Goodness of fit via phi-divergences: a new family of test statistics

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- joint work with Leah Jager,
  Grinnell College
- Talk at Northwest Probability Seminar
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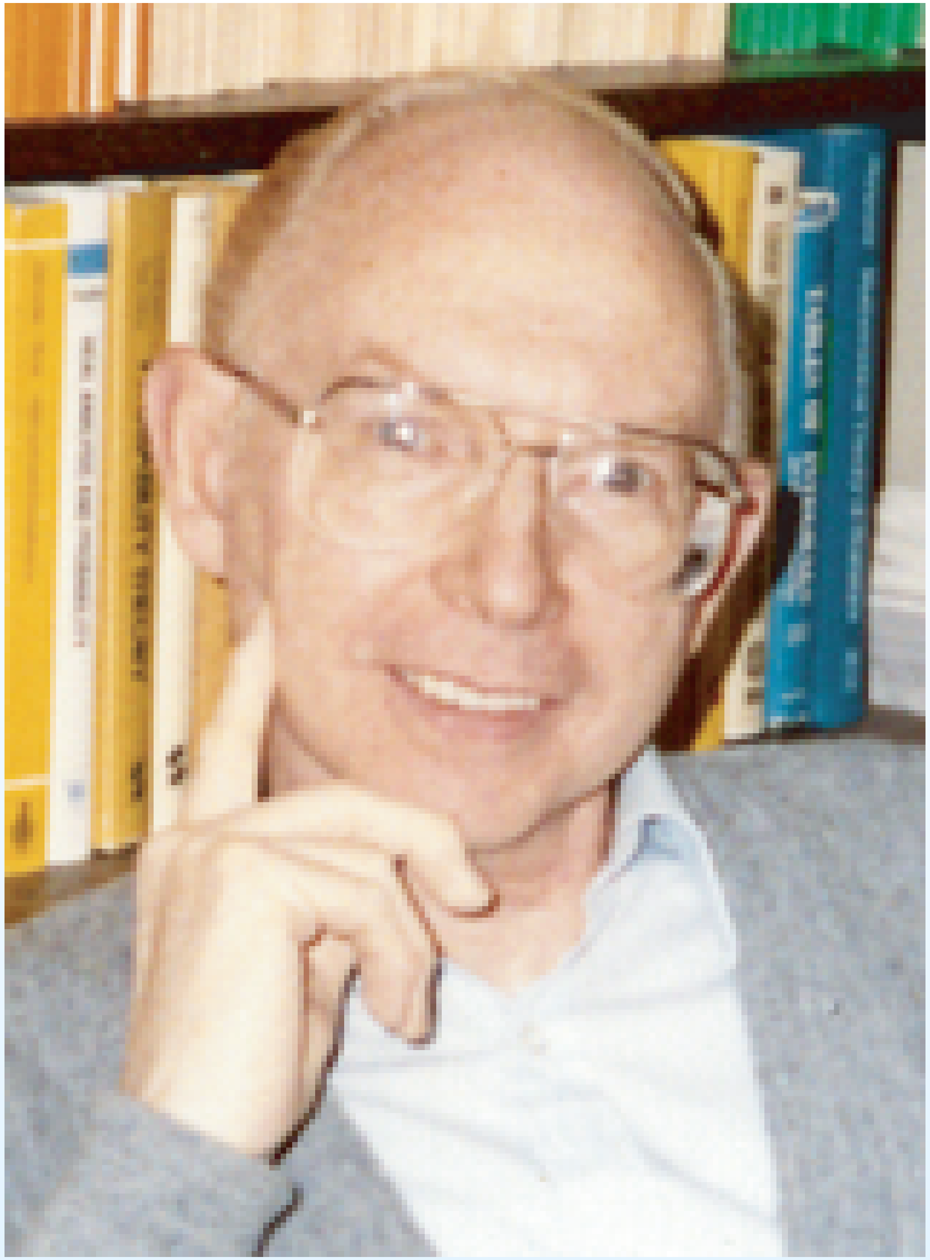
- Problems and questions

Ron Pyke at Oberwolfach, 1971



Ron Pyke

Bill Birnbaum and Ron Pyke,
IMS Summer Research Institute, Bloomington, 1974

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- Pyke, Ronald (1970). *Asymptotic results for rank statistics*. in *Nonparametric Techniques in Statistical Inference*, pp. 21 - 37. M. L. Puri, ed. Cambridge University Press.

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- Pyke, Ronald (1972). *Empirical processes*. in *Jeffrey-Williams Lectures: 1968-1972*, 13-43. Canadian Mathematical Congress, Montreal.

ASYMPTOTIC RESULTS FOR RANK STATISTICS†

RONALD PYKE‡

1 INTRODUCTION

I appreciate the opportunity to speak at this Symposium. When I accepted Professor Puri's invitation to give an expository paper, I did so because I believe that an expository paper is a *good thing*, in the spirit of Sellar and Yeatman's dichotomization of British history, 1960. It is clear however that a complete exposition of the enormous literature which exists today on limit theorems for rank statistics should not be attempted within a short paper. Moreover, the book by Hájek and Sidak (1967) already provides an excellent coverage of most of this literature. I shall instead simply attempt to impart some personal comments on the philosophy of limit theory and to illustrate by means of two examples one particular approach to asymptotic results for rank statistics.

The significant role of limit theory in Nonparametric Inference, and hence the importance of a limit theorist or 'limitor', is empirically verified by the following statement:

'64% of all papers in Nonparametric Inference
are concerned primarily with asymptotic results.' (1)

This finding is based on the personal assignment of weights 0, $\frac{1}{2}$ and 1 to the random [*sic*] sample of 33 papers listed in the program of this Symposium. The weights were assigned according to a paper's proportional concern with asymptotic results. The sample average of these weights was 21/33. Although the announced finding (1) is based on a very quick and subjective analysis, I am sure it truly reflects the high concentration of limit theory in our subject. I suspect if one made a more careful and objective evaluation of this concentration, based say on the printed proceedings of this Symposium, a higher estimate would be obtained. This emphasis on asymptotic results is of course a quality of statistical literature as a whole, and not just of Nonparametric Inference.

'Practical problems are finite; tractable problems are infinite.'

† An invited paper presented at the First International Symposium on Nonparametric Techniques in Statistical Inference, Indiana University, Bloomington, Ind., 1–6 June 1969.

‡ The research described herein was supported in part by the National Science Foundation under G-5719.

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- Break hypotheses down into family of pointwise hypotheses: $H_x : F(x) = F_0(x)$ versus $K_x : F(x) \neq F_0(x)$
- $H = \cap_x H_x$, $K = \cup_x K_x$

- Likelihood ratio statistic for testing H_x versus K_x:

$$\begin{aligned} \lambda_n(x) &= \frac{\sup_{F(x)} L_n(F(x))}{L_n(F_0(x))} = \frac{L_n(\mathbb{F}_n(x))}{L_n(F_0(x))} \\ &= \frac{\mathbb{F}_n(x)^{n\mathbb{F}_n(x)}(1-\mathbb{F}_n(x))^{n(1-\mathbb{F}_n(x))}}{F_0(x)^{n\mathbb{F}_n(x)}(1-F_0(x))^{n(1-\mathbb{F}_n(x))}} \\ &= \left(\frac{\mathbb{F}_n(x)}{F_0(x)}\right)^{n\mathbb{F}_n(x)} \left(\frac{1-\mathbb{F}_n(x)}{1-F_0(x)}\right)^{n(1-\mathbb{F}_n(x))} \end{aligned}$$

- Thus

$$\log \lambda_n(x) = n\mathbb{F}_n(x) \log\left(\frac{\mathbb{F}_n(x)}{F_0(x)}\right) + n(1-\mathbb{F}_n(x)) \log\left(\frac{1-\mathbb{F}_n(x)}{1-F_0(x)}\right) = nK(\mathbb{F}_n(x), F_0(x))$$

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- $K(u,v) \equiv u\log\left(\frac{u}{v}\right) + (1-u)\log\left(\frac{1-u}{1-v}\right)$,
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  Bernoulli(u), Bernoulli(v)
- Berk-Jones (1979) test statistic:

$$R_n \equiv \sup_x n^{-1}\log\lambda_n(x) = \sup_x K(\mathbb{F}_n(x), F_0(x))\,.$$

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  - Einmahl and McKeague (2002): integral version of R_n
  - Donoho and Jin (2002): supremum version of Anderson-Darling statistic
    with comparison to Berk - Jones statistic R_n

2. A new family of statistics via phi-divergences

- For $s \in \mathbb{R}$, $x \geq 0$ define

$$\phi_s(x) = \begin{cases} \frac{1-s+sx-x^s}{s(1-s)}, & s \neq 0, 1 \\ x \log x - x + 1, & s = 1 \\ -\log x + x - 1, & s = 0\,. \end{cases}$$

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- Then define

$$K_s(u, v) = v\phi_s(u/v) + (1-v)\phi_s((1-u)/(1-v))\,.$$

- Special cases:

$$K_1(u,v) = K(u,v)$$
$$= u\log(u/v) + (1-u)\log((1-u)/(1-v))$$
$$K_0(u,v) = K(v,u)$$
$$K_2(u,v) = \frac{1}{2}\frac{(u-v)^2}{v(1-v)}$$
$$K_{-1}(u,v) = K_2(v,u) = \frac{1}{2}\frac{(u-v)^2}{u(1-u)}$$
$$K_{1/2}(u,v) = 2\{(\sqrt{u}-\sqrt{v})^2 + (\sqrt{1-u}-\sqrt{1-v})^2\}$$
$$= 4\{1-\sqrt{uv}-\sqrt{(1-u)(1-v)}\}.$$

