type, all depending on an additional parameter which assumes a particular value for the problem in question. The relationship between this idea and dynamic programming, which is a technique for dealing with problems in which many decisions must be made, often sequentially, to maximise or minimise a quantity of interest, is deferred until the end of the book.

In its entirety the book comprises twelve chapters, of which the first reviews methods currently used to discuss the linear wave equation, and the second the WKB method. Invariant imbedding is only introduced in Chapter 3, and it is then applied to the wave equation in Chapter 4, where the convergence of solutions is investigated along with scattering problems and wave equations with source terms.

Chapters 5 to 8 discuss splitting of the wave function, quasilinearisation, the time dependent wave equation and some asymptotics. Chapter 9 introduces operator techniques and multidimensional imbedding, while Chapter 10 presents a review of variational principles.

Included in Chapter 11, which introduces and deals with some aspects of dynamic programming, are short sections on the Green's function, the Sturm–Liouville equation, the Fredholm resolvent, quasilinearisation and the matrix Riccati equation. Finally, Chapter 12 addresses the general problem of approximation, with sections on quadrature, weighting coefficients, splines viewed from several different points of view, and some other topics.

The book offers a good general review of the fundamental ideas underlying invariant imbedding applied to the wave equation together with various applications, though never in sufficient detail for the work to be really self-contained. Despite its reasonably comprehensive coverage of issues related to the wave equation, the book conveys the impression of being both a little dated and incomplete. Perhaps this is due in part to the choice of title, which suggests a wider interpretation of wave propagation than can be achieved by studying the wave equation alone. There is, for example, no mention of any of the recent work on semilinear and quasilinear hyperbolic equations and systems to which the wave equation sometimes provides a linear approximation. In addition to this, the absence of any reference more recent than 1979 serves to reinforce the impression that the development of the book stopped long before its publication.

The main contribution made by this book is probably to be found in the path that it charts through one aspect of the work of a most gifted mathematician, carried out over a period of more than two decades.

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EMPIRICAL PROCESSES WITH APPLICATIONS TO STATISTICS
(Wiley Series in Probability and Mathematical Statistics)

By GALEN R. SHORACK and JON A. WELLNER: pp. 938. £57.45. (John Wiley & Sons Ltd, 1986)

Empirical distributions, processes, etc. ('empiricals') are of great importance in probability theory and (non-parametric) statistics, and have been much studied. The present work gives a long and detailed account of the theory by two leading contributors on the subject.

The basic framework is of a law F, from which (real-valued) random variables $X_1, X_2, \ldots$ are sampled independently. Then $F_n := n^{-1} \Sigma_1^n \delta_{x_i}$ is the empirical distribution based on the sample $(X_1, \ldots, X_n)$. The Glivenko–Cantelli theorem tells us that $F_n(.) \to F(.)$ uniformly a.s., while Donsker's theorem tells us that $\sqrt{n}(F_n(.) - F(.))$ converges weakly to a limiting Gaussian process (zero mean with covariance $F(s)(1 - F(t))$, $s \leqslant t$). The most basic case is that when the law F is U, the uniform distribution on [0, 1] (honoured in the authors' 'Uniform Song' before the preface); much of the theory can be reduced to (or based on) the uniform case, when the above limit process is the Brownian bridge.

After an introductory Chapter 1 surveying the contents of the book, the authors deal (Chapter 2) with preliminaries (including the results for partial sums, of which the empirical results are analogues), discussing weak convergence, Skorohod embedding, etc. Results quoted without proof include the Skorohod–Dudley–Wichura and Komlós–Major–Tusnády ('KMT') theorems. The main text starts with Chapter 3, where the main objects of study—the uniform empirical distribution function $\mathbb{G}_n$, empirical process $\mathbb{U}_n$, quantile process $\mathbb{V}_n$, weighted empirical process $\mathbb{W}_n$, sequential empirical process $\mathbb{K}_n$, and the authors' preferred technical tool the 'special Skorohod construction'—are introduced. Contiguous alternatives of various kinds are studied in Chapter 4, following Le Cam, Hajek & Šidák, etc. In addition to the basic Kolmogorov–Smirnov statistics of supremum type, various integral statistics (Cramér–von Mises, etc.) are considered in Chapter 5. Martingale methods (of Aalen, Rebolledo and others) are introduced in Chapter 6, and applied to censored data (Kaplan–Meier product-limit estimator) in Chapter 7, following Gill and others. Representations in terms of exponential random variables and Poisson processes follow in Chapter 8. Various exact distributions are derived in Chapter 9, notably by combinatorial methods, following Csáki, etc. 'Nearly linear bounds' for $\mathbb{G}_n$ follow in Chapter 10, and are applied to give, for instance, various upper and lower class results. One may wish to give more weight to some parts of [0, 1] than to others, which may be done by introducing a suitable weight-function q. Weighted analogues of the results above, due to Chibisov, O'Reilly and others, follow in Chapter 11.

