x)\mathbb{P}(Y \le y)$.

For a real random variable Z with probability distribution μ on $\mathbb{R}$, take $X = g(Z)$ and $Y = h(Z)$, where g and h are nondecreasing and left-continuous. Then by Hoeffding's covariance representation,

$$\mathrm{Cov}\bigl(g(Z), h(Z)\bigr) = \iint K_{(g(Z),h(Z))}(x, y)\, dx\, dy.$$

Furthermore,

$$K_{(g(Z),h(Z))}(x, y) = \mathbb{P}\bigl(g(Z) \le x, h(Z) \le y\bigr) - \mathbb{P}\bigl(g(Z) \le x\bigr)\mathbb{P}\bigl(h(Z) \le y\bigr)$$

and by considering the generalized inverses $g^{-1}(x) = \inf\{z : g(z) \ge x\}$ and h^{-1} of g and h respectively, it follows that

$$\begin{aligned} K_{(g(Z),h(Z))}(x, y) &= \mathbb{P}\bigl(Z \le g^{-1}(x), Z \le h^{-1}(y)\bigr) - \mathbb{P}\bigl(Z \le g^{-1}(x)\bigr)\mathbb{P}\bigl(Z \le h^{-1}(y)\bigr) \\ &= K_{(Z,Z)}\bigl(g^{-1}(x), h^{-1}(y)\bigr) =: K_Z\bigl(g^{-1}(x), h^{-1}(y)\bigr), \end{aligned}$$

where $K_Z = K_\mu$. Hence, by change of variables,

$$\mathrm{Cov}\bigl(g(Z), h(Z)\bigr) = \iint K_Z\bigl(g^{-1}(x), h^{-1}(y)\bigr)\, dx\, dy = \iint K_Z(u, v)\, dg(u)\, dh(v).$$

Considering differences of monotone functions and restricting to absolutely continuous functions now gives Menz and Otto's covariance identity: for $g(Z)$ and $h(Z)$ in L_2 and absolutely continuous,

$$\mathrm{Cov}\bigl(g(Z), h(Z)\bigr) = \iint g'(x)K_Z(x, y)h'(y)\, dx\, dy.$$

Hoeffding's covariance representation has been extended to a multi-dimensional setting, by considering the so-called cumulant between several random variables rather than the covariance (Block and Fang [4]). However, it remains unclear how such extension of Hoeffding's lemma could be used to generalize our results, for instance concerning L_p-Poincaré inequalities, to a multivariate setting. It is also worth mentioning that the fact that the covariance representation (2.5) of Lemma 2.1 is valid for any probability measure μ is specific to dimension one. More precisely, Bobkov, Götze and Houdré [7] proved that, in dimension greater than two, a covariance identity of the form of (2.5) implies that μ is Gaussian. Bobkov, Götze and Houdré [7] also proved some genuine concentration inequalities from such covariance identities. Further extensions of covariance inequalities and related deviation inequalities for infinitely divisible and stable random vectors have been obtained in Houdré [18] and Houdré and Marchal [19].

We will also need the following formulas, which are in fact special instances of the previous covariance representation formula.

Lemma 2.2 (Corollary 2.1, Saumard and Wellner [31]). *For an absolutely continuous function $h \in L_1(F)$,*

$$F(z)\int_{\mathbb{R}} h\,dF - \int_{-\infty}^{z} h\,dF = \int_{\mathbb{R}} K_\mu(z,y)h'(y)\,dy \tag{2.6}$$

and

$$-\big(1-F(z)\big)\int_{\mathbb{R}} h\,dF + \int_{(z,\infty)} h\,dF = \int_{\mathbb{R}} K_\mu(z,y)h'(y)\,dy. \tag{2.7}$$

We are now able to give a proof of Theorem 2.1.

Proof. Using the notation of Theorem 2.1, Lemma 2.1 yields

$$\begin{aligned}
\big|\mathrm{Cov}(g,h)\big| &= \left|\iint g'(x)K_\mu(x,y)h'(y)\,dx\,dy\right| \\
&= \left|\int g'(x)\left(\int K_\mu(x,y)h'(y)\,dy\right)dx\right| \\
&\le \int \big|g'(x)\big|\left|\int K_\mu(x,y)h'(y)\,dy\right|dx \\
&= \int \big|g'(x)\big|\frac{|\int K_\mu(x,y)h'(y)\,dy|}{f(x)}f(x)\,dx \\
&\le \sup_{a<x\le m}\frac{|\int_{\mathbb{R}} K_\mu(x,y)h'(y)\,dy|}{f(x)}\int_{(-\infty,m]}\big|g'\big|\,dF \\
&\quad + \sup_{m<x<b}\frac{|\int_{\mathbb{R}} K_\mu(x,y)h'(y)\,dy|}{f(x)}\int_{(m,\infty)}\big|g'\big|\,dF.
\end{aligned}$$

Now, by using (2.6) and (2.7), we get

$$\begin{aligned}
&\sup_{a<x\le m}\frac{|\int_{\mathbb{R}} K_\mu(x,y)h'(y)\,dy|}{f(x)}\int_{(-\infty,m]}\big|g'\big|\,dF \\
&\quad + \sup_{m<x<b}\frac{|\int_{\mathbb{R}} K_\mu(x,y)h'(y)\,dy|}{f(x)}\int_{(m,\infty)}\big|g'\big|\,dF \\
&\le \max\left\{\sup_{a<x\le m}\frac{|\frac{\int_{-\infty}^{x} h\,dF}{F(x)} - \mathbb{E}[h]|}{f(x)/F(x)},\ \sup_{m<x<b}\frac{|\frac{\int_{(x,\infty)} h\,dF}{1-F(x)} - \mathbb{E}[h]|}{f(x)/(1-F(x))}\right\}\int_{\mathbb{R}}\big|g'\big|\,dF
\end{aligned}$$

$$\le \big(\mathrm{Is}(\mu)\big)^{-1} \max\left\{ \sup_{a<x\le m}\left|\frac{\int_{-\infty}^{x} h\,dF}{F(x)} - \mathbb{E}[h]\right|, \sup_{m<x<b}\left|\frac{\int_{(x,\infty)} h\,dF}{1-F(x)} - \mathbb{E}[h]\right|\right\} \cdot \int_{\mathbb{R}} |g'|\,dF.$$

by using (1.3) in the last line. □

Remark 2.1. The proof of Theorem 2.1 allows to give other variants of covariance inequalities. Indeed, we have

$$\begin{aligned}\big|\mathrm{Cov}(g,h)\big| &\le \int \big|g'(x)\big| \frac{|\int K_\mu(x,y)h'(y)\,dy|}{f(x)} f(x)\,dx \\ &\le \sup_{x\in\mathbb{R}}\left\{\frac{|\int K_\mu(x,y)h'(y)\,dy|}{f(x)}\right\}\int_{\mathbb{R}}\big|g'(x)\big| f(x)\,dx.\end{aligned}$$

