Strong log-concavity of Chernoff's density:

connections and open problems

Jon A. Wellner

University of Washington, Seattle

Seminaire, Institut de Mathématiques de Toulouse

1 March 2012

Seminaire 1, Toulouse: Tuesday

http://www.stat.washington.edu/jaw/RESEARCH/TALKS/talks.html

Seminaire 2, Toulouse: today

Based on joint work with:

- Fadoua Balabdaoui
- Werner Ehm

Seminaire 3, Monday, 5 March:

Nonparametric estimation of log-concave densities

Seminaire 4, Thursday, 8 March:

A local maximal inequality under uniform entropy

Outline

- A Strong - log-concave densities on $\mathbb{R}$
- B Bondesson's Harmonicallly Completely
  Monotone classes: HM_∞
  - ▷ B1. A product representation for the standard normal density
  - ▷ B2. Strong log-concavity and HM_∞
- C: Pólya frequency functions, the class PF_∞.
  - ▷ C1. Is Chernoff's density strongly log-concave?
  - ▷ C2. Sums of independent exponential random variables
  - ▷ C3. Interpolation: Gaussian to Chernoff and beyond ...
- D: Questions and problems

A. Strong - log-concave densities on $\mathbb{R}$ and $\mathbb{R}^d$

- $h : \mathbb{R}^d \to \overline{\mathbb{R}}$ is *strongly convex* if there exists a constant $c > 0$ such that

$$h(\theta x + (1-\theta)y) \le \theta h(x) + (1-\theta)h(y) - \frac{1}{2}c\theta(1-\theta)\|x-y\|^2$$

for all $x, y \in \mathbb{R}^d$, $\theta \in (0,1)$.

- Equivalent to convexity of

$$h(x) - \frac{1}{2}c\|x\|^2$$

for some $c > 0$.

- Replacing h by $-\log f$ leads to a definition of *strong log-concavity* of a (density) function: $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ is strongly log-concave if and only if

$$-\log f(x) - \frac{1}{2}c\|x\|^2$$

is convex for some $c > 0$.

- Defining $-\log g(x) \equiv -\log f(x) - (1/2)c\|x\|^2$, it is easily seen that f is strongly log-concave if and only if

$$f(x) = g(x)\exp(-(1/2)c\|x\|^2)$$

  for some $c > 0$ and log-concave function g.

- Thus if $f \in C^2(\mathbb{R}^d)$, a sufficient condition for strong log-concavity is: $\text{Hess}(-\log f)(x) \geq cI_d$ for all $x \in \mathbb{R}^d$ and some $c > 0$ where I_d is the $d \times d$ identity matrix.

B. Bondesson's Harmonicallly Completely Monotone classes: HM_∞

Let $\phi(x) = (2\pi)^{-1/2}\text{exp}(-x^2/2)$ be the standard normal density.

Question: Can we write

$$\phi(x) = (1/2)g(x)g(-x) \tag{1}$$

for some "interesting" functions g. Of course this holds almost trivially for g Gaussian (and hence $g \in PF_\infty$) If

$$g(x) = (8\pi)^{1/4}\phi\left(\frac{x}{\sqrt{2}}\right),$$

then (1) holds.

Can (1) hold for (non-Gaussian) $g \in HM_\infty$?

Definition. A density f on $(0,\infty)$ is hyperbolically completely monotone, and we write $f \in HM_\infty$ if

$$H(w) \equiv f(uv)f(u/v)$$

is a completely monotone function of $w \equiv (v + v^{-1})/2$ for every u.

Theorem. (Bondesson, 1992). The class HM_∞ consists of all densities of the form

$$f(x) = Cx^{\beta-1}h_1(x)h_2(1/x) \tag{2}$$

where $\beta \in \mathbb{R}$ and

$$h_j(x) = \exp\left\{-b_j x + \int \log\left(\frac{y+1}{y+x}\right) d\Gamma_j(y)\right\} \tag{3}$$

for $b_j \geq 0$ and measures Γ_j on $[1,\infty)$ satisfying $\int (1+y)^{-1} d\Gamma_j(y) < \infty$.

- If $Y \sim f_Y$ on $(0, \infty)$,
  then $X \equiv \log Y \sim f_X(x) = f_Y(e^x)e^x$ on $\mathbb{R}$.

- If $X \sim f_X$ on $\mathbb{R}$,
  then $Y = e^X \sim f_Y(y) = f_X(\log y)y^{-1}$ on $(0, \infty)$.

- Thus if $f_Y \in HM_\infty$, $X \equiv \log Y$ has density f_X of the form

$$f_X(x) = Ce^{\beta x}\tilde{h}_1(x)\tilde{h}_2(-x) \equiv C\tilde{g}_1(x)\tilde{g}_2(-x) \tag{4}$$

  where, for $j = 1, 2$,

$$\begin{aligned} \tilde{g}_j(x) &\equiv e^{\beta_j x}\tilde{h}_j(x) = e^{\beta_j x}h_j(e^x), & (5) \\ \tilde{h}_j(x) &= h_j(e^x) & (6) \\ &= \exp\left(-b_j e^x + \int_1^\infty \log\left(\frac{v+1}{v+e^x}\right) d\Gamma_j(v)\right), & (7) \end{aligned}$$

  and $\beta_1 - \beta_2 \equiv \beta$.

Proposition 1. (Bondesson, 1992). The log-normal density on $(0,\infty)$ given by

$$f(y) = Cy^{\mu/\sigma^2-1}\text{exp}\left(-\frac{(\text{log}y)^2}{2\sigma^2}\right)$$

is hyperbolically completely monotone; i.e. $f \in HM_\infty$.

Alternative statement: $\phi \in \log(HM_\infty)$.