- The new family of statistics:

$$S_n(s) = \begin{cases} \sup_{x\in\mathbb{R}} K_s(\mathbb{F}_n(x), F_0(x)), & s \geq 1 \\ \sup_{x\in[X_{(1)},X_{(n)})} K_s(\mathbb{F}_n(x), F_0(x)), & s < 1\,, \end{cases}$$

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- Thus, with $F_0(x) = x$,

$$S_n(1) = R_n, \qquad S_n(0) = \text{"reversed" Berk-Jones} \equiv \widetilde{R}_n$$

$$S_n(2) = \frac{1}{2} \sup_{x\in\mathbb{R}} \frac{(\mathbb{F}_n(x) - x)^2}{x(1-x)},$$

$$S_n(-1) = \frac{1}{2} \sup_{x\in[X_{(1)},X_{(n)})} \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n(x)(1-\mathbb{F}_n(x))}$$

$$S_n(1/2)$$

$$= 4 \sup_{x\in[X_{(1)},X_{(n)})} \{1 - \sqrt{\mathbb{F}_n(x)x} - \sqrt{(1-\mathbb{F}_n(x))(1-x)}\}$$

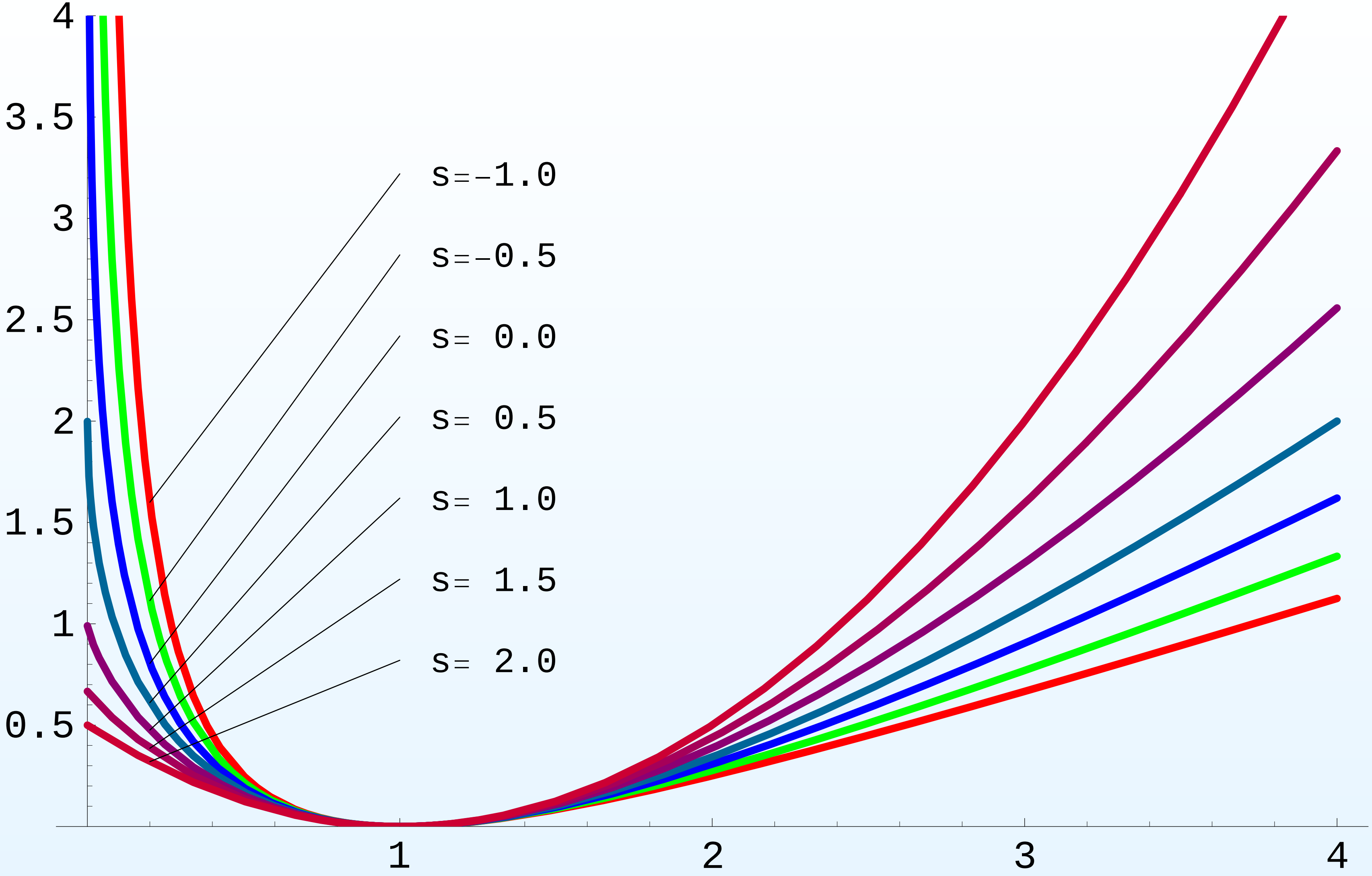

Fig. 1: $\phi_s(x)$, $s \in \{-1, -0.5, 0.0, 0.5, 1.0, 1.5, 2.0\}$

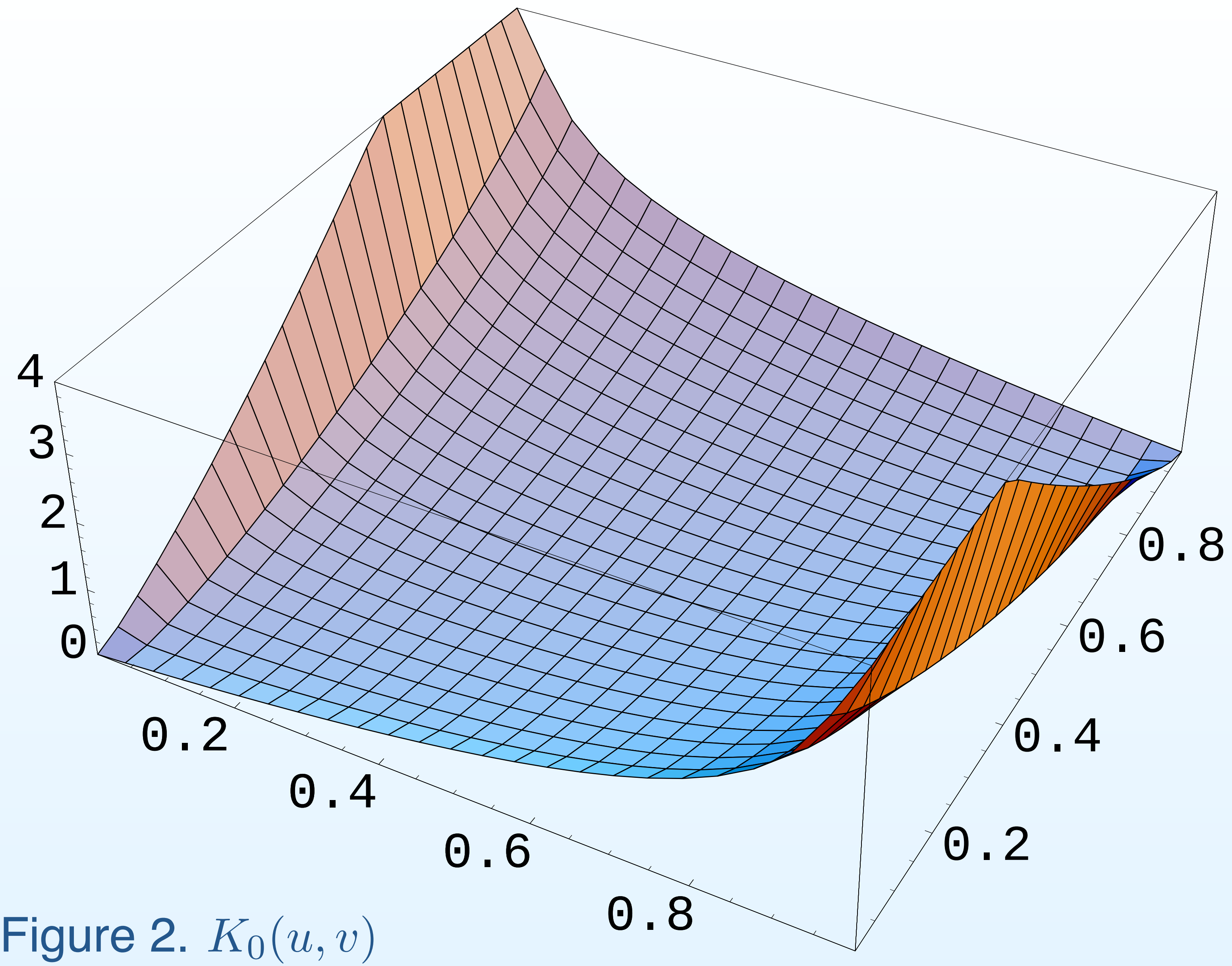

Figure 2. $K_0(u, v)$

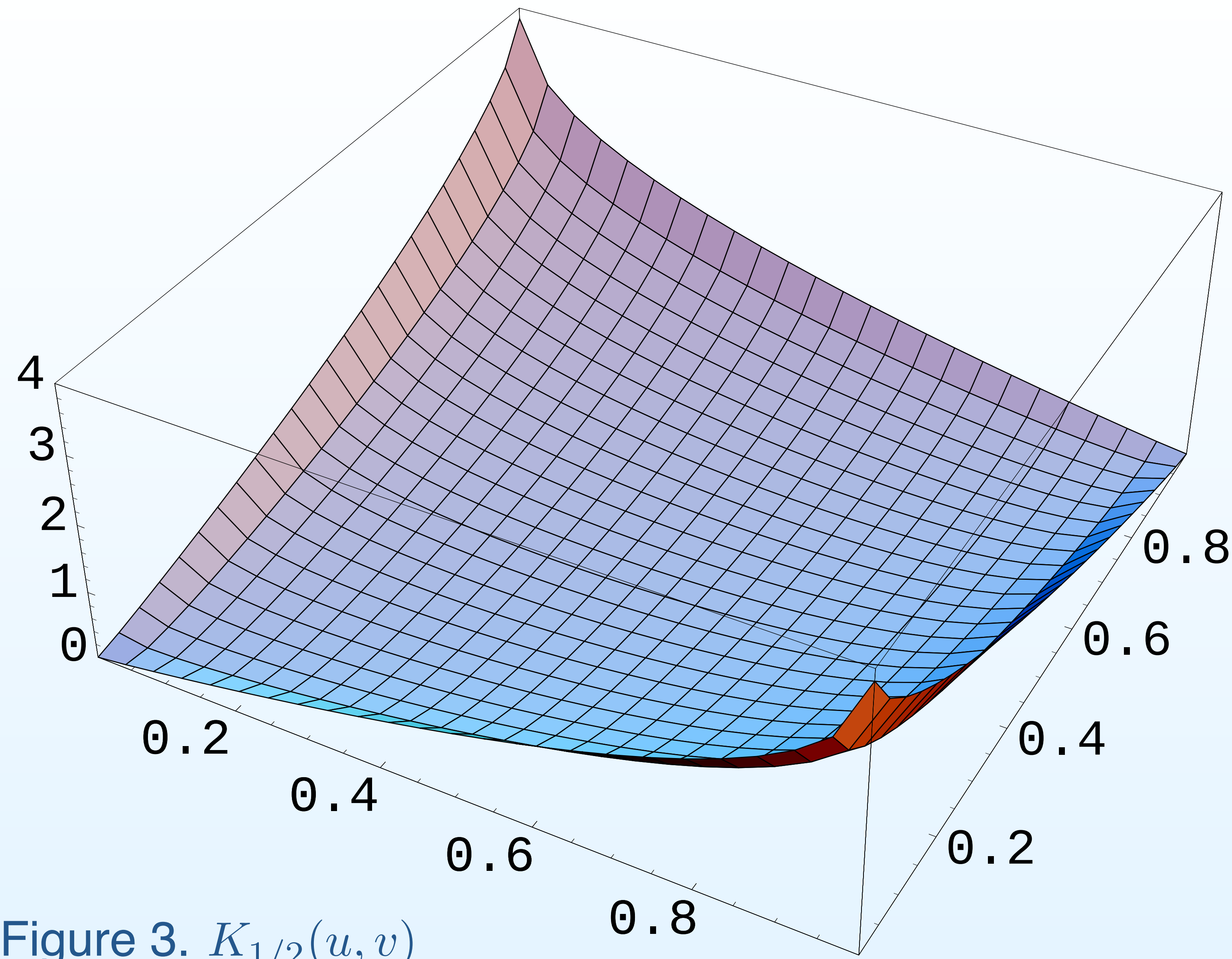

Figure 3. $K_{1/2}(u, v)$

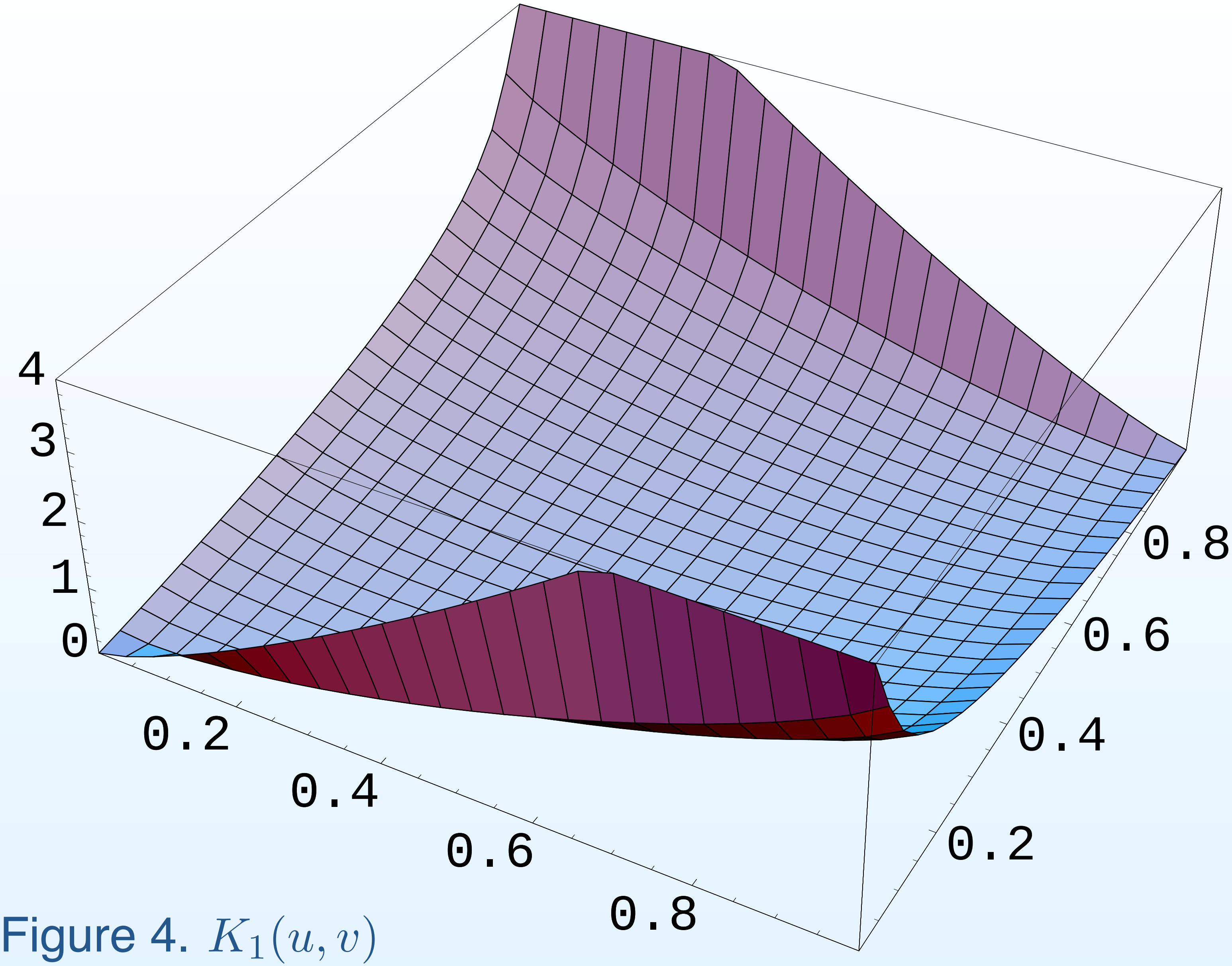

Figure 4. $K_1(u, v)$

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- For $n \geq 3000$, asymptotic theory via Jaeschke (1979) and Eicker (1979) (cf. SW p. 597 - 615), together with

$$K_s(u,v) \approx 2^{-1}(u-v)^2/[v(1-v)]$$

so

$$nK_s(\mathbb{F}_n(x),x) \approx \frac{1}{2}\frac{n(\mathbb{F}_n(x)-x)^2}{x(1-x)} \equiv \frac{1}{2}\mathbb{Z}_n(x)^2$$

where

$$\mathbb{Z}_n(x) \equiv \frac{\sqrt{n}(\mathbb{F}_n(x)-x)}{\sqrt{x(1-x)}} \rightarrow_{f.d.} \frac{\mathbb{U}(x)}{\sqrt{x(1-x)}} \equiv \mathbb{Z}(x)$$

with $\mathbb{U}$ a standard Brownian bridge process on $[0,1]$.

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- $\mathbb{X} \equiv$ Ornstein-Uhlenbeck process on $(-\infty, \infty)$