Then, using Lemma 2.2, we get

$$\big|\mathrm{Cov}(g,h)\big| \le \begin{cases} \sup_{x\in(a,b)}\left\{\dfrac{|F(x)\int_{\mathbb{R}} h\,dF - \int_{-\infty}^{x} h\,dF|}{f(x)}\right\}\displaystyle\int_{\mathbb{R}}|g'|\,dF, \\ \sup_{x\in(a,b)}\left\{\dfrac{|(1-F(x))\int_{\mathbb{R}} h\,dF - \int_{(x,\infty)} h\,dF|}{f(x)}\right\}\displaystyle\int_{\mathbb{R}}|g'|\,dF. \end{cases} \tag{2.8}$$

The latter covariance inequality generalizes Menz and Otto's covariance inequality (1.7) for strictly log-concave measures to any measure with positive density on the real line. Indeed, let us write $f(x) = \exp(-\varphi(x))$, with $\varphi : \mathbb{R} \to \mathbb{R}$. If f is strictly log-concave, and φ' is absolutely continuous, so that φ is convex and $\varphi'' > 0$ on $\mathbb{R}$, then by Lemma 2.2 and Corollary 2.3 of Saumard and Wellner [31], we have

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left\{\frac{|F(x)\int_{\mathbb{R}} h(y)f(y)\,dy - \int_{-\infty}^{x} h(y)f(y)\,dy|}{f(x)}\right\} \\ &\quad= \sup_{x\in\mathbb{R}}\left\{\frac{|\int K_\mu(x,y)h'(y)\,dy|}{f(x)}\right\} \\ &\quad= \sup_{x\in\mathbb{R}}\left\{\frac{|\int K_\mu(x,y)\frac{h'(y)}{\varphi''(y)}\varphi''(y)\,dy|}{f(x)}\right\} \\ &\quad\le \sup_{x\in\mathbb{R}}\left\{\frac{\sup_{y\in\mathbb{R}}|\frac{h'(y)}{\varphi''(y)}|\int K_\mu(x,y)\varphi''(y)\,dy}{f(x)}\right\} \\ &\quad= \sup_{y\in\mathbb{R}}\left|\frac{h'(y)}{\varphi''(y)}\right|.\end{aligned}$$

Applying the latter bound in inequality (2.8) indeed yields the asymmetric Brascamp–Lieb inequality presented in Menz and Otto [25]. Thus, the covariance inequality (2.8) may be viewed as a generalization of Menz and Otto's result in dimension one.

By setting, for $k \in \overline{\mathbb{R}}$,

$$T_k h(x) = \mathbf{1}_{(a,k]}(x) \frac{1}{F(x)} \int_{(a,k]} h \, dF + \mathbf{1}_{(k,b)}(x) \frac{1}{1 - F(x)} \int_{(k,b)} h \, dF,$$

Inequality (2.3) of Theorem 2.1 yields,

$$\left|\mathrm{Cov}(g, h)\right| \le \mathrm{Is}(\mu)^{-1} \left\|g'\right\|_1 \|T_m h_0\|_\infty.$$

It is straightforward to see that $\|T_m h_0\|_\infty \le \|h_0\|_\infty$, so that we recover inequality (2.2).

We will say that a measure μ satisfies a $L_1 - L_\infty$ covariance inequality with constant $r > 0$, if, for every $g \in L_\infty(F)$ and $h \in L_1(F)$ with g and h absolutely continuous,

$$\left|\mathrm{Cov}_\mu\big(g(X), h(X)\big)\right| \le r \left\|g'\right\|_1 \|T_m h_0\|_\infty. \tag{2.9}$$

In this case, we denote $\mathrm{Cov}_{1,\infty}(\mu)$ the smallest constant achieving a $L_1 - L_\infty$ covariance inequality for the measure μ. In other words, if (2.9) is valid for every $g \in L_\infty(F)$ and $h \in L_1(F)$, with g and h absolutely continuous, then $\mathrm{Cov}_{1,\infty}(\mu) \le r$. We have the following optimality result.

Proposition 2.1. *Let μ be a probability measure with a positive density f on $\mathbb{R}$ and cumulative distribution F. If μ satisfies a $L_1 - L_\infty$-covariance inequality then its isoperimetric constant is finite and*

$$\frac{1}{\mathrm{Is}(\mu)} \le \mathrm{Cov}_{1,\infty}(\mu).$$

Furthermore, if μ has a finite isoperimetric constant, then μ satisfies a $L_1 - L_\infty$-covariance inequality with constant $\mathrm{Is}(\mu)^{-1}$. *In other words, the inverse of the isoperimetric constant* $\mathrm{Is}(\mu)^{-1}$ *is the optimal constant achieving inequality* (2.9) *when available*,

$$\frac{1}{\mathrm{Is}(\mu)} = \mathrm{Cov}_{1,\infty}(\mu).$$

Proof. The second part simply corresponds to Theorem 2.1. For the first part, we will use Lemma 2.3 below that is due to Bobkov and Houdré [8]. Indeed, consider an absolutely continuous function $g \in L_1(\mu)$. As the function $\mathrm{sign}(g - \mathrm{med}(g)) = \mathbf{1}_{\{g - \mathrm{med}(g) \ge 0\}} - \mathbf{1}_{\{g - \mathrm{med}(g) \le 0\}}$, we deduce from Lemma 2.3 that there exists a sequence of Lipschitz functions h_n on $\mathbb{R}$ with values in $[-1, 1]$ such that $h_n \to \mathrm{sign}(g - \mathrm{med}(g))$ pointwise as $n \to \infty$. Hence, the dominated convergence theorem and identity (1.6) give

$$\mathrm{Cov}_\mu\big(g(X), h_n(X)\big) \to \mathbb{E}\big|g - \mathrm{med}(g)\big| \qquad \text{as } n \to \infty.$$

Furthermore, again by dominated convergence,

$$\max\left\{\sup_{a<x\leq m}\left|\frac{\int_{-\infty}^{x} h_n\,dF}{F(x)} - \mathbb{E}[h_n]\right|, \sup_{b>x>m}\left|\frac{\int_{x}^{+\infty} h_n\,dF}{1-F(x)} - \mathbb{E}[h_n]\right|\right\}$$
$$\leq \left\|h_n - \mathbb{E}[h_n]\right\|_\infty \leq 1 + \left|\mathbb{E}[h_n]\right| \to 1 \qquad \text{as } n \to \infty.$$