B1. A product representation for the standard normal density

Consequence: (Bondesson, 1992)

$$\begin{aligned}\phi(z) &= \frac{1}{\sqrt{2\pi}}\exp\left(\int_0^\infty \left\{\log\left(\frac{e^s+1}{e^s+e^z}\right)+\log\left(\frac{e^s+1}{e^s+e^{-z}}\right)\right\}ds\right)\\ &= \frac{1}{2}g(z)g(-z)\end{aligned} \tag{8}$$

where

$$\begin{aligned}g(z) &\equiv (2/\pi)^{1/4}\exp(z)\exp\left(\int_0^\infty \log\left(\frac{e^s+1}{e^s+e^z}\right)ds\right)\\ &= (2/\pi)^{1/4}\exp\left(\frac{\pi^2}{12}+z+\int_{-e^z}^0 \frac{\log(1-t)}{t}dt\right)\end{aligned} \tag{9}$$

is log-concave, integrable, and $g \in \log(HM_\infty)$.

What is the measure $\Gamma = \Gamma_1 = \Gamma_2$? In fact $d\Gamma(y) = y^{-1}dy$!

Is this g in PF_∞?

Proof of Proposition 1. Note that

$$\frac{1}{y}\left\{\frac{1}{x+y}-\frac{1}{x^2}\frac{1}{y+x^{-1}}\right\}=\frac{1}{y}\left\{\frac{x^2(y+x^{-1})-(y+x)}{x^2(y+x)(y+x^{-1})}\right\}$$

$$=\frac{x^2-1}{x^2(y+x)(y+x^{-1})}=\frac{x^2-1}{x^2(1+xy)(1+y/x)}$$

$$=\frac{1}{(1+xy)(1+y/x)}-\frac{1/x^2}{(1+xy)(1+y/x)}$$

$$=\frac{1}{x}\left\{\frac{1}{y+x^{-1}}-\frac{1}{y+x}\right\}.$$

Thus it follows that

$$\int_1^\infty\left\{\frac{1}{x+y}-\frac{1}{x^2}\frac{1}{y+x^{-1}}\right\}\frac{1}{y}dy=\frac{1}{x}\int_1^\infty\left\{\frac{1}{y+x^{-1}}-\frac{1}{y+x}\right\}dy$$

$$=\frac{1}{x}\{\log(y+1/x)-\log(y+x)\}\Big|_1^\infty=\frac{1}{x}\log x \qquad (10)$$

Also note that the integral on the right side in (10) can be rewritten as

$$\int_1^\infty \left\{\frac{1}{y+x^{-1}} - \frac{1}{y+x}\right\} dy = \int_1^\infty \frac{y+x-(y+x^{-1})}{(y+x^{-1})(y+x)} dy$$
$$= (x-x^{-1}) \int_1^\infty \frac{1}{y^2+(x+x^{-1})y+1}\, dy = (x-x^{-1}) \frac{x \log x}{x^2-1}$$
$$= \log x;$$

this rewrite makes the integrability completely clear. Integrating the resulting identity

$$\frac{\log x}{x} = \int_1^\infty \left\{\frac{1}{x+y} - \frac{1}{x^2}\frac{1}{y+x^{-1}}\right\} \frac{1}{y} dy$$

with respect to x on both sides over $[1,z]$ and using Fubini's theorem (permitted because the integrand is non-negative) yields

$$\begin{aligned}
\frac{1}{2}(\log z)^2 &= \int_1^z \left\{ \int_1^\infty \left\{ \frac{1}{x+y} - \frac{1}{x^2}\frac{1}{y+x^{-1}} \right\} \frac{1}{y} dy \right\} dx \\
&= \int_1^\infty \left\{ \int_1^z \left\{ \frac{1}{x+y} - \frac{1}{x^2}\frac{1}{y+x^{-1}} \right\} dx \right\} \frac{1}{y} dy \\
&= \int_1^\infty \left\{ \log(y+x) + \log(y+x^{-1}) \right\} \Big|_1^z \frac{1}{y} dy \\
&= -\int_1^\infty \left\{ \log\left(\frac{y+1}{y+z}\right) + \log\left(\frac{y+1}{y+z^{-1}}\right) \right\} \frac{1}{y} dy.
\end{aligned}$$

Making the change of variable of integration $y = e^s$ gives

$$\frac{1}{2}(\log z)^2 = -\int_0^\infty \left\{ \log\left(\frac{e^s+1}{e^s+z}\right) + \log\left(\frac{e^s+1}{e^s+z^{-1}}\right) \right\} ds,$$

Changing the variable z to e^x then yields the claim:

$$-\frac{1}{2}x^2 = \int_0^\infty \left\{ \log\left(\frac{e^s+1}{e^s+e^x}\right) + \log\left(\frac{e^s+1}{e^s+e^{-x}}\right) \right\} ds.$$

The claimed identity involving g follows by direct substitution. To see the second form of g, note that

$$\begin{aligned}
\int_1^\infty \log\left(\frac{y+1}{y+z}\right)\frac{1}{y}dy &= \int_1^\infty \log\left(\frac{1+1/y}{1+z/y}\right)\frac{1}{y}dy \\
&= \int_0^1 \log\left(\frac{1+t}{1+tz}\right)\frac{1}{t}dt \\
&\quad \text{by the change of variables } \; t = 1/y \\
&= \int_0^1 \log(1+t)\frac{1}{t}dt - \int_0^1 \log(1+tz)\frac{1}{t}dt \\
&= \frac{\pi^2}{12} + \int_{-z}^0 \log(1-s)\frac{1}{s}ds \\
&\quad \text{by the change of variable } \; -s = tz.
\end{aligned}$$

To see that g is log-concave, note that

$$-\log g(z) = -\frac{\pi^2}{12} - z - \int_{-e^z}^0 \frac{\log(1-t)}{t}dt,$$

and hence

$$(-\log g)'(z) = -1 + \frac{\log(1+e^z)}{-e^z} \cdot (-e^z) = -1 + \log(1+e^z),$$

$$(-\log g)''(z) = \frac{e^z}{1+e^z} \geq 0.$$

Integrability of g follows from

$$-\log g(z) \sim \begin{cases} -z & \text{as} \quad z \to -\infty, \\ -z + z^2/2 & \text{as} \quad z \to \infty, \end{cases}$$

where we used

$$\int_{-y}^{0} \frac{\log(1-t)}{t} dt \sim \begin{cases} 0 & \text{as } y \searrow 0, \\ -(1/2)(\log y)^2 & \text{as } y \to \infty. \end{cases}$$

B2. Strong log-concavity and HM_∞

Sufficient conditions for $f \in \log(HM_\infty)$ to be strongly log-concave?