$$Cov[\mathbb{X}(s), \mathbb{X}(t)] = \exp(-|t - s|)$$

$$\mathbb{X}(t) \stackrel{d}{=} e^{-t}\mathbb{B}(e^{2t}) = \frac{\mathbb{B}(e^{2t})}{\sqrt{e^{2t}}}$$

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- Thus we can represent $\mathbb{Z}$ in terms of $\mathbb{X}$:

$$\begin{aligned} \mathbb{Z}(x) \quad &\equiv \quad \frac{\mathbb{U}(x)}{\sqrt{x(1-x)}} \stackrel{d}{=} \sqrt{\frac{1-x}{x}}\mathbb{B}\left(\frac{x}{1-x}\right) \\ &\stackrel{d}{=} \quad \mathbb{X}\left(\frac{1}{2}\log\frac{x}{1-x}\right) \end{aligned}$$

- Thus for $0 < d_n < e_n < 1$, with $a_n \equiv e_n(1 - d_n)/[d_n(1 - e_n)]$,

$$\|\mathbb{Z}\|_{d_n}^{e_n} \equiv \sup_{d_n \le x \le e_n} |\mathbb{Z}(x)| \stackrel{d}{=} \|\mathbb{X}\|_{2^{-1}\log(d_n/(1-d_n))}^{2^{-1}\log(e_n/(1-e_n))}$$

$$\stackrel{d}{=} \|\mathbb{X}\|_0^{2^{-1}\log a_n} \quad \text{by stationarity of } \mathbb{X}$$

$$\stackrel{d}{=} \left\|\frac{\mathbb{B}(t)}{\sqrt{t}}\right\|_1^{a_n}$$

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- If $d_n = 1/n$, $e_n = 1 - 1/n$, then $a_n = n^2(1-1/n)^2 \sim n^2$, and

$$2^{-1}\log a_n \sim \log n$$

- $b(t) \equiv \sqrt{2 \log \log t}$

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- $E_v(x) = \exp(-\exp(-x))$, $x \in \mathbb{R}$
- **Theorem** (Darling and Erdös, 1956).

$$b(t) \left\| \frac{\mathbb{B}^+(t)}{\sqrt{t}} \right\|_1^t - c(t) \to_d Y \sim E_v, \quad \text{as} \quad t \to \infty$$

$$b(t) \left\| \frac{\mathbb{B}(t)}{\sqrt{t}} \right\|_1^t - c(t) \to_d \max\{Y_1, Y_2\} \sim E_v^2, \quad \text{as} \quad t \to \infty$$

where Y_1, Y_2 are independent, $Y_j \sim E_v$, $j = 1, 2$.