Now, the conclusion simply follows from (1.4). □

Remark 2.2. In dimension $d \geq 1$, if a measure μ has a finite isoperimetric constant, Inequality (1.4) combined with Hölder's inequality implies that for any $g \in L_1(\mu)$ locally Lipschitz and any $h \in L_\infty(\mu)$,

$$\left|\mathrm{Cov}(g,h)\right| \leq \left(\mathrm{Is}(\mu)\right)^{-1}\|\nabla g\|_1\|h_0\|_\infty.$$

Now, if a measure μ satisfies for any $g \in L_1(\mu)$ locally Lipschitz and any $h \in L_\infty(\mu)$,

$$\left|\mathrm{Cov}(g,h)\right| \leq r\|\nabla g\|_1\|h_0\|_\infty,$$

for some finite constant $r > 0$, then by the same arguments as in the proof of Proposition 2.1 and in particular by the use of Lemma 2.3, μ has a finite isoperimetric constant.

Lemma 2.3 (Bobkov and Houdré [8], Lemma 3.5). *For any Borel set $A \subset X$ with $0 < \mu(A) < 1$, there exists a sequence of Lipschitz functions f_n on $\mathbb{R}$ with values in $[0,1]$ such that $f_n \to \mathbf{1}_{\mathrm{clos}(A)}$ pointwise as $n \to \infty$, and $\limsup_{n\to\infty}\mathbb{E}|f_n'| \leq \mu^+(A)$.*

3. $L_p - L_q$ covariance inequalities and L_p-Poincaré inequalities

Let us begin this section by deriving the following $L_p - L_q$ covariance inequalities, that generalize Theorem 2.1.

Theorem 3.1. *Let μ be a probability measure with a positive density f on $\mathbb{R}$ and cumulative distribution F. Let $g \in L_p(F)$ and $h \in L_q(F)$, $p^{-1} + q^{-1} = 1$, $p \in [1, +\infty)$. Assume also that g and h are absolutely continuous. Then we have,*

$$\left|\mathrm{Cov}(g,h)\right| \leq \left[\mathrm{Is}(\mu)\right]^{-1}\left\|g'\right\|_p\left\|T_m(h_0)\right\|_q. \tag{3.1}$$

Consequently, we also have

$$\left|\mathrm{Cov}(g,h)\right| \leq p\left[\mathrm{Is}(\mu)\right]^{-1}\left\|g'\right\|_p\|h_0\|_q. \tag{3.2}$$

Before proving Theorem 3.1, we note that Inequality (3.2) is a consequence of Inequality (3.1) applied together with the following weighted Hardy type inequality, that is of independent interest.

Theorem 3.2 (Generalized Hardy Inequality). *Let F be a continuous distribution on $\mathbb{R}$, with $a = \inf\{x : F(x) > 0\}$ and $b = \sup\{x : F(x) < 1\}$. For a function $h \in L_p(F)$, $1 < p < \infty$ and $k \in \overline{\mathbb{R}}$,*

$$\begin{aligned}\|T_k(h)\|_p^p &= \int_{a<x\le k} \left| \frac{\int_{-\infty}^x h\,dF}{F(x)} \right|^p dF(x) + \int_{b>x\ge k} \left| \frac{\int_x^\infty h\,dF}{1-F(x)} \right|^p dF(x) \\ &\le \left(\frac{p}{p-1}\right)^p \|h\|_p^p.\end{aligned} \tag{3.3}$$

In particular,

$$\int_{\mathbb{R}} \left| \frac{1}{F(x)} \int_{(-\infty,x]} h(y)\,dF(y) \right|^p dF(x) \le \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}} |h(x)|^p\,dF(x)$$

and

$$\int_{\mathbb{R}} \left| \frac{1}{1-F(x)} \int_{(x,\infty)} h(y)\,dF(y) \right|^p dF(x) \le \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}} |h(x)|^p\,dF(x).$$

The proof of Theorem 3.2 can be found below. The fact that Hardy-type inequalities naturally come into play here is appealing, since it is well-known from the work of Miclo [26] and Bobkov and Götze [6] – see also Ané et al. [3], Chapter 6 and Bobkov and Zegarlinski [10], Chapters 4 and 5 – that Hardy-type inequalities can be used to have access to sharp constants in Poincaré and log-Sobolev inequalities on the real line and also in the discrete setting.

It is also worth noting that Inequality (3.2) of Theorem 3.1 induces the celebrated Cheeger's inequality as a corollary and thus in particular gives a new proof of it, avoiding the classical use of the co-area formula.

Corollary 3.1 (Cheeger's inequality). *Let μ be a probability measure with a positive density f on $\mathbb{R}$ and cumulative distribution F. Let $g \in L_2(F)$, absolutely continuous. Then we have,*

$$\operatorname{Var}(g) \le 4[\operatorname{Is}(\mu)]^{-2} \|g'\|_2^2. \tag{3.4}$$

Consequently, if λ_1 denotes the best constant in the Poincaré inequality, it follows that

$$(\operatorname{Is}(\mu))^2/4 \le \lambda_1.$$

Proof. Simply take $g = h$ and $p = 2$ in Inequality (3.2). □

Proof of Theorem 3.1. The case where $p = 1$ and $q = +\infty$ is given by Theorem 2.1. Let us assume that $p, q \in (1, +\infty)$. We have

$$\begin{aligned}|\operatorname{Cov}(g,h)| &\le \int_{(a,b)} |g'(x)| \frac{|\int_{\mathbb{R}} K_\mu(x,y) h'(y)\,dy|}{f(x)} f(x)\,dx \\ &= \int_{(a,b)} |g'(x)| \left(\frac{|F(x)\int_{\mathbb{R}} h\,dF - \int_{-\infty}^x h\,dF|}{f(x)} 1_{x\le m} \right.\end{aligned}$$

$$
\begin{aligned}
&\quad + \frac{|(1-F(x))\int_{\mathbb{R}} h\,dF - \int_x^\infty h\,dF|}{f(x)} 1_{x\ge m}\Big) f(x)\,dx \\
&\le \|g'\|_p \bigg(\int_{a<x\le m} \frac{|F(x)\int_{\mathbb{R}} h\,dF - \int_{-\infty}^x h\,dF|^q}{f^q(x)}\,dF \\
&\quad + \int_{b>x\ge m} \frac{|(1-F(x))\int_{\mathbb{R}} h\,dF - \int_x^\infty h\,dF|^q}{f^q(x)}\,dF\bigg)^{1/q} \\
&= \|g'\|_p \bigg(\int_{a<x\le m} \frac{|\int_{\mathbb{R}} h\,dF - \frac{\int_{-\infty}^x h\,dF}{F(x)}|^q}{f^q(x)/F^q(x)}\,dF \\
&\quad + \int_{b>x\ge m} \frac{|\int_{\mathbb{R}} h\,dF - \frac{\int_x^\infty h\,dF}{1-F(x)}|^q}{f^q(x)/(1-F(x))^q}\,dF\bigg)^{1/q} \\
&\le \mathrm{Is}(\mu)^{-1}\|g'\|_p \|T_m(h_0)\|_q.
\end{aligned}
$$