Proposition: Suppose that f_X is given by (4) - (6).
A. If $b_1 > 0$ and $b_2 > 0$, then f_X is strongly log-concave for any (all) measures Γ_1 and Γ_2.
B. Suppose $b_1 = 0$ or $b_2 = 0$ and $d\Gamma_j(v) = v^{-1} r_j(v) dv$ for $j = 1, 2$ where (at least one of) r_1 and r_2 satisfy:
(i) $r_j(y) \ge 0$ all $y \in [1, \infty)$ with strict inequality for some $y > 0$;
(ii) r_j is non-decreasing.
Then $v_j(x) \equiv (-\log g_j)''(x) \ge v_j(0) > 0$ for all $x \ge 0$ and hence f_X is strongly log-concave with $(-\log f_X)''(x) \ge \max\{v_1(0), v_2(0)\} > 0$.

Proof of Proposition: Since Γ_j has density $\gamma_j(y) = y^{-1} r_j(y)$,

$$
\begin{aligned}
(-\log g_j)''(x) &= \int_0^\infty \frac{e^{w-x}}{(1+e^{w-x})^2} e^w \gamma_j(e^w)\,dw \\
&= \int_0^\infty \frac{e^{w-x}}{(1+e^{w-x})^2} r_j(e^w)\,dw \quad \text{if} \quad \gamma(y) = y^{-1} r_j(y) \\
&= \int_{-x}^\infty \frac{e^z}{(1+e^z)^2} r_j(e^{z+x})\,dz \\
&= E r_j(e^{Z+x}) 1\{Z \ge -x\} \equiv v(x)
\end{aligned}
$$

where $Z \sim$ standard logistic with density $e^z/(1+e^z)^2$. Then it follows that

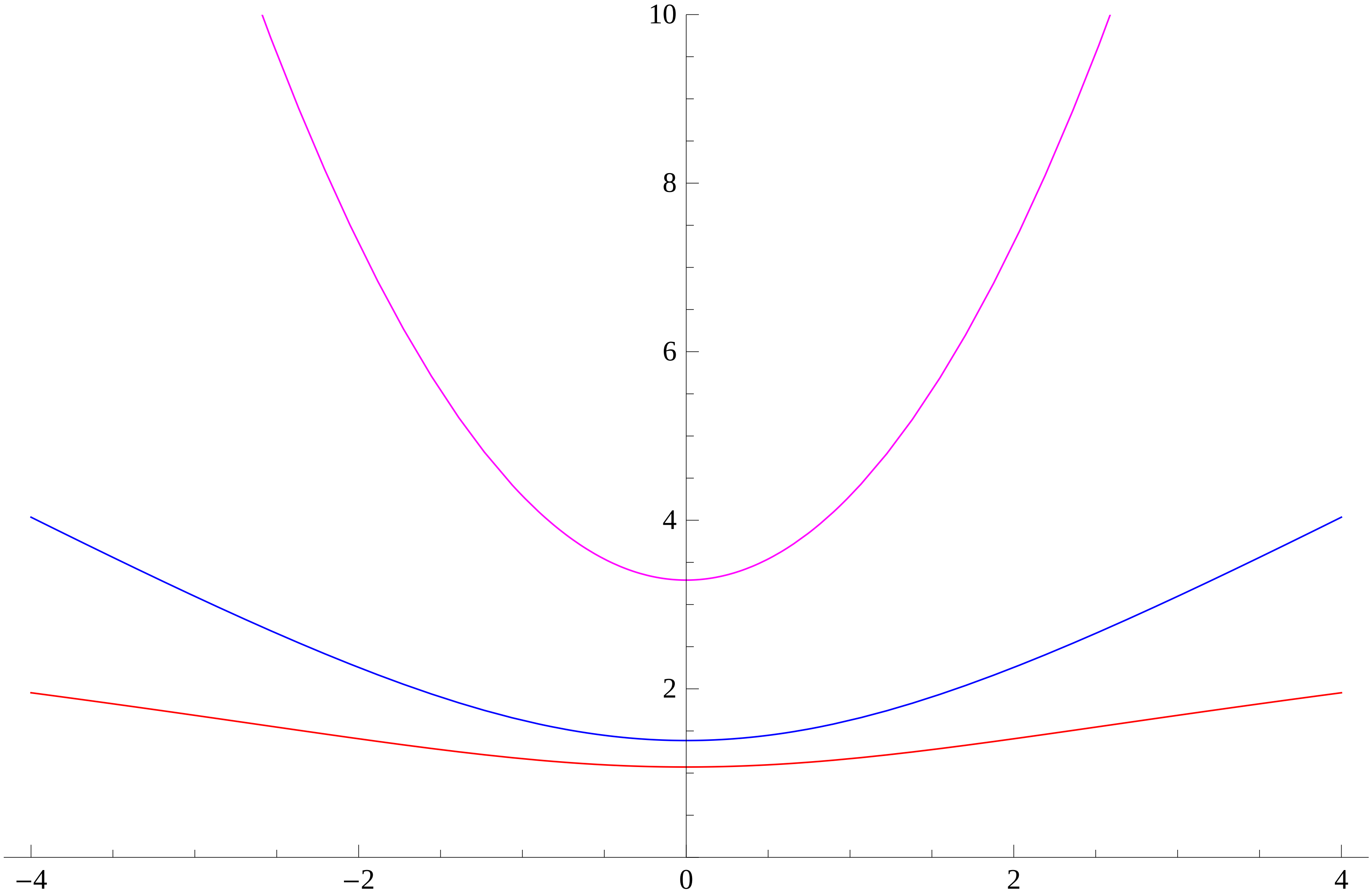
$$
\begin{aligned}
v_j(x) &= E r_j(e^{Z+x}) 1\{Z + x \ge 0\} \\
&= E r_j(e^{Z+x}) 1_{[-x,0]}(Z) + E r_j(e^{Z+x}) 1_{(0,\infty)}(Z) \\
&\ge E r_j(e^{Z+x}) 1_{(0,\infty)}(Z) \ge E r_j(e^{Z}) 1_{(0,\infty)}(Z) = v_j(0) > 0.
\end{aligned}
$$

Thus the density f_X is strongly log-concave.