- Let $r_n \equiv \log_2 n + (1/2) \log_3 n - (1/2) \log(4\pi)$.

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- **Theorem.** If $F = F_0$, the uniform distribution on $[0,1]$, then for $-1 \leq s \leq 2$

$$nS_n(s) - r_n \to_d Y_4$$

where $P(Y_4 \leq x) = \exp(-4\exp(-x))$.

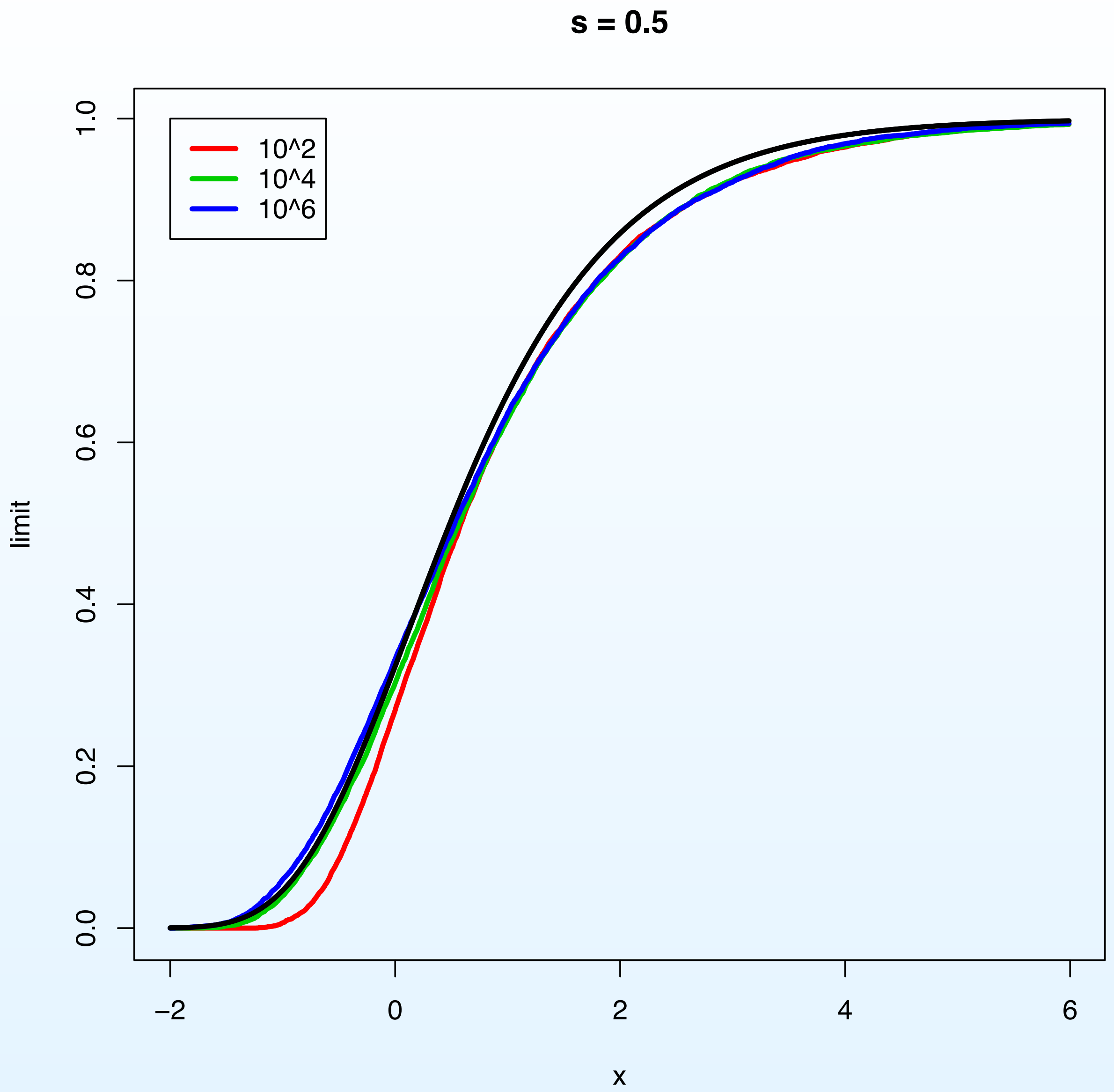

Figure 5. $P(nS_n(1/2) - r_n \leq x)$ for $n = 10^2$, 10^4, 10^6 and $P(Y_4 \leq x)$

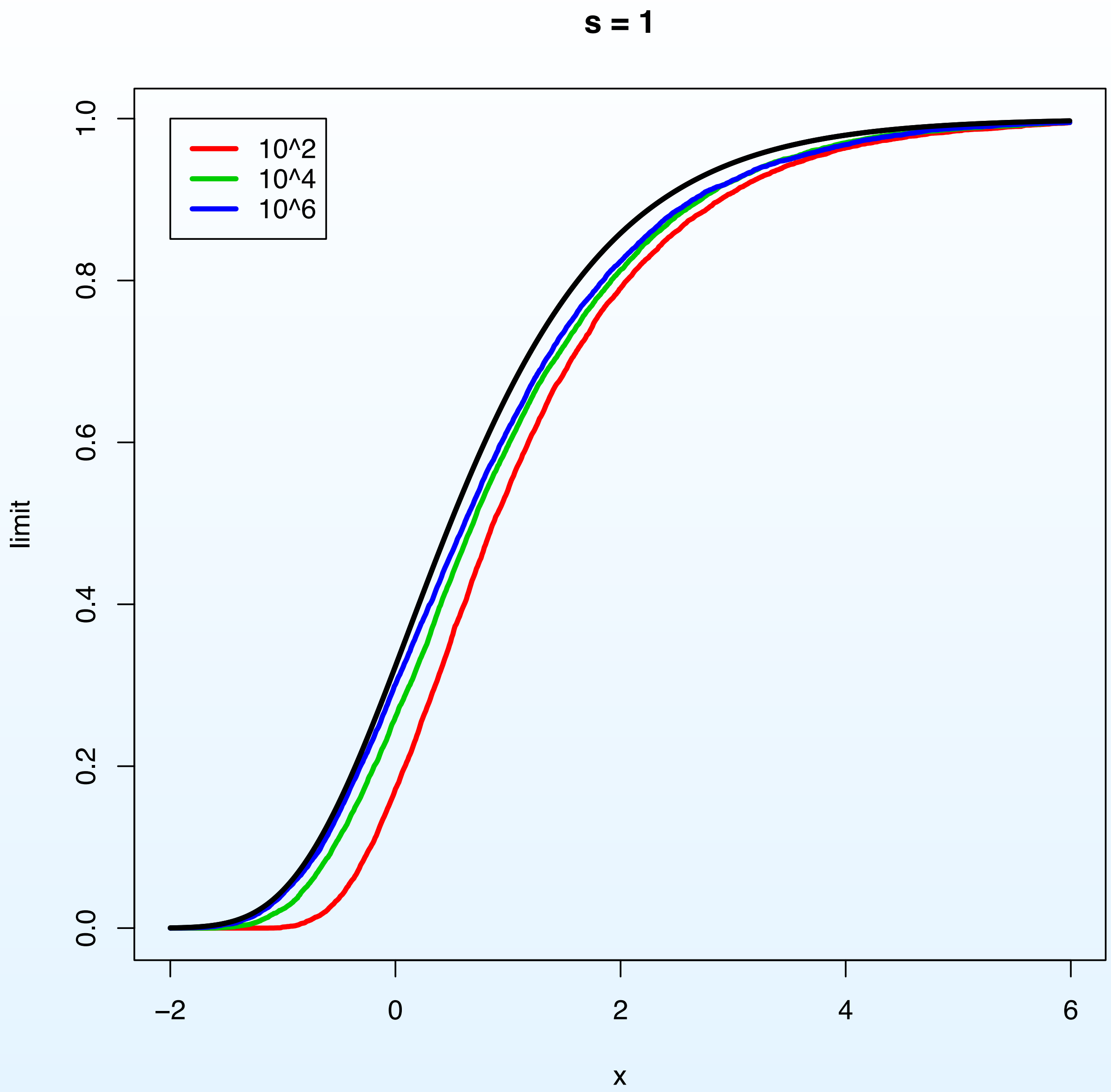

Figure 6. $P(nS_n(1) - r_n \leq x)$ for $n = 10^2$, 10^4, 10^6 and $P(Y_4 \leq x)$

- $b_n \equiv \sqrt{2\log_2 n}, \quad c_n \equiv b_n^2 + (1/2)\{\log_3 n - \log(4\pi)\}$

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- $q_n^{(3)}(\alpha) = x_{\alpha,n} + c_n^2/(2b_n^2)$