Hence, inequality (3.1) is proved. To prove inequality (3.2), simply combine inequality (3.1) with inequality (3.3) of Theorem 3.2. □

Proof of Theorem 3.2. It suffices to prove the inequalities for $h \ge 0$. From now on, we assume that $h \ge 0$. To prove Inequality (3.3) we first define the functions δ_x and $F_{x,s}$: for $x \in (a,b)$,

$$
\delta_x(t) := 1_{[x,\infty)}(k)\frac{1_{(-\infty,x]}(t)}{F(x)} + 1_{(-\infty,x)}(k)\frac{1_{(x,\infty)}(t)}{1-F(x)}, \qquad \text{for } t \in \mathbb{R},
$$

and, for $t \in \mathbb{R}$, $s > 0$,

$$
F_{x,s}(t) := F^s(t)1_{[x,\infty)}(k)1_{(-\infty,x]}(t) + \big(1-F(t)\big)^s 1_{(-\infty,x)}(k)1_{(x,\infty)}(t).
$$

It follows that

$$
\begin{aligned}
\|T_k h\|_p^p &= \int_{a<x\le k}\left(\frac{\int_{-\infty}^x h\,dF}{F(x)}\right)^p dF(x) + \int_{b>x>k}\left(\frac{\int_x^\infty h\,dF}{1-F(x)}\right)^p dF(x) \\
&= \int_{x\in\mathbb{R}}\left(\int_{t\in\mathbb{R}} h(t)\delta_x(t)\,dF(t)\right)^p dF(x).
\end{aligned}
$$

Now, by Hölder's inequality,

$$
\begin{aligned}
&\left(\int_{t\in\mathbb{R}} h(t)\delta_x(t)\,dF(t)\right)^p \\
&\quad \le \int_{t\in\mathbb{R}}\big[h(t)\delta_x(t)F_{x,s}(t)\big]^p\,dF(t)\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q}.
\end{aligned}
$$

The use of Fubini's theorem then gives

$$\begin{aligned}
&\int_{x\in\mathbb{R}}\left(\int_{t\in\mathbb{R}} h(t)\delta_x(t)\,dF(t)\right)^p dF(x)\\
&\quad\le \int_{x\in\mathbb{R}}\int_{t\in\mathbb{R}}\big[h(t)\delta_x(t)F_{x,s}(t)\big]^p\,dF(t)\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x)\\
&\quad= \int_{t\in\mathbb{R}} h^p(t)\left(\int_{x\in\mathbb{R}}\big[\delta_x(t)F_{x,s}(t)\big]^p\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x)\right)dF(t).
\end{aligned}$$

In order to conclude, it suffices to prove that for an appropriate choice of $s>0$,

$$\int_{x\in\mathbb{R}}\big[\delta_x(t)F_{x,s}(t)\big]^p\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x) \le \left(\frac{p}{p-1}\right)^p.$$

But we find that

$$\begin{aligned}
&\int_{x\in\mathbb{R}}\big[\delta_x(t)F_{x,s}(t)\big]^p\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x)\\
&\quad= \int_{x\in\mathbb{R}}\left(1_{[x,\infty)]}(k)\frac{1_{(-\infty,x]}(t)}{F(x)} + 1_{(-\infty,x)}(k)\frac{1_{(x,\infty)}(t)}{1-F(x)}\right)^p\\
&\qquad\cdot\left(F^s(t)1_{[x,\infty)}(k)1_{(-\infty,x]}(t) + \big(1-F(t)\big)^s 1_{(-\infty,x)}(k)1_{(x,\infty)}(t)\right)^p\\
&\qquad\cdot\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x)\\
&\quad= \int_{x\in\mathbb{R}}\left(1_{[x,\infty)}(k)\frac{1_{(-\infty,x]}(t)}{F^p(x)}F^{sp}(t) + 1_{(-\infty,x)}(k)\frac{1_{(x,\infty)}(t)}{(1-F(x))^p}\big(1-F(t)\big)^{sp}\right)\\
&\qquad\cdot\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x)\\
&\quad= \int_{x\in\mathbb{R}} 1_{[x,\infty)}(k)\frac{1_{(-\infty,x]}(t)}{F^p(x)}F^{sp}(t)\cdot\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x)\\
&\qquad+ \int_{x\in\mathbb{R}} 1_{(-\infty,x)}(k)\frac{1_{(x,\infty)}(t)}{(1-F(x))^p}\big(1-F(t)\big)^{sp}\cdot\left(\int_{u\in\mathbb{R}}\big[F_{x,s}(u)\big]^{-q}\,dF(u)\right)^{p/q} dF(x)\\
&\quad= \mathbf{1}_{[t\le k]}F(t)^{sp}\int_{x=t}^{k} F(x)^{-p}\left(\int_{u=-\infty}^{x} F(u)^{-sq}\,dF(u)\right)^{p/q} dF(x)\\
&\qquad+ \mathbf{1}_{[t>k]}\big(1-F(t)\big)^{sp}\int_{x=k}^{t}\big(1-F(x)\big)^{-p}\left(\int_{u=x}^{\infty}\big(1-F(u)\big)^{-sq}\,dF(u)\right)^{p/q} dF(x)\\
&\quad= \mathbf{1}_{[t\le k]}(1-sq)^{-p/q}F^{sp}(t)\int_{x=t}^{k} F(x)^{\frac{p}{q}-sp-p}\,dF(x)
\end{aligned}$$