Example. $b_1 = b_2 = 0$ and $r(y) = (\log y)^c$ with $c > 0$; thus $d\Gamma(y) = y^{-1}(\log y)^c dy$. Then

$$\begin{aligned}(-\log g)''(x) &\equiv v(x) \\ &= \int_1^\infty \frac{ve^x}{(v+e^x)^2} d\Gamma(v) \equiv v(x) \\ &= Er(e^{Z+x}) 1\{Z \ge -x\} \\ &= E(Z+x)^c 1\{Z \ge -x\} \sim x^c \quad \text{as} \quad x \to \infty.\end{aligned}$$

$$w(x) \equiv v(x) + v(-x) = (-\log f)''(x) \quad \text{for} \quad f(x) = (1/2)g(x)g(-x).$$



$w(x) = (-\log f)''(x)$, $c = 1/2$ (red), $c = 1$ (blue), $c = 2$ (purple)

C: Pólya frequency functions, the class PF_∞.

C1: Is Chernoff's density strongly log-concave?

Recall that From Chernoff (1964)

$$f(z) = \frac{1}{2} g(z) g(-z) \equiv \frac{1}{2} g_1(z) g_1(-z).$$

From Groeneboom (1885, 1989), Daniels and Skyrme (1985)

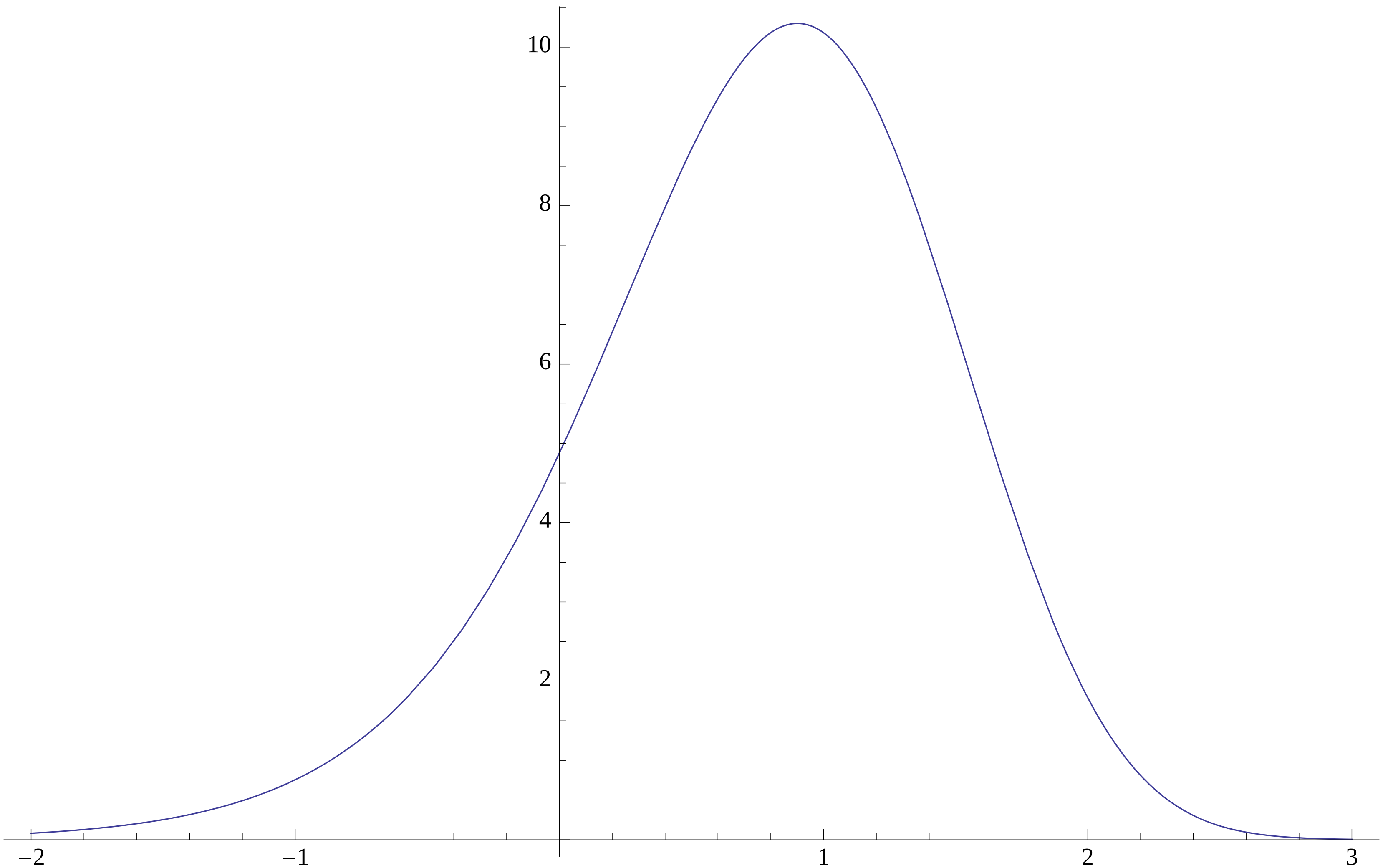
$$\widehat{g}_c(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda s} g_c(s) ds = \frac{2^{1/3} c^{-1/3}}{Ai(i(2c^2)^{-1/3}\lambda)}.$$