The KMT results mentioned above are applied in Chapter 12 to give 'Hungarian constructions' of $\mathbb{K}_n$, $\mathbb{U}_n$, $\mathbb{V}_n$, etc; these are applied to laws of the iterated logarithm, oscillations of the empirical process, etc. Further topics treated include (Chapters 16–26) normalised empiricals, indexing by intervals and functions, quantiles, L-statistics, ranks, spacings, tests for symmetry, large deviations, the non-identically distributed case, and empiricals on more general spaces. There is a long appendix on inequalities (emphasised throughout the text: 'good inequalities are the key to strong theorems', as the authors say in their preface), and another on martingales and counting processes. There is a bibliography of some five hundred items.

The treatment is detailed, even encyclopaedic, and the book contains a massive amount of information. It pre-supposes 'a standard graduate course in probability and some exposure to non-parametric statistics'. Much of the contents might be described as statistically motivated probability theory: statistical applications are discussed, and some statistical tables given, but the statistical motivation the reader will need to undertake the study of this book must be found elsewhere. The authors make an effort to supply heuristic understanding of their (often very technical) arguments, frequently giving several proofs of major results by different methods.

The book contains so much that seems almost churlish to turn to what it omits. One major technical omission is a proof of the KMT results, of which the authors make such extensive use (they stress that their aim is to teach the reader how KMT theory can be applied). There is no mention of the non-independent case; as the authors remark in their preface, the literature here (on mixing, etc.) is large. However, in a work of nearly a thousand pages, space could perhaps have been found for a brief sketch of the mixing theory, with references. The great bulk of the text is devoted to the real-valued case; more general spaces are given only a brief mention in the last chapter. In particular, probability on Banach spaces is barely touched on, despite its important links with empiricals. See for instance §6 of Kuelbs [4] for an application to weighted empiricals in the context of Chapter 11. Also not developed here is the $\mathbb{R}^d$-valued case. In this connection, we mention that the KMT theory has recently been extended to the higher-dimensional case by Berger [1]. (The KMT theorems are stated without proof, not only here, but in the other most relevant text, that of M. Csörgő & Révész [2]. A textbook synthesis, in one and higher dimensions, seems overdue.) There is also no mention of empirical characteristic functions and processes (S. Csörgő, M. B. Marcus & W. Philipp, etc.). The bibliography, though large, is not complete, particularly from 1984 onwards. Among authors who have made distinguished recent contributions and are not cited, we mention R. F. Bass, J. Beck, P. Deheuvels, R. S. Ellis, J. H. J. Einmahl, J. Hawkes, V. I. Kolčinskii, M. Ledoux and S. R. S. Varadhan. A number of important recent references – including R. M. Dudley's lectures [3] – are cited only in preprint form, despite having appeared in 1983–85.

Turning to more mundane matters: there is an index of notation, but this does not contain the special symbols $\mathbb{U}_n$, $\mathbb{V}_n$, etc. needed to write this review. There are both short (1-page) and long (18-page) tables of contents, but even so this is not an easy book to find one's way around in: addition of chapter and section numbers to the titles at the tops of pages would save endless thumbing in following up cross-references. In the foreign (particularly French) language material in the bibliography there is hardly an accent correctly in place. Unfortunately, proper names are also affected, with Lévy, Rényi, Loève and Cramér among the sufferers (though not, apparently, Csörgő!).

In sum: the book does not attempt a complete coverage of empiricals, but rather of the real-valued independent case. Here it gives a thorough and well-organised account, which will be welcomed as a work of reference by many probabilists and statisticians.

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