$$
\begin{aligned}
&+ \mathbf{1}_{[t>k]}(1-sq)^{-p/q}\big(1-F(t)\big)^{sp}\int_{x=k}^{t}\big(1-F(x)\big)^{\frac{p}{q}-sp-p}\,dF(x)\\
&= \mathbf{1}_{[t\le k]}\frac{(1-sq)^{-p/q}}{1+p/q-sp-p}F^{sp}(t)\big[F(k)^{1+p/q-sp-p}-F(t)^{1+p/q-sp-p}\big]\\
&\quad + \mathbf{1}_{[t>k]}\frac{(1-sq)^{-p/q}}{1+p/q-sp-p}\big(1-F(t)\big)^{sp}\big[\big(1-F(k)\big)^{1+p/q-sp-p}-\big(1-F(t)\big)^{1+p/q-sp-p}\big]\\
&= \mathbf{1}_{[t\le k]}\frac{(1-sq)^{-p/q}}{sp}\left[1-\frac{F^{sp}(t)}{F(k)^{sp}}\right]\\
&\quad + \mathbf{1}_{[t>k]}\frac{(1-sq)^{-p/q}}{sp}\left[1-\frac{(1-F(t))^{sp}}{(1-F(k))^{sp}}\right] \qquad \text{since } 1+p/q-p=0\\
&= \mathbf{1}_{[t\le k]}\frac{(1-sq)^{-p/q}}{sp}(t)\left[1-\frac{F^{sp}(t)}{F(k)^{sp}}\right]\\
&\quad + \mathbf{1}_{[t>k]}\frac{(1-sq)^{-p/q}}{sp}\left[1-\frac{(1-F(t))^{sp}}{(1-F(k))^{sp}}\right]\le\frac{(1-sq)^{-p/q}}{sp}.
\end{aligned}
$$

Choosing $s=(p-1)/p^2$

$$
1-sq=sp=\frac{p-1}{p}>0,
$$

the conclusion for T_k follows. □

L_p-Poincaré inequalities are an essential tool of functional analysis in relation with the concentration of measure phenomenon. In particular, the pathbreaking work of Milman [27] shows that under convexity assumptions that include the log-concave setting, the exponential concentration property is equivalent to any L_p-Poincaré inequality for $p\in[1,+\infty]$. Theorem 3.1 induces the following sharp L_p-Poincaré inequalities.

Theorem 3.3. *Let μ be a probability measure of finite isoperimetric constant with a positive density f on $\mathbb{R}$ and cumulative distribution F. Let $p\in[1,+\infty)$ and $u\in L_p(F)$ be absolutely continuous. Then we have,*

$$
\big\|u-\mathbb{E}[u]\big\|_p\le 2p\big[\mathrm{Is}(\mu)\big]^{-1}\big\|u'\big\|_p. \tag{3.5}
$$

If in addition $\mathbb{E}[\mathrm{sign}((u-\mathbb{E}[u]))|u-\mathbb{E}[u]|^{p-1}]=0$ (for instance, if p is an even integer and $\mathbb{E}[(u-\mathbb{E}[u])^{p-1}]=0$), then

$$
\big\|u-\mathbb{E}[u]\big\|_p\le p\big[\mathrm{Is}(\mu)\big]^{-1}\big\|u'\big\|_p. \tag{3.6}
$$

Furthermore, if p is an odd integer and $\mathbb{E}[u^p]=0$, then

$$
\|u\|_p\le 2p\big[\mathrm{Is}(\mu)\big]^{-1}\big\|u'\big\|_p. \tag{3.7}
$$

Finally, when in addition $\mathbb{E}[\mathrm{sign}(u^p)] = 0$, *we get*

$$\|u\|_p \le p\big[\mathrm{Is}(\mu)\big]^{-1}\big\|u'\big\|_p. \tag{3.8}$$

Inequality (3.6) is a L_p version of Cheeger's inequality (which is recovered by taking $p = 2$ in (3.6) since in this case the condition $\mathbb{E}[(u - \mathbb{E}[u])^{p-1}] = 0$ is satisfied for any u), see Corollary 3.1 above. Some general relations between optimal constants in L_p-Poincaré inequalities are studied in Chapter 2 of Bobkov and Zegarlinski [10], in connection also with constants in some log-Sobolev type inequalities. Estimates of these optimal constants through the derivation of Muckenhoupt type criteria are further obtained in the framework of Orlicz spaces for dimension one in Bobkov and Zegarlinski [10], Chapter 5.

Closer to our results, the following L_p-Poincaré inequality is obtained in Bobkov and Houdré [8], Remark 7.2: if $p \ge 1$ then

$$\|u\|_p \le 4\sqrt{6}p\big[\mathrm{Is}(\mu)\big]^{-1}\|\nabla u\|_p,$$

where μ is a product probability measure, for instance on $\mathbb{R}^n$, $n \ge 1$. However, even in the case where $n = 1$, it seems that the arguments used in Bobkov and Houdré [8] do not allow to get rid of the constant $4\sqrt{6}$, at least in a simple way.

The Laplace distribution on $\mathbb{R}$, $\varepsilon(dx) = 1/2 \cdot \exp(-|x|)\,dx$, is known to give equality in Cheeger's inequality, see, for instance, Fougères [14]. In this case, the isoperimetric constant is equal to one, $\mathrm{Is}(\varepsilon) = 1$. Moreover, by applying inequalities (3.6) and (3.8) for the exponential measure to monomials $u(x) = x^k$ and letting k be odd and go to infinity, we see that actually, inequalities (3.6) and (3.8) are optimal in the sense that the ratios between the respective left- and right-hand sides tend to one as k goes to infinity.

However, we still do not know if the factors of 2 in inequalities (3.5) and (3.7) are necessary or not.

Proof. Let us begin with the proof of Inequality (3.5). Define $g = u - \mathbb{E}[u]$, $h = \mathrm{sign}(u - \mathbb{E}[u])$ and take a sequence h_n of Lipschitz functions with values in $[-1, 1]$ approximating the function h (this sequence exists via Lemma 2.3 above). Note that since $(p-1)q = p$, we have $|g|^{p-1} \in L_q$. By the dominated convergence theorem,

$$\mathrm{Cov}\big(g, h_n|g|^{p-1}\big) \to \mathrm{Cov}\big(g, h|g|^{p-1}\big) = \mathbb{E}\big[\big|u - \mathbb{E}[u]\big|^p\big]$$

and

$$\big\|h_n|g|^{p-1} - \mathbb{E}\big[h_n|g|^{p-1}\big]\big\|_q \to \big\|h|g|^{p-1} - \mathbb{E}\big[h|g|^{p-1}\big]\big\|_q$$

as $n \to \infty$. Furthermore, Inequality (3.2) of Theorem 3.1 yields

$$\mathrm{Cov}\big(g, h_n|g|^{p-1}\big) \le p\big[\mathrm{Is}(\mu)\big]^{-1}\big\|g'\big\|_p\big\|h_n|g|^{p-1} - \mathbb{E}\big[h_n|g|^{p-1}\big]\big\|_q$$

and taking the limit on both sides we obtain,

$$\big\|u - \mathbb{E}[u]\big\|_p^p \le p\big[\mathrm{Is}(\mu)\big]^{-1}\big\|g'\big\|_p\big\|h|g|^{p-1} - \mathbb{E}\big[h|g|^{p-1}\big]\big\|_q. \tag{3.9}$$