From Salmassi (1999) and Merkes and Salmassi (1997)

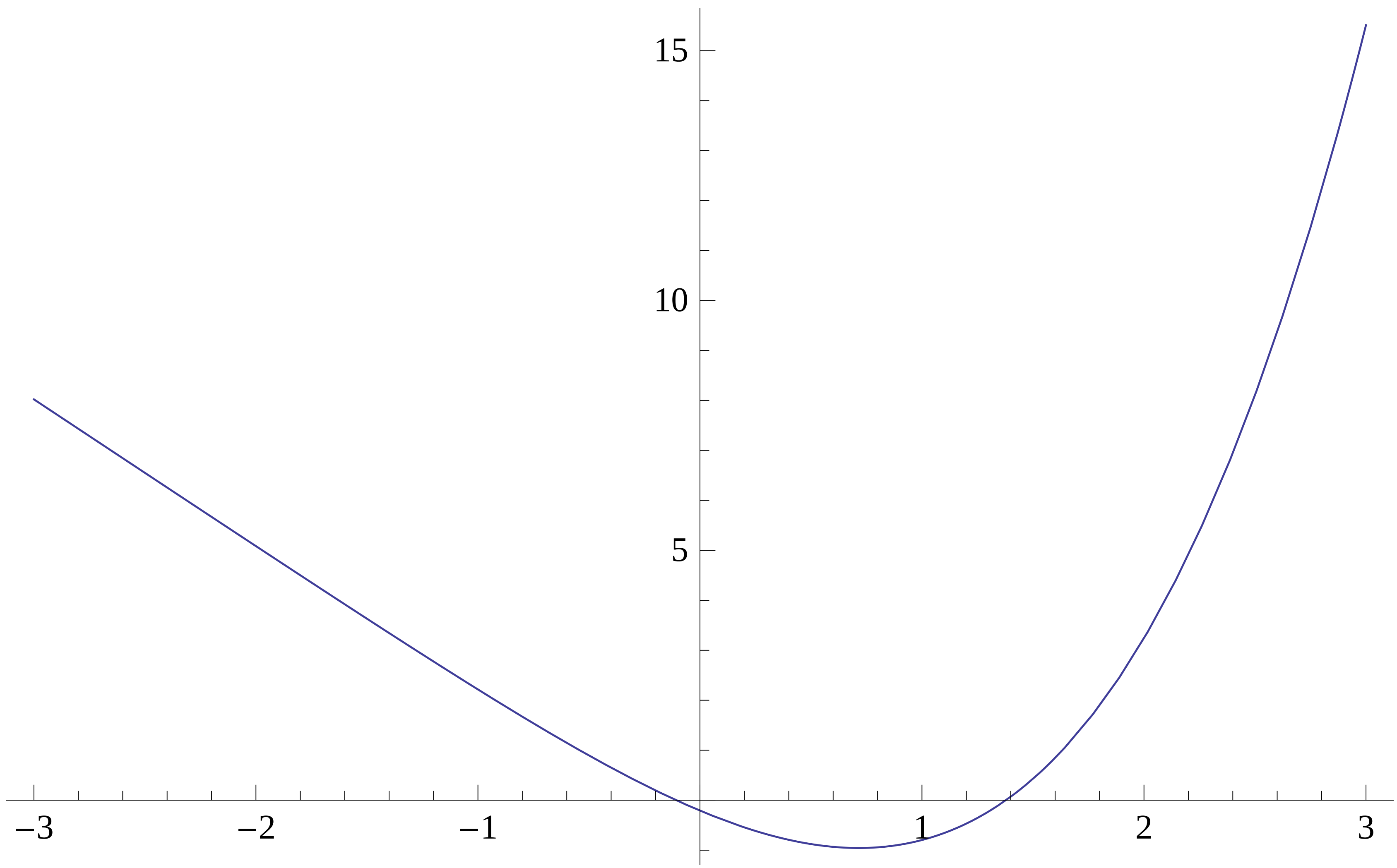
$$Ai(z) = Ai(0) e^{-\nu z} \prod_{j=1}^{\infty} (1 + z/a_j) \exp(-z/a_j)$$

where $\{-a_k\}$ are the zeros of the Airy function Ai (so that $a_k > 0$ for each k). where $Ai(0) = c_1 = 1/(3^{2/3}\Gamma(2/3)) \approx 0.35503$, and