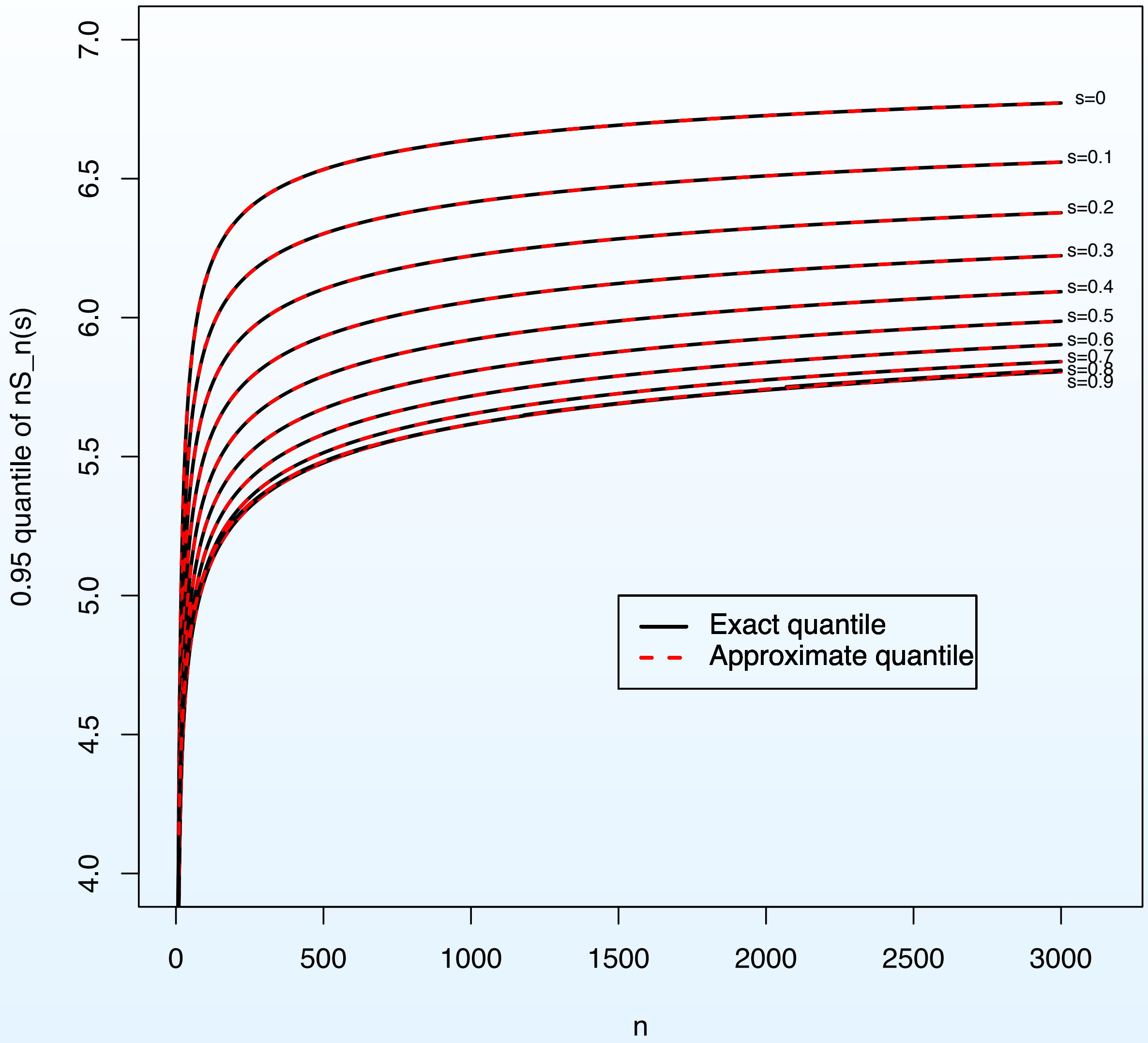

Figure 7. Exact and approximate $.95$ quantiles of $nS_n(s)$, $10 \leq n \leq 3000$.

4. Limit theory under alternatives and power

- **Theorem 1.** If $X_1, \ldots, X_n$ are i.i.d. $F \in K$ and $0 < s < 1$, then

$$S_n(s) \rightarrow_{a.s.} \sup_{0<x<1} K_s(F(x), x) \equiv S_\infty(s, F). \quad (1)$$

- **Theorem 2.** If $X_1, \ldots, X_n$ are i.i.d. $F \in K$ and $s > 1$, then (1) holds if and only if

$$\int_0^1 \{F^{-1}(u)(1 - F^{-1}(u))\}^{(1-s)/s} du < \infty.$$

- Poisson boundary distributions

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- **Theorem: (Berk & Jones, 1979).** If $F(x) = 1/(1 + \log(1/x))$, and $X_1, \ldots, X_n$ are i.i.d. F, then

$$R_n = S_n(1) \rightarrow_d 1/U \stackrel{d}{=} \sup_{t>0} \frac{\mathbb{N}(t)}{t}$$

where $U \sim U(0,1)$, $\mathbb{N}$ is a standard Poisson process.

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- Generalization: let

$$F_s(x) = \begin{cases} (1 + \frac{x^{1-s}-1}{s-1})^{-1/s}, & 1 < s < \infty, \\ (1 + \log(1/x))^{-1}, & s = 1, \\ (1 - s(x^{s-1} - 1))^{1/s}, & s < 0. \end{cases}$$

- **Theorem.** (Poisson boundaries for $s \geq 1$ and $s < 0$).
  (i) Fix $s \geq 1$ and suppose that $X_1, \ldots, X_n$ are i.i.d. F_s. Then

$$S_n(s) \to_d \frac{1}{s} \left(\sup_{t>0} \frac{\mathbb{N}(t)}{t} \right)^s \stackrel{d}{=} \frac{1}{sU^s}$$

  (ii) Fix $s < 0$ and suppose that $X_1, \ldots, X_n$ are i.i.d. F_s. Then

$$S_n(s) \to_d \frac{1}{1-s} \left(\sup_{t \geq S_1} \frac{t}{\mathbb{N}(t)} \right)^{-s}$$

  where $S_1 = E_1$ is the first jump point of $\mathbb{N}$.

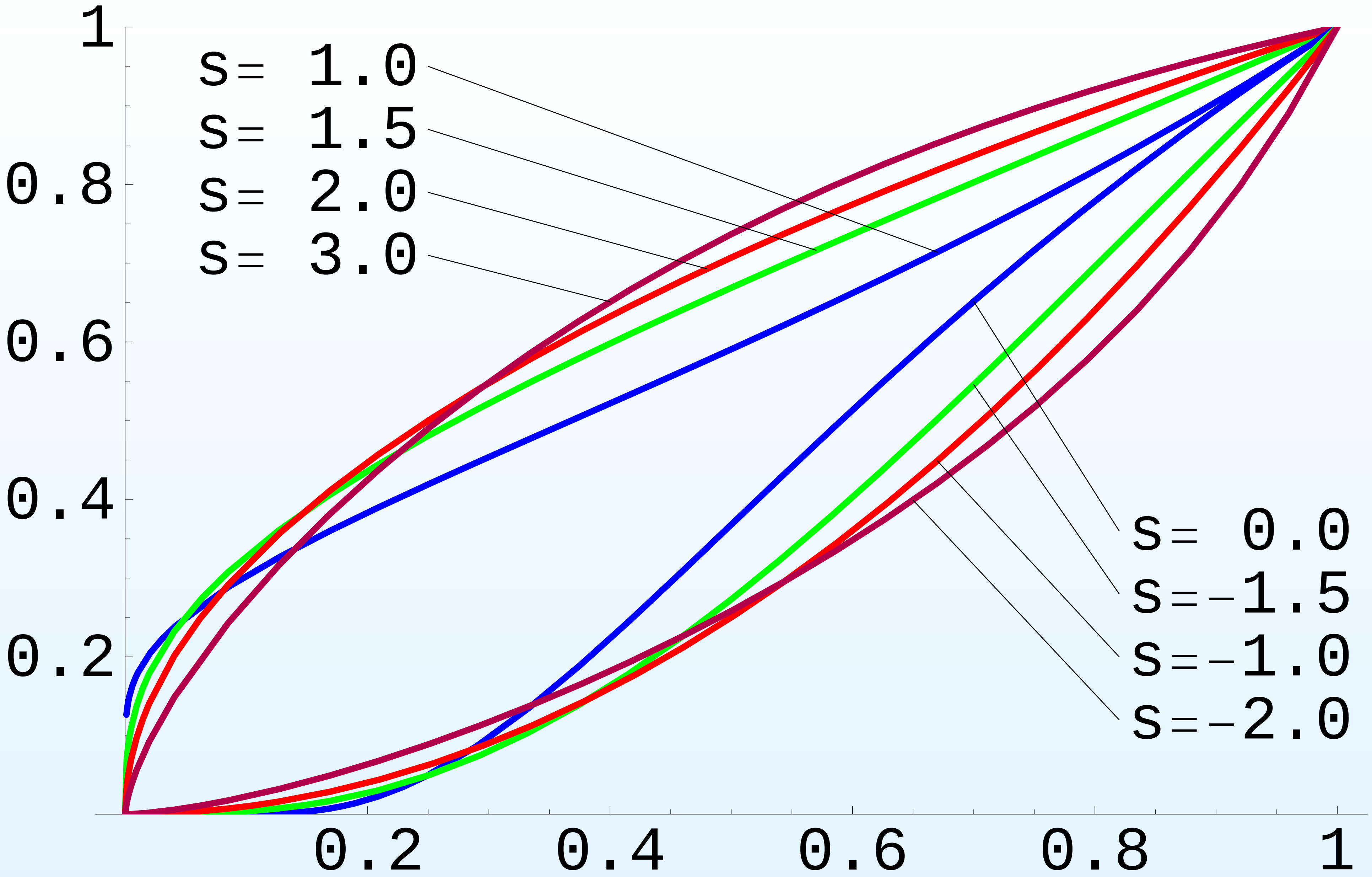

Figure 7. Poisson boundary distribution functions,
$s \in \{1, 1.5, 2, 3\} \cup \{-2, -1.0, -1.5, 0.0\}$.