Now,

$$
\begin{aligned}
&\big\|h|g|^{p-1} - \mathbb{E}\big[h|g|^{p-1}\big]\big\|_q \\
&\quad \le \big\|h|g|^{p-1}\big\|_q + \big|\mathbb{E}\big[h|g|^{p-1}\big]\big| \\
&\quad \le \mathbb{E}\big[\big|u - \mathbb{E}[u]\big|^{(p-1)q}\big]^{1/q} + \big|\mathbb{E}\big[h|g|^{p-1}\big]\big| \\
&\quad = \big\|u - \mathbb{E}[u]\big\|_p^{p-1} + \big|\mathbb{E}\big[\operatorname{sign}\big(u - \mathbb{E}[u]\big)\big|u - \mathbb{E}[u]\big|^{p-1}\big]\big| && (3.10) \\
&\quad \le 2\big\|u - \mathbb{E}[u]\big\|_p^{p-1}. && (3.11)
\end{aligned}
$$

Inequality (3.6) then follows from combining (3.9) and (3.10). Inequality (3.5) is deduced by combining (3.9) and (3.11).

Let us prove Inequalities (3.7) and (3.8). Take $g = u^p$ and a sequence h_n of Lipschitz functions with values in $[-1, 1]$ approximating the function $h = \operatorname{sign}(g)$ (this sequence exists via Lemma 2.3 above). By the dominated convergence theorem,

$$
\operatorname{Cov}(g, h_n) \to_{n\to\infty} \operatorname{Cov}(g, h) = \mathbb{E}\big[|u|^p\big]. \tag{3.12}
$$

Furthermore, Inequality (3.2) of Theorem 3.1 with $p = 1$ and $q = +\infty$ yields

$$
\begin{aligned}
\operatorname{Cov}(g, h_n) &\le \big[\operatorname{Is}(\mu)\big]^{-1}\|g'\|_1\big(1 + \big|\mathbb{E}\big[\operatorname{sign}\big(u^p\big)\big]\big|\big) \\
&= p\big[\operatorname{Is}(\mu)\big]^{-1}\big\|u'u^{p-1}\big\|_1\big(1 + \big|\mathbb{E}\big[\operatorname{sign}\big(u^p\big)\big]\big|\big) \\
&\le p\big[\operatorname{Is}(\mu)\big]^{-1}\|u'\|_p\big\|u^{p-1}\big\|_q\big(1 + \big|\mathbb{E}\big[\operatorname{sign}\big(u^p\big)\big]\big|\big) && (3.13) \\
&\le 2p\big[\operatorname{Is}(\mu)\big]^{-1}\|u'\|_p\big\|u^{p-1}\big\|_q, && (3.14)
\end{aligned}
$$

where the Inequality (3.13) comes from the use of Hölder's inequality. Now, noticing that $\|u^{p-1}\|_q = \|u\|_p^{(p-1)/p}$ and combining (3.12) with (3.13) and (3.14) we get the results. □

One can notice that Theorem 7.1 of Bobkov and Houdré [8] establishes more general Poincaré type inequalities for some Orlicz spaces defined with product probability measures. L_p-norms are special cases of Orlicz norms considered in Bobkov and Houdré [8]. It seems that these Orlicz norms are outside of the scope of our techniques, that are based on covariance inequalities established in Section 3. However, taking advantage of the dimension one, we can easily provide the following Hardy type and Poincaré type inequalities for Orlicz norms.

Take μ a probability measure on $\mathbb{R}$ with density p with respect to the Lebesgue measure and denote by m the median of μ. Let N be a differentiable convex function and assume furthermore that N is a Young function, meaning that N is even, nonnegative, $N(0) = 0$ and $N(x) > 0$ for $x \neq 0$. Assume also

$$
C_N = \sup \frac{xN'(x)}{N(x)} < +\infty \tag{3.15}
$$

and set the Orlicz norm associated to N (N-norm),

$$\|f\|_N = \inf\left\{\lambda > 0 : \mathbb{E}\left[N\left(\frac{f}{\lambda}\right)\right] \le 1\right\} < +\infty. \tag{3.16}$$

Proposition 3.1. *For a smooth function f with finite N-norm, the following Hardy-type inequality holds,*

$$\left\|f - f(m)\right\|_N \le C_N \operatorname{Is}^{-1}(\mu)\left\|f'\right\|_N. \tag{3.17}$$

Consequently,

$$\left\|f - \mathbb{E}[f]\right\|_N \le 2C_N \operatorname{Is}^{-1}(\mu)\left\|f'\right\|_N. \tag{3.18}$$

One can notice that by taking $N = |\cdot|^p$, $p \ge 1$, for which $C_N = p$, Inequality (3.18) generalizes Inequality (3.5). However, it seems that a generalization of Inequality (3.6) is not directly accessible by our arguments in the context of Orlicz norms.

Proof. Let us prove first Inequality (3.17). We can assume without loss of generality that f has a compact support, that $f(m) = 0$ and that $\|f\|_N = 1$. In particular, $\lim_{x\to-\infty} N(f(x))F(x) = 0$ and $\lim_{x\to\infty} N(f(x))(1 - F(x)) = 0$. It follows that

$$\begin{aligned}
\mathbb{E}\big[N(f)\big] &= \int N\big(f(x)\big)p(x)\,dx \\
&= \big[N(f)F\big]_{-\infty}^{m} - \int_{-\infty}^{m} f'(x)N'\big(f(x)\big)F(x)\,dx \\
&\quad + \big[N(f)(F-1)\big]_{m}^{+\infty} - \int_{m}^{+\infty} f'(x)N'\big(f(x)\big)\big(1-F(x)\big)\,dx \\
&= \int \big(-f'(x)\mathbf{1}_{(-\infty,m]}(x) + f'(x)\mathbf{1}_{[m,\infty)}(x)\big) \\
&\quad \cdot N'\big(f(x)\big)\min\left\{\frac{F(x)}{p(x)}, \frac{1-F(x)}{p(x)}\right\}d\mu(x).
\end{aligned}$$