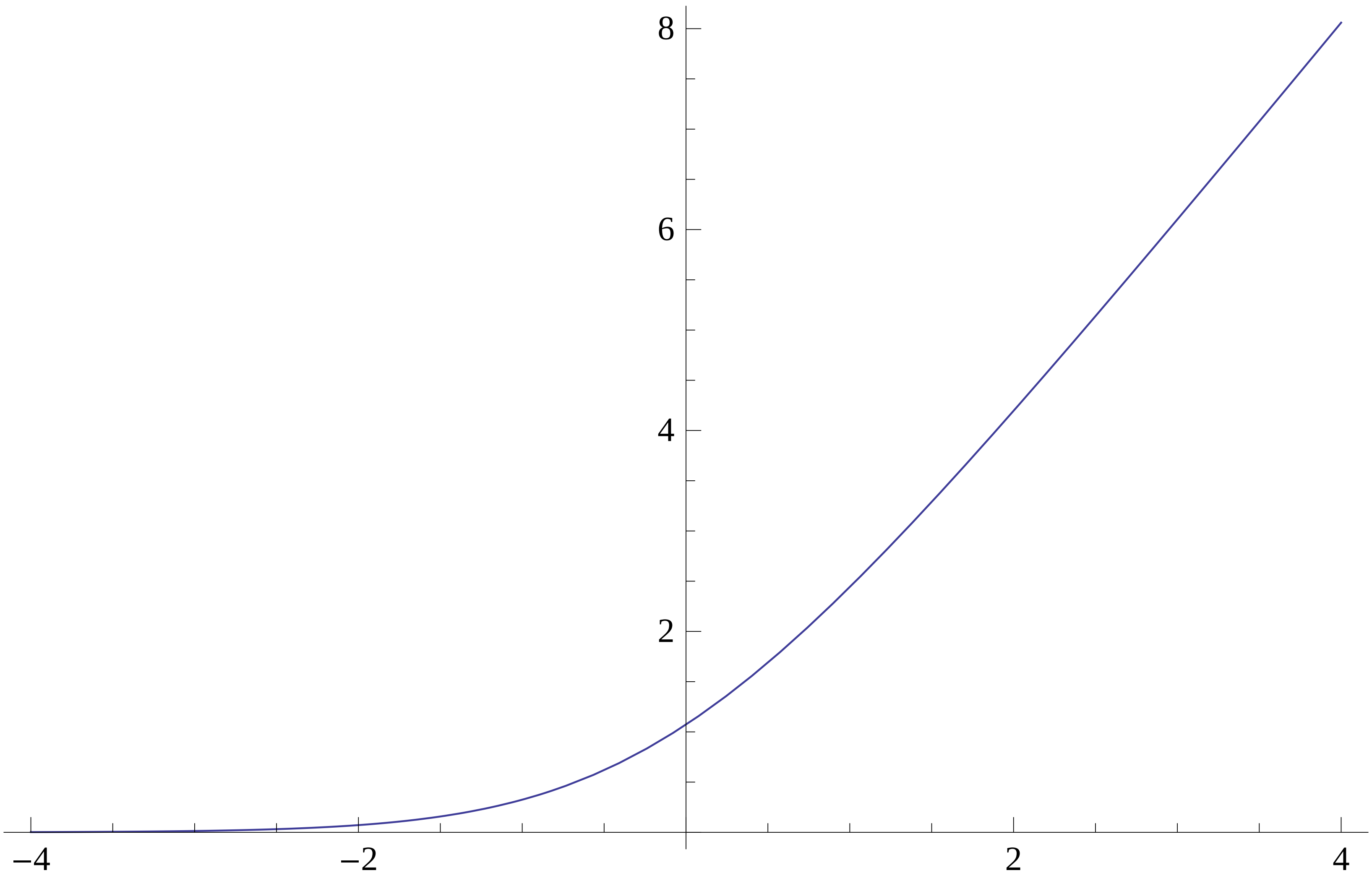
$$\nu = -Ai'(0)/Ai(0) = 3^{1/3}\Gamma(2/3)/\Gamma(1/3) \approx .729011\ldots.$$



The function g_1



$-\log(g_1)$


8
6
4
2
−4
−2
2
4

C2. Sums of independent exponential random variables

Suppose that $X \sim \text{Exp}(\lambda)$. Then $f(x) = \lambda e^{-\lambda x} 1_{(0,\infty)}(x)$ and

$$\begin{aligned} Ee^{sX} &= \int_0^\infty e^{sx} \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-(\lambda - s)x} dx \\ &= \frac{\lambda}{\lambda - s} = \frac{1}{1 - \frac{s}{\lambda}} \end{aligned}$$

for $s < \lambda$. Thus

$$E\exp(s(X - 1/\lambda)) = \exp(-s/\lambda) \frac{1}{1 - s/\lambda} = \frac{1}{(1 - s/\lambda)e^{s/\lambda}}.$$

Now suppose that $X_i \sim \text{Exp}(1/b_i)$ for $i = 1, 2, \ldots$ are independent. Thus

$$E\exp(-s(X_i - b_i)) = \frac{1}{(1 + b_i s)e^{-b_i s}},$$

and

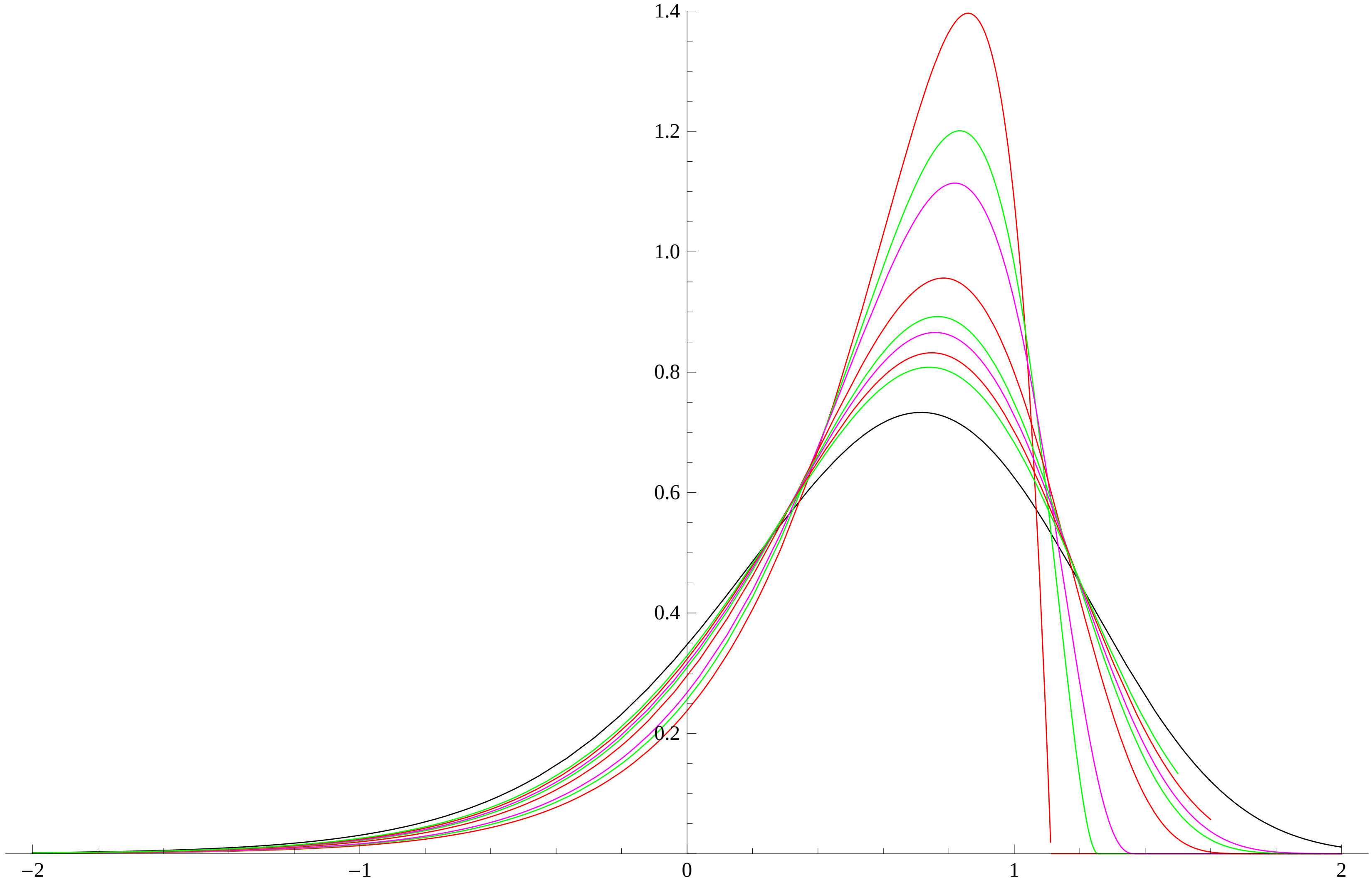
$$E\exp(-s\sum_{i=1}^{\infty}(X_i - b_i)) = E\left\{\prod_{i=1}^{\infty} e^{-s(X_i-b_i)}\right\} = \prod_{i=1}^{\infty} Ee^{-s(X_i-b_i)}$$
$$= \frac{1}{\prod_{i=1}^{\infty}(1+b_i s)e^{-b_i s}}.$$