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- test $H \,:\, G = \Phi$, the standard $N(0,1)$ d.f. versus $H_1 : G = (1-\epsilon)\Phi + \epsilon\Phi(\cdot - \mu)$, and, in particular, against

$$H_1^{(n)} : G = (1-\epsilon_n)\Phi + \epsilon_n\Phi(\cdot - \mu_n)$$

for $\epsilon_n = n^{-\beta}$, $\quad \mu_n = \sqrt{2r\log n}$
$1/2 < \beta < 1$, $0 < r < 1$.

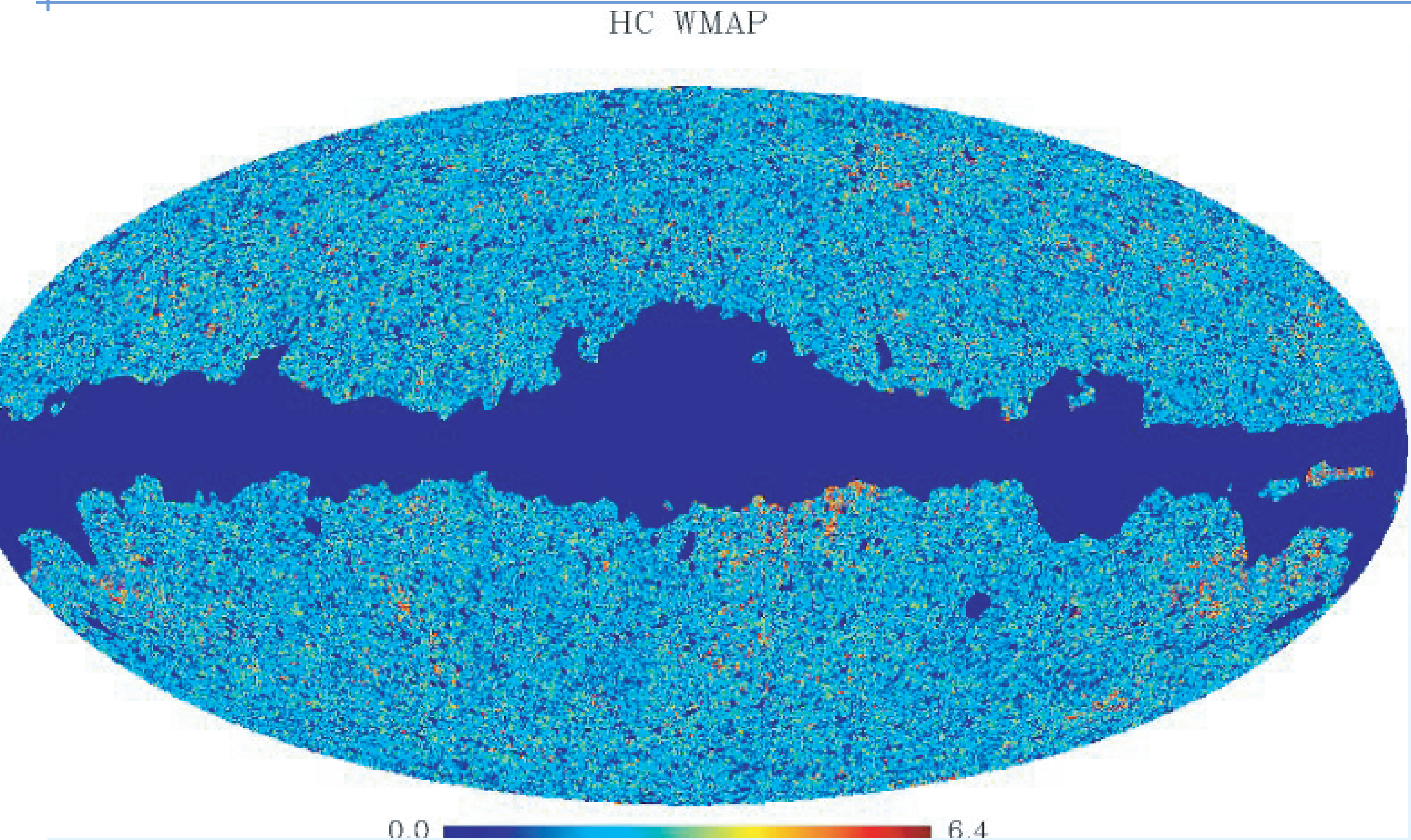
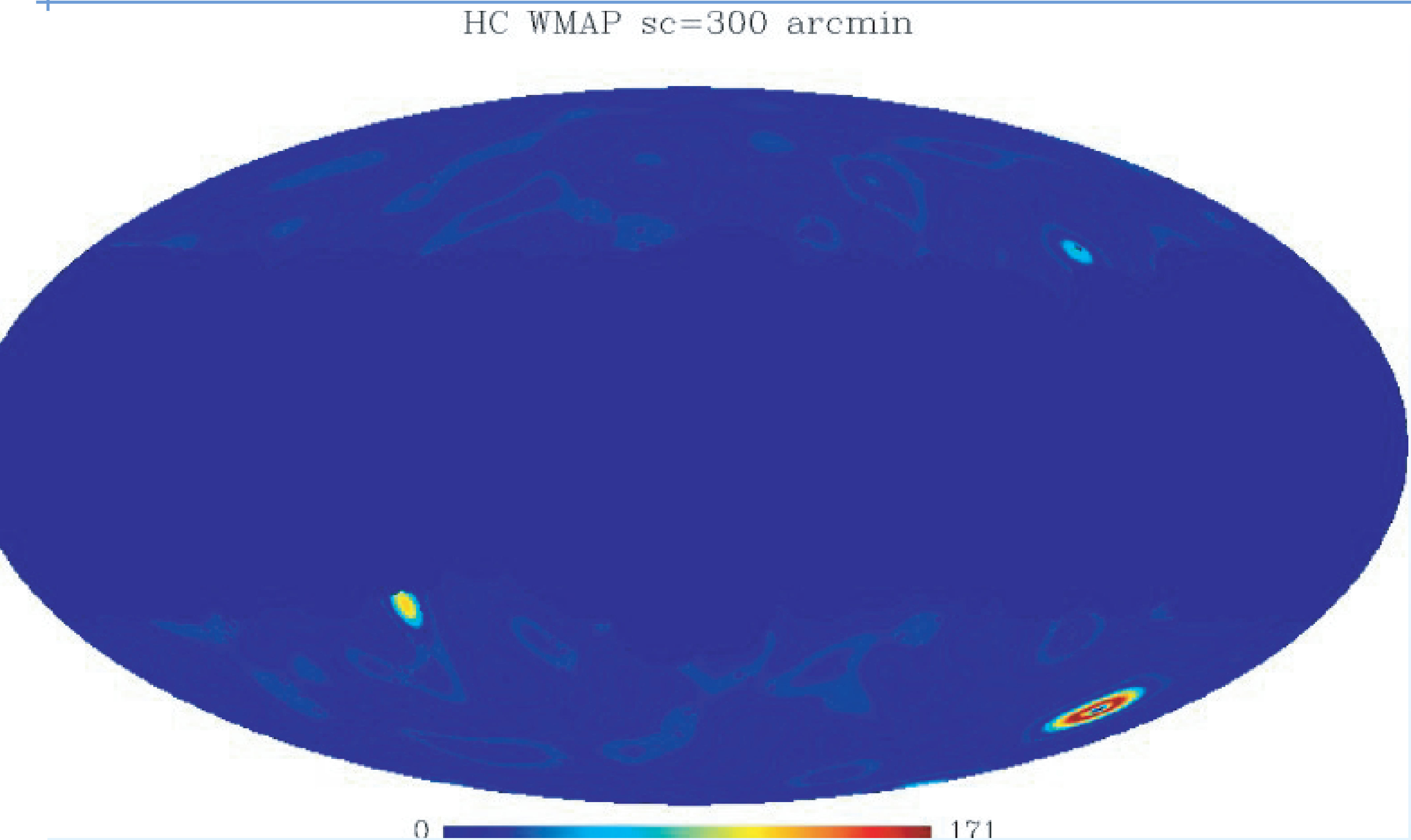


Figure 6.



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- transform to $X_i \equiv 1 - \Phi(Y_i) \in [0,1]$ i.i.d.