Now, by Lemma 2.1 of Bobkov and Houdré [8], if

$$\int N(g)\,d\mu \le \int N(f)\,d\mu$$

then

$$\int N'(f)g\,d\mu \le \int N'(f)f\,d\mu.$$

This directly implies that, in general,

$$\|f\|_N \int N'\left(\frac{f}{\|f\|_N}\right)g\,d\mu \le \|g\|_N \int N'\left(\frac{f}{\|f\|_N}\right)f\,d\mu.$$

Then, by setting $g = (-f'\mathbf{1}_{(-\infty,m]} + f'\mathbf{1}_{[m,\infty)}) \min\{\frac{F}{p}, \frac{1-F}{p}\}$, it follows that

$$\mathbb{E}\big[N(f)\big] - N\big(f(m)\big) = \int N'(f) g \, d\mu \le \|g\|_N \int N'(f) f \, d\mu.$$

Since $\min\{\frac{F}{p}, \frac{1-F}{p}\} \le \mathrm{Is}^{-1}(\mu)$ almost everywhere and N is even, we have $\|g\|_N \le \mathrm{Is}^{-1}(\mu)\|f'\|_N$. In addition,

$$\begin{aligned}\int N'(f) f \, d\mu &= \int N'(f) f \mathbf{1}_{f \le 0} + N'(f) f \mathbf{1}_{f>0} \, d\mu \\ &= \int -f N'(-f) \mathbf{1}_{f \le 0} + N'(f) f \mathbf{1}_{f>0} \, d\mu \\ &\le C_N \int N(f) \, d\mu = C_N.\end{aligned}$$

This concludes the proof of Inequality (3.17). Then Inequality (3.18) is deduced from (3.17) by noticing that

$$\big\|f - \mathbb{E}[f]\big\|_N \le \big\|f - f(m)\big\|_N + \big\|\mathbb{E}\big[f - f(m)\big]\big\|_N \le 2\big\|f - f(m)\big\|_N. \qquad \square$$

To conclude this section, we combine the L_p-Poincaré inequality (3.5) with the covariance inequality given in Theorem 3.1. We obtain the following covariance inequality, which can be compared to inequality (1.8) – which is a consequence of asymmetric Brascamp–Lieb inequality – for strongly log-concave measures.

Proposition 3.2. *Let μ be a probability measure with a positive density f on $\mathbb{R}$ and cumulative distribution F. Let $g \in L_p(F)$ and $h \in L_q(F)$, $p^{-1} + q^{-1} = 1$, $p, q \in (1, +\infty)$. Assume also that g and h are absolutely continuous. Then we have,*

$$\big|\mathrm{Cov}(g,h)\big| \le 2(p+q)\big[\mathrm{Is}(\mu)\big]^{-2}\big\|g'\big\|_p\big\|h'\big\|_q. \tag{3.19}$$

Notice that using Hölder's inequality together with Inequality (3.5) of Theorem 3.3 already yields,

$$\begin{aligned}\big|\mathrm{Cov}(g,h)\big| &\le \big\|g - E_F[g]\big\|_p\big\|h - E_F[h]\big\|_p \\ &\le 4pq\big[\mathrm{Is}(\mu)\big]^{-2}\big\|g'\big\|_p\big\|h'\big\|_q \\ &= 4(p+q)\big[\mathrm{Is}(\mu)\big]^{-2}\big\|g'\big\|_p\big\|h'\big\|_q.\end{aligned}$$

Proof. This simply follows by combining Theorem 3.1 and Inequality (3.5) of Theorem 3.3 with $u = h$. $\square$

4. Moment inequalities and concentration

We investigate here other consequences of covariance inequalities of the forms obtained in Sections 2 and 3. For a measure μ on $\mathbb{R}$, we will say in this section that μ satisfies a (p,q) covariance inequality, $p,q \geq 1$ and $p^{-1}+q^{-1}=1$, with constant $c_p > 0$ if for any absolutely continuous functions $g \in L_p$ and $h \in L_q$,

$$\left|\mathrm{Cov}(g,h)\right| \leq c_p \left\|g'\right\|_p \|h_0\|_q. \tag{4.1}$$

We proved that one can always take $c_p = p[\mathrm{Is}(\mu)]^{-1}$ (the case where $\mathrm{Is}(\mu)=0$ corresponding to $c_p = +\infty$ and being uninformative). From such covariance inequalities, we can control the growth of moments of μ.

Proposition 4.1. *Assume that μ satisfies a (p,q) covariance inequality with constant $c_p > 0$ and let X be a random variable with distribution μ. Then*

$$\left\|X - \mathbb{E}[X]\right\|_p := \left\{\mathbb{E}\left[\left|X-\mathbb{E}[X]\right|^p\right]\right\}^{1/p} \leq 2c_p. \tag{4.2}$$

For X a real random variable, define the Orlicz norm $\|X\|_{\Psi_1}$ by (3.16) with $N(v) \equiv \Psi_1(v) \equiv \exp(|v|)-1$. Note that applying (3.15) with $N=\Psi_1$ gives $C_{\Psi_1}=+\infty$ so that Proposition 3.1 does not apply to the Ψ_1-norm. From Proposition 4.1, note also that if there exists $c \in \mathbb{R}_+$ such that $c_p \leq cp$ then the moments of X grow linearly and

$$\left\|X-\mathbb{E}[X]\right\|_{\Psi_1} \leq C \sup_{p>1} \frac{\|X-\mathbb{E}[X]\|_p}{p} \leq C \sup_{p>1} \frac{c_p}{p},$$

for a numerical constant $C>0$. Saumard and Wellner [30] Appendix A show that the inequality holds with $C=4$. If in addition $\sup_{p>1} p^{-1}c_p$ is finite then μ achieves a sub-exponential concentration inequality. In particular, we recover the well-known fact that if $\mathrm{Is}(\mu)>0$ then all moments of X are finite, they grow linearly, X has a finite exponential moment and achieves a sub-exponential concentration inequality. More precisely, in this case,

$$\left\|X-\mathbb{E}[X]\right\|_p \leq p\left[\mathrm{Is}(\mu)\right]^{-1} \quad \text{and} \quad \left\|X-\mathbb{E}[X]\right\|_{\Psi_1(\mu)} \leq C\left[\mathrm{Is}(\mu)\right]^{-1}.$$

Proof. It suffices to prove (4.2). The rest is simple and/or well known and thus left to the reader. From (4.1) applied with $g(X)=X-\mathbb{E}[X]$ and $h(X)=\mathrm{sign}(X-\mathbb{E}[X])|X-\mathbb{E}[X]|^{p-1}$, we get

$$\begin{aligned}\left\|X-\mathbb{E}[X]\right\|_p &= \left|\mathrm{Cov}_\mu(g,h)\right| \\ &\leq c_p\|h_0\|_q \leq 2c_p\|h\|_q \\ &\leq 2c_p\left\|X-\mathbb{E}[X]\right\|_p^{1-1/p}\end{aligned}$$

and thus (4.2) is proved. □

We have the following moment comparison theorem for measures with finite isoperimetric constant.