This is exactly the form of one term of the reciprocal of the bilateral Laplace or Fourier transform in Schoenberg's theorem characterizing PF_∞ if we require $\sum_1^\infty b_j^2 < \infty$. In the context of Chernoff's distribution $b_j = 1/a_j$ where $\{-a_j\}$ are the zeros of the Airy function, with $a_j \sim ((3/2)\pi(k-1/4))^{2/3}$.

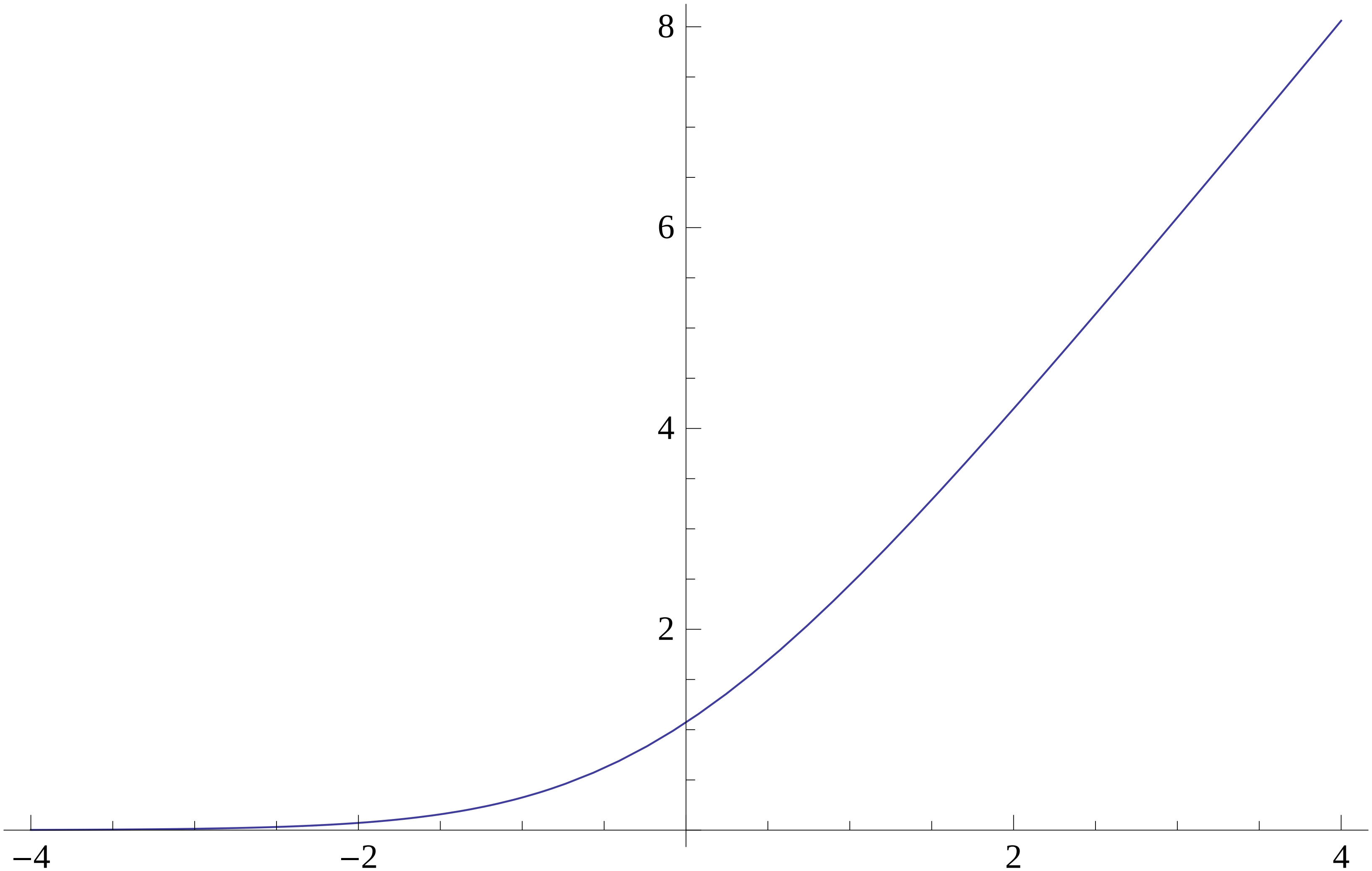
This leads to consideration of the distributions of

$$\sum_{j=1}^{m}(X_j - b_j) \qquad \text{and} \qquad \sum_{j=1}^{m} X_j \equiv Y_m.$$

What do we know about the distribution of Y_m?