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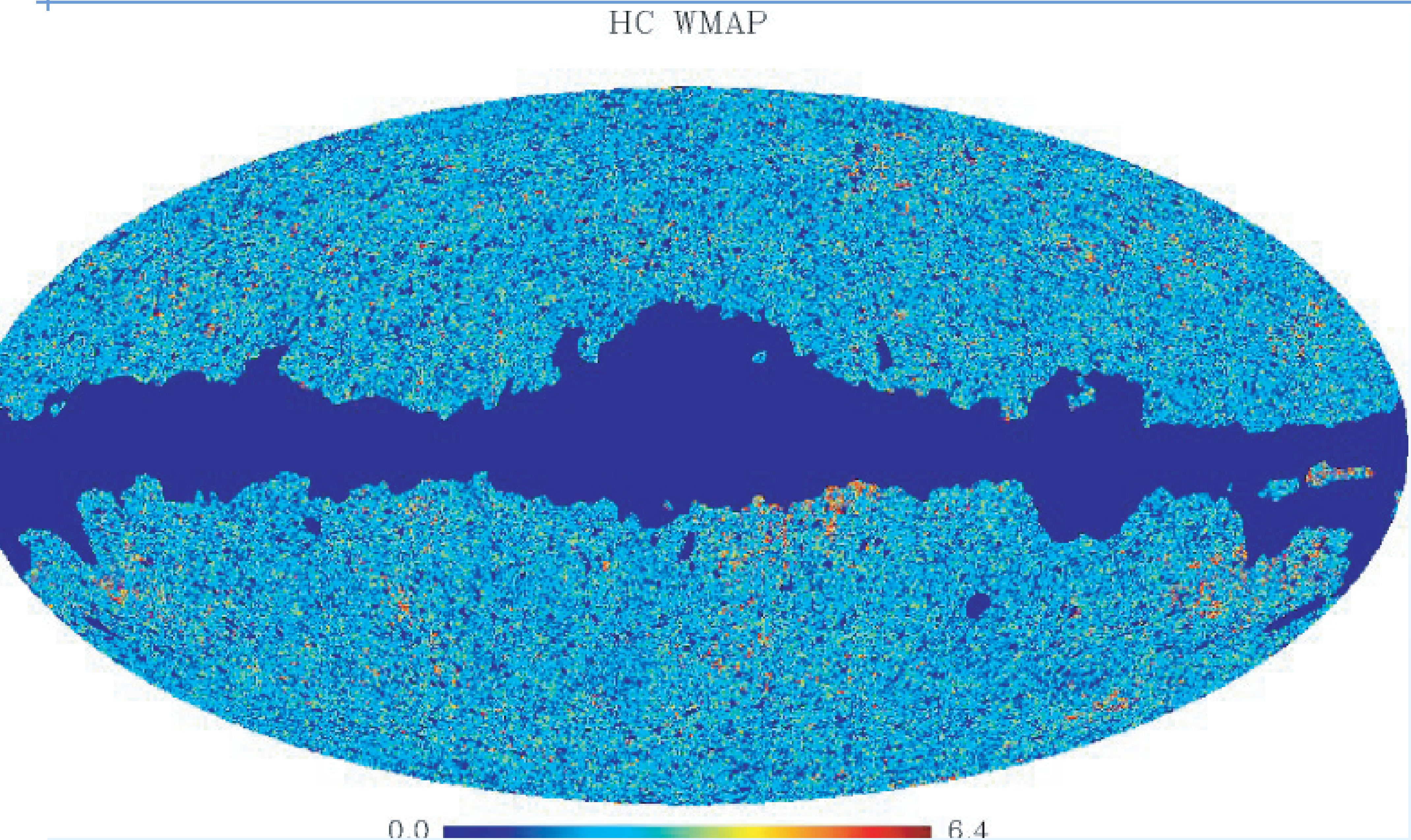
- Test statistics: Donoho-Jin: Berk-Jones $R_n = S_n(1)$ and

$$HC_n^* \equiv \sup_{X_{(1)} \le x < X_{([n/2])}} \frac{\sqrt{n}(\mathbb{F}_n(x) - x)}{\sqrt{x(1-x)}}$$
$$\equiv \text{Tukey's “higher criticism statistic”}$$



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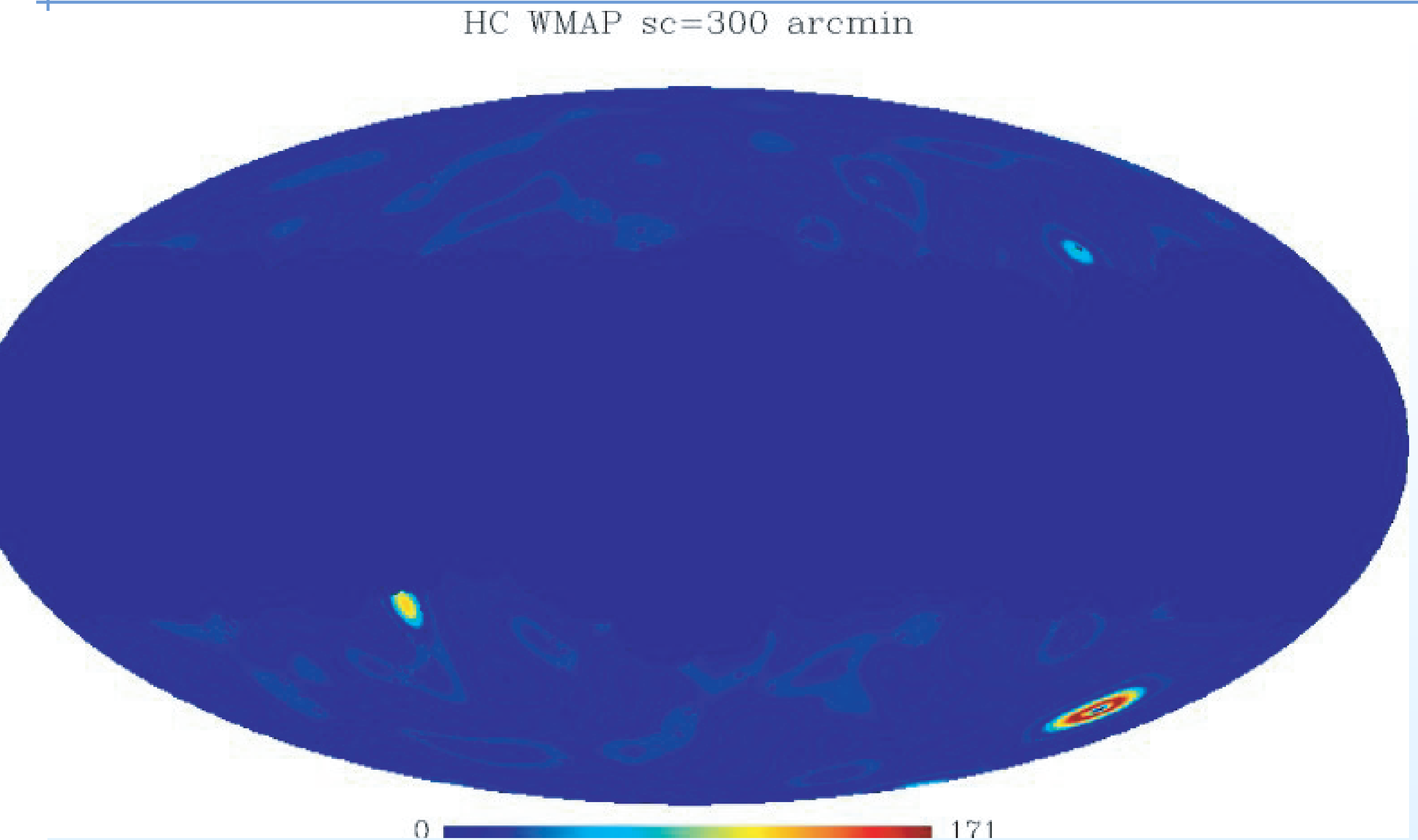

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- Define the optimal detection boundary $\rho^*(\beta)$ by

$$\rho^*(\beta) = \begin{cases} \beta - 1/2, & 1/2 < \beta \leq 3/4 \\ (1 - \sqrt{1-\beta})^2, & 3/4 < \beta < 1 \end{cases}$$

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- **Theorem:** (Jager - Wellner, 2006). For $r > \rho^*(\beta)$ the tests based on $S_n(s)$ with $-1 \leq s \leq 2$ are size and power consistent for testing H_0 versus $H_1^{(n)}$.