Proposition 4.2. *Let X be a real centered random variable of distribution μ with finite isoperimetric constant and density f on $\mathbb{R}$. Then, for any $p > 1$,*

$$\|X\|_{p+1} \le \left(\frac{p^2}{(p-1)\operatorname{Is}(\mu_{\|X\|_p})} \right)^{1/(p+1)} \|X\|_p,$$

where μ_c is the distribution associated to the density $f_c(x) = cf(cx)$.

To the best of our knowledge, the result of Proposition 4.2 is new. Its proof is based on the covariance inequality (3.2) of Theorem 3.1. On the other hand, Proposition 4.2 is apparently considerably weaker than Theorem 8.1 of Bobkov and Houdré [8] who considered the following setting: suppose that $v_1, \dots, v_n$ are vectors in a Banach space B, and let $\xi_1, \dots, \xi_n$ be i.i.d. real-valued random variables with $E\xi_1 = 0$, $\xi_1 \neq 0$ a.s., and with law μ such that $\operatorname{Is}(\mu) > 0$. Then, with $S \equiv \| \sum_{j=1}^n \xi_j v_j \|_B$ and N a Young function satisfying $K_N \equiv \|\Lambda\|_N < \infty$ where Λ is the standard Laplace distribution on $\mathbb{R}$, they show that

$$\|S\|_N \le \left(2 + \frac{8\sqrt{3}K_N}{\operatorname{Is}(\mu)\mathbb{E}|\xi_1|} \right) \|S\|_1. \tag{4.3}$$

Specializing this to the case $n = 1$ and $N(x) = |x|^p$ with $p \ge 1$ yields

$$\|\xi_1\|_p \le \left(2 + \frac{8\sqrt{3}\Gamma(p+1)}{\operatorname{Is}(\mu)\mathbb{E}|\xi_1|} \right) \|\xi_1\|_1.$$

Since our Proposition 4.2 does not allow $p = 1$ and since it does not yield a Khintchine-Kahane type result as in (4.3), it is evidently much weaker.

Proof. We use Inequality (3.2) and take $g(x) = \operatorname{sign}(x)|x|^p$, $h(x) = x$ and the value q in Inequality (3.2) to be equal to the value of p in Theorem 4.2. Note that we have $g'(x) = p|x|^{p-1}$ if $x \neq 0$. This gives

$$\begin{aligned}
\mathbb{E}\big[|X|^{p+1}\big] &\le \frac{p^2}{p-1}\big[\operatorname{Is}(\mu)\big]^{-1}\mathbb{E}\big[\big(|X|^{p-1}\big)^{\frac{p}{p-1}}\big]^{1-1/p}\mathbb{E}\big[|X|^p\big]^{1/p} \\
&= \frac{p^2}{p-1}\big[\operatorname{Is}(\mu)\big]^{-1}\mathbb{E}\big[|X|^p\big].
\end{aligned}$$

Now the conclusion follows by remarking that by homogeneity, $\|X\|_p[\operatorname{Is}(\mu_{\|X\|_p})]^{-1} = [\operatorname{Is}(\mu)]^{-1}$. □

For log-concave measures, Theorem 4.2 has the following corollary.

Corollary 4.1. *Let X be a real centered log-concave random variable. Then, for any $p \geq 2$,*

$$\|X\|_{p+1} \leq \left(\frac{\sqrt{3}p^2}{p-1}\right)^{1/(p+1)} \|X\|_p. \tag{4.4}$$

In particular,

$$\mathbb{E}\big[|X|^3\big] \leq 4\sqrt{3}\mathbb{E}\big[X^2\big]^{3/2}. \tag{4.5}$$

Proof. Simply apply Theorem 4.2 combined with the following bound on the isoperimetric constant of a log-concave variable (see for instance Bobkov and Ledoux [5], Appendix B.2): $\mathrm{Is}(\mu)^{-1} \leq \sqrt{3\,\mathrm{Var}(X)}$, which implies $\mathrm{Is}(\mu_{\|X\|_p})^{-1} \leq \sqrt{3}$ for any $p \geq 2$. □

It is well known (Borell [11]; see also Latała and Wojtaszczyk [23], Adamczak et al. [2], Latała [21]) that for Z a real log-concave variable and $p \geq q \geq 2$,

$$\|Z\|_p \leq C\frac{p}{q}\|Z\|_q, \tag{4.6}$$

where $C = 3$ works in general but can be sharpened to $C = 1$ for symmetric measures and $C = 2$ for centered measures. Considering a centered log-concave measure we can compare inequality (4.4) to inequality (4.6) with p replaced by $p + 1$ and q replaced by p in (4.6). Then inequality (4.4) can be rewritten as

$$\|X\|_{p+1} \leq \left(\frac{p}{p+1}\right)\left(\frac{\sqrt{3}p^2}{p-1}\right)^{1/(p+1)} \frac{p+1}{p}\|X\|_p \equiv C_p\frac{p+1}{p}\|X\|_p,$$

where

$$C_p = \left(\frac{p}{p+1}\right)\left(\frac{\sqrt{3}p^2}{p-1}\right)^{1/(p+1)}.$$

The latter constant C_p is a strictly decreasing quantity of p with $C_2 = 2^{5/3}/3^{5/6} \approx 1.27091$ and $C_\infty = 1$. In this special case, inequality (4.4) indeed provides a sharpening of the constant in inequality (4.6).

Moment comparisons for log-concave vectors is a theme of active research, related to conjectures on concentration of log-concave measures and involves comparisons of weak and strong moments (Paouris [28], Latała [20,21], Adamczak et al. [1,2], Latała and Strzelecka [22]).

Bubeck and Eldan [12] proved an inequality related to (4.5), with the absolute moment $\mathbb{E}[|X|^3]$ in (4.5) replaced by the smaller third moment $\mathbb{E}[X^3]$ and with $4\sqrt{3}$ replaced by 2. Their original proof of this result is based on a rather deep result of Fradelizi and Guédon [15] describing the extremal points of a convex set of log-concave measures. In comparison, the derivation of Theorem 4.2 and Corollary 4.1 from the covariance inequality (3.2) is much easier.

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