Finite m approximations of $\tilde{g}$, $m \in \{2, 3, 4, 10, 20, 30, 60, 120\}$



Second derivative, $(-\log(g_1))''(x)$

Proposition 1. (Harrison, 1990) If $X_i \sim \text{Exp}(1/b_i) \equiv \text{Exp}(\lambda_i)$ for $i \geq 1$ be independent. Then the density f_m of Y_m is given by

$$f_m(t) = \sum_{j=1}^{m} \lambda_j \exp(-\lambda_j t) \prod_{i \neq j} \frac{\lambda_i}{\lambda_i - \lambda_j} 1_{(0,\infty)}(t).$$

Define

$$v_m(t) \equiv (-\log f_m)''(t).$$

Conjecture: v_m is convex (and decreasing) for every m.

Proposition 2. v_2 is convex.

Proof: By Harrison's formula

$$\begin{aligned} f_2(x) &= \lambda_1 e^{-\lambda_1 x} \frac{\lambda_2}{\lambda_2 - \lambda_1} + \lambda_2 e^{-\lambda_2 x} \frac{\lambda_1}{\lambda_1 - \lambda_2} \\ &= \frac{\lambda_1 \lambda_2}{\lambda_2 - \lambda_1} e^{-\lambda_1 x} \left(1 - e^{-(\lambda_2 - \lambda_1)x}\right), \end{aligned}$$

and hence

$$-\log f_2(x) = -\log\left(\frac{\lambda_1\lambda_2}{\lambda_2-\lambda_1}\right) + \lambda_1 x - \log\left(1 - e^{-(\lambda_2-\lambda_1)x}\right).$$

Thus we compute

$$\begin{aligned}(-\log f_2)'(x) &= \lambda_1 - \frac{(\lambda_2-\lambda_1)e^{-(\lambda_2-\lambda_1)x}}{1-e^{-(\lambda_2-\lambda_1)x}} \\ &= \lambda_1 + (\lambda_2-\lambda_1)\left(1 - \frac{1}{1-e^{-(\lambda_2-\lambda_1)x}}\right),\end{aligned}$$

and

$$(-\log f_2)''(x) = (\lambda_2-\lambda_1)^2 \frac{e^{-(\lambda_2-\lambda_1)x}}{(1-e^{-(\lambda_2-\lambda_1)x})^2} \geq 0$$

for all $x > 0$ (the ≥ 0 part we know already from log-concavity of the two (different) exponential densities and preservation of log-concavity under convolution). Now set $c \equiv \lambda_2 - \lambda_1$ and consider

$$h(x) \equiv h_c(x) = e^{-cx}/(1 - e^{-cx})^2.$$

Now

$$h'(x) = -ce^{-cx}\left\{\frac{1 - e^{-cx} + 2e^{-cx}}{(1 - e^{-cx})^3}\right\} = -ce^{-cx}\frac{1 + e^{-cx}}{(1 - e^{-cx})^3}.$$

Thus h is decreasing. Furthermore,

$$\begin{aligned} h''(x) &= c^2 e^{-cx}\frac{1 + e^{-cx}}{(1 - e^{-cx})^3} + c^2 e^{-2cx}\frac{1}{(1 - e^{-cx})^3} + 3c^2 e^{-2cx}\frac{1 + e^{-cx}}{(1 - e^{-cx})^4} \\ &= c^2 e^{-cx}\frac{1 + 4e^{-cx} + e^{-2cx}}{(1 - e^{-cx})^4} \\ &> 0. \end{aligned}$$

Thus $(-\log f_2)''$ is convex. □

C3. Interpolation: Gaussian to Chernoff and beyond ...

Question: Does there exist a natural family of densities f_γ with $\gamma \in [1/2, 1)$ satisfying:

- $f_{1/2} = \phi$ and $f_{2/3} =$ Chernoff's density f_1.
- f_γ is strongly log-concave for each γ.
- With $\beta \equiv 1 - \gamma$ we have
$$(-\log f_\gamma)''(x) \sim \left(\frac{1}{\beta} - 1\right) x^{\frac{1}{\beta}-2} \quad \text{as} \quad x \to \infty.$$
- $$f_\gamma(x) \sim D_\gamma x^\kappa \exp(-\beta x^{1/\beta}) \quad \text{as} \quad x \to \infty.$$
  where $\kappa = \kappa_\gamma = (1 - 2\beta)/\beta$.

Such a family would interpolate between the Gaussian density ϕ (for $\gamma = 1/2$), Chernoff's density (for $\gamma = 2/3$), and yield increasingly light-tailed densities in the range $2/3 < \gamma < 1$. It

C3. Interpolation: Gaussian to Chernoff and beyond ...

would also correspond to a "full range" of the densities of the form

$$f(x) = \frac{1}{2} g(x) g(-x) \quad \text{with} \quad g \in PF_{\infty}$$

On-going work with Werner Ehm ... !

D: Questions and problems

- Is Chernoff's density strongly log-concave?
- is $v_m \equiv (-\log f_m)''$ convex for every m and $0 < \lambda_1 < \lambda_2 < \cdots$?
- Is there a "natural family" interpolating between standard normal and Chernoff?

- Are the densities corresponding to limit distributions for nonparametric estimators of convex functions (as will be discussed in my talk on Monday) log-concave or strongly log-concave?

Merci!