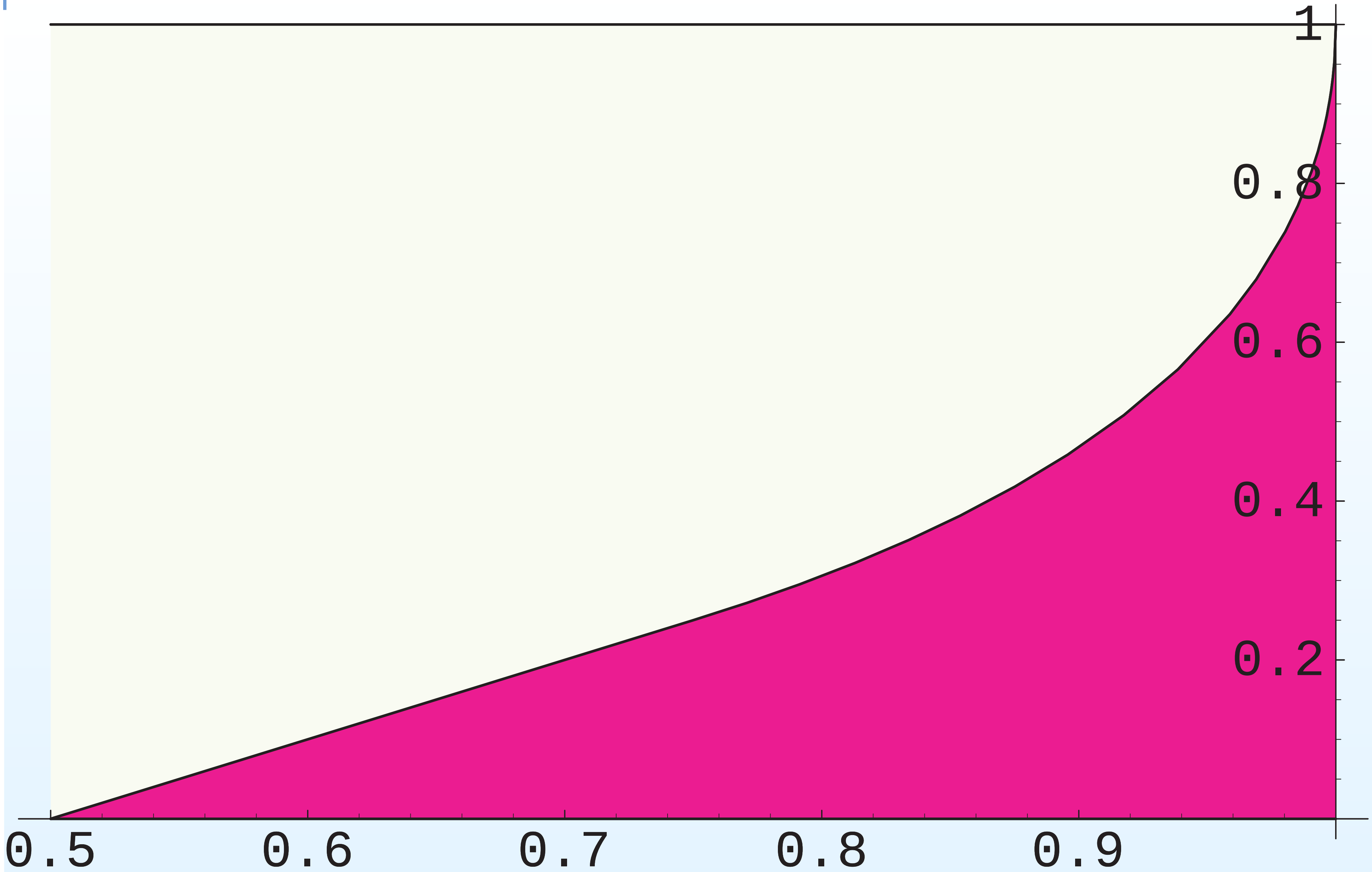

Figure 8. Detection boundary: $r > \rho^*(\beta)$ detectable

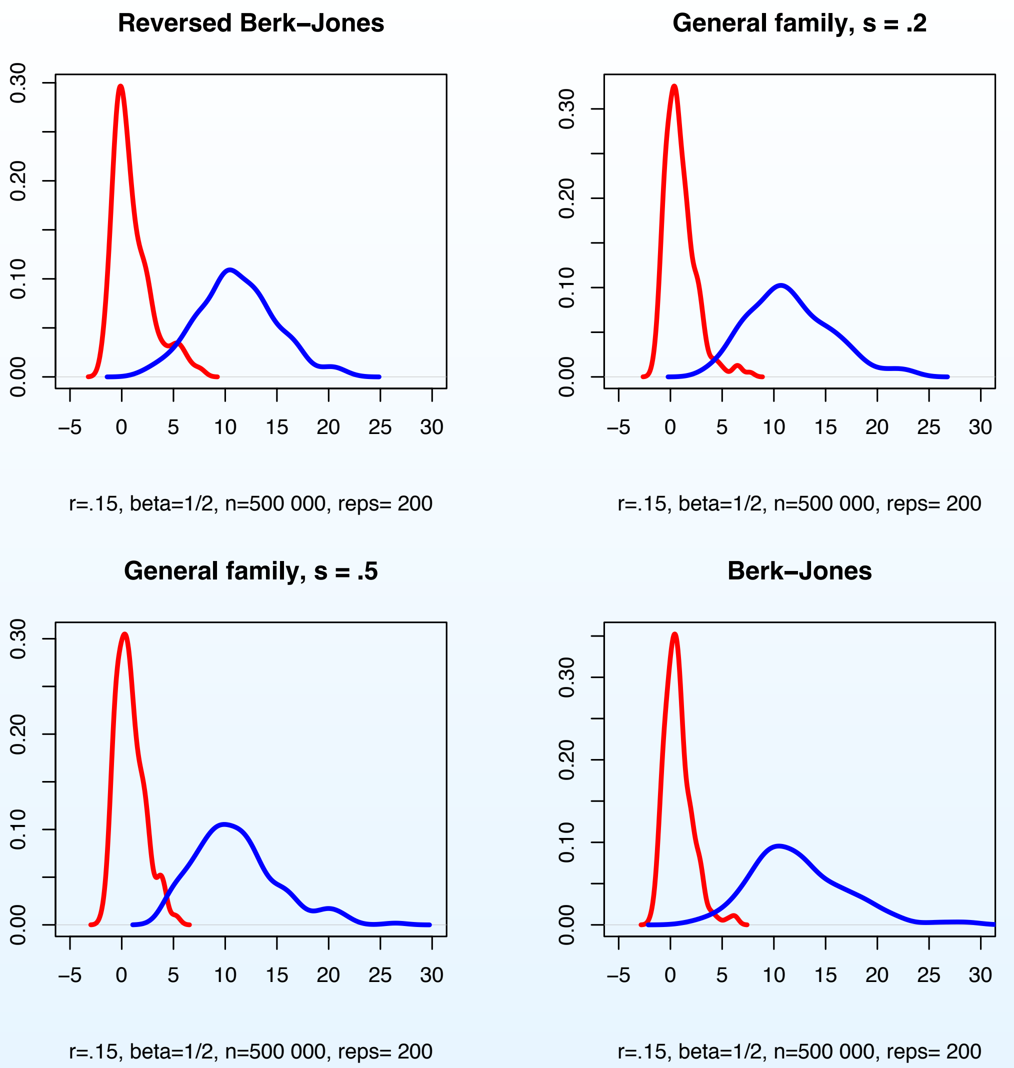

Figure 9. Separation plots: $n = 5 \times 10^5$, $r = .15$, $\beta = 1/2$
Smoothed histograms of reps $= 200$ of the statistics under the null hypothesis and the the alternative hypothesis

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- Can the statistics $S_n(s)$ be used to estimate ϵ_n in the two point normal mixture model of Ingster - Donoho - Jin? (Meinhausen and Rice (2006); Cai, Jin, and Low (2006))

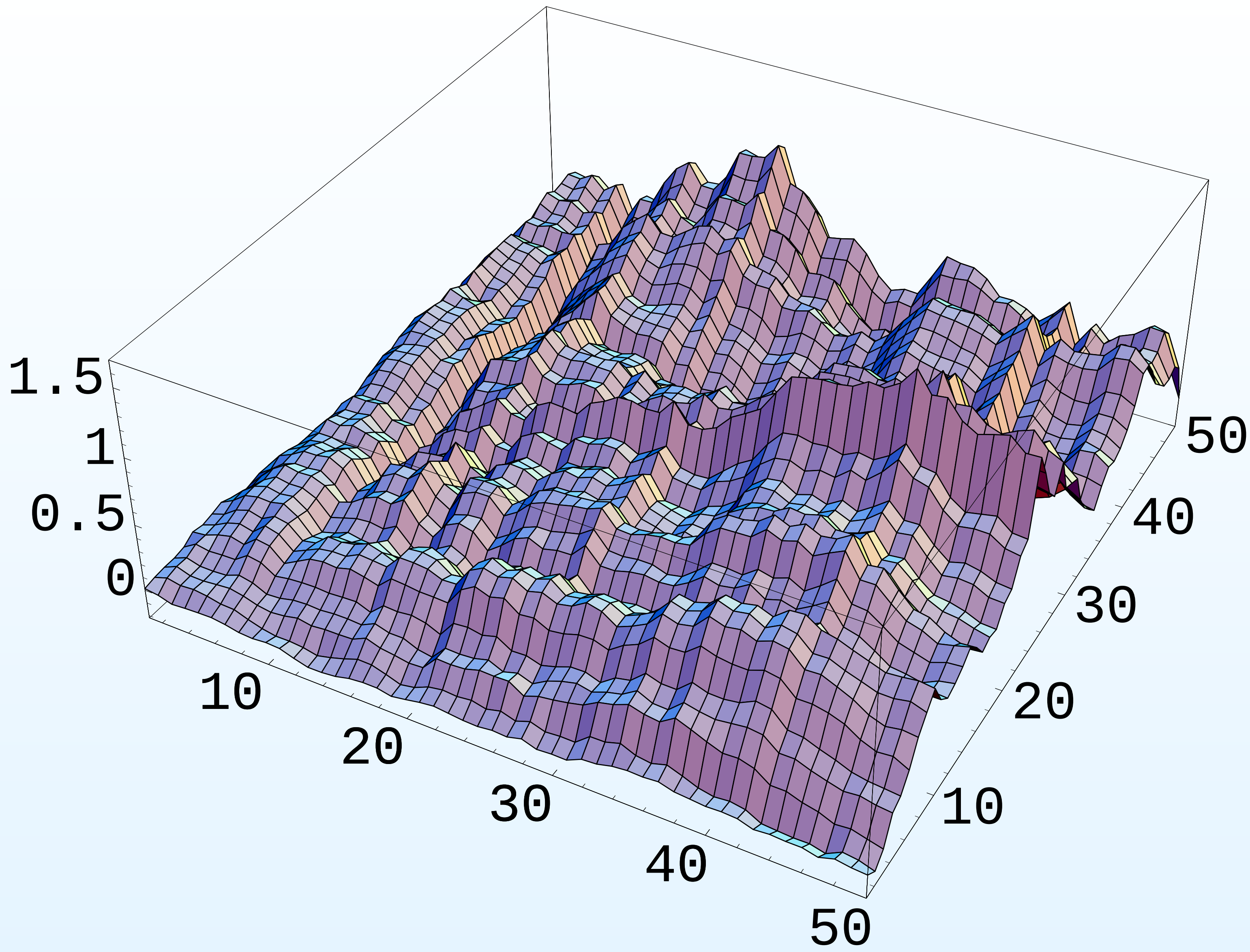

Figure 9. partial sum approximation of a Brownian sheet

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