A Law of the Iterated Logarithm for Grenander's Estimator

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Outline

- 1: Introduction: classical LIL's.
- 2: Grenander's estimator: the MLE of a decreasing density.
- 3: Chernoff's distribution and tail behavior.
- 4: A LIL for Grenander's estimator.
- 5: Summary; further questions and open problems.

1. Introduction: Classical LIL's

- Setting: $X_1, X_2, \ldots, X_n, \ldots$ i.i.d. random variables with $\mu = E(X_1) = 0$, $Var(X_1) = \sigma^2$.
- $S_n \equiv \sum_{i=1}^n X_i = n\overline{X}_n$.
- Then, with $Z \sim N(0,1)$,

$$\frac{S_n}{\sqrt{n}} = \sqrt{n}(\overline{X}_n - \mu) \rightarrow_d \sigma Z$$

Theorem. (Hartman and Wintner (1941)).

$$\limsup_{n\rightarrow\infty} \frac{S_n}{\sqrt{2n\text{loglog}n}} = +\sigma \quad \text{a.s.,}$$

$$\liminf_{n\rightarrow\infty} \frac{S_n}{\sqrt{2n\text{loglog}n}} = -\sigma \quad \text{a.s..}$$

Moreover, Strassen (1964) showed that

$$\frac{S_n}{\sqrt{2n\text{loglog}n}} \rightsquigarrow [-\sigma, \sigma] \quad \text{a.s.} \ldots \text{and} \ldots$$

with

$$\mathbb{S}_n(t) \equiv \begin{cases} \frac{S_k}{\sqrt{2n \mathrm{loglog} n}} & \text{at } t = k/n, \\ \text{linearly interp.} & \text{on } \ k/n \le t \le (k+1)/n, \ k = 0, 1, \ldots, n-1, \end{cases}$$

we have

$$\mathbb{S}_n \rightsquigarrow \sigma \mathcal{K} \quad \text{a.s.}$$

where

$$\mathcal{K} \equiv \left\{ f \in C[0,1] : \ f(t) = \int_0^t \dot{f}(s) ds, \ \text{some } \dot{f}, \ \int_0^1 \dot{f}^2(s) ds \le 1 \right\}.$$

Corollary.

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \mathrm{loglog} n}} = \limsup_{n\to\infty} \mathbb{S}_n(1) = \sigma \sup_{f \in \mathcal{K}} f(1) = \sigma \cdot 1$$

where the supremum in the last line is achieved at $f(t) = t$, $0 \le t \le 1$.

Connection with exponential bounds:

Suppose that $X_i \sim N(0,1)$ for $1 \le i \le n$.

- $n^{-1/2} S_n \sim N(0,1)$, $\sigma = 1$, and

$$P\left(n^{-1/2} S_n > x\right) = 1 - \Phi(x) \le x^{-1}\phi(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.$$

- Thus with $b_n \equiv (2\log\log n)^{1/2}$, and $\delta > 0$,

$$\begin{aligned} & P(n^{-1/2} S_n > (1+\delta)(2\log\log n)^{1/2}) \\ \le \ & \frac{1}{\sqrt{2\pi}} \frac{1}{(1+\delta) b_n} \exp(-(1+\delta)^2 \log\log n) \\ \equiv \ & \frac{C}{b_n} (\log n)^{-(1+\delta)^2} \\ \sim \ & \frac{C}{b_{n_k}} (k \log \alpha)^{-(1+\delta)^2}, \quad n_k \equiv \lfloor \alpha^k \rfloor, \ \alpha > 1. \end{aligned}$$

- Therefore, by the Borell-Cantelli lemma,

$$P(n_k^{-1/2} S_{n_k} > (1+\delta)(2\log\log n_k)^{1/2} \ \text{ i.o.}) = 0.$$

2. Grenander's estimator

- $X_1, X_2, \ldots, X_n$ are i.i.d. with density f on $\mathbb{R}^+ \equiv [0, \infty)$.
- Then Grenander's estimator $\widehat{f}_n$ of f is the Maximum Likelihood Estimator of f_0 over the class of decreasing densities on $\mathbb{R}^+$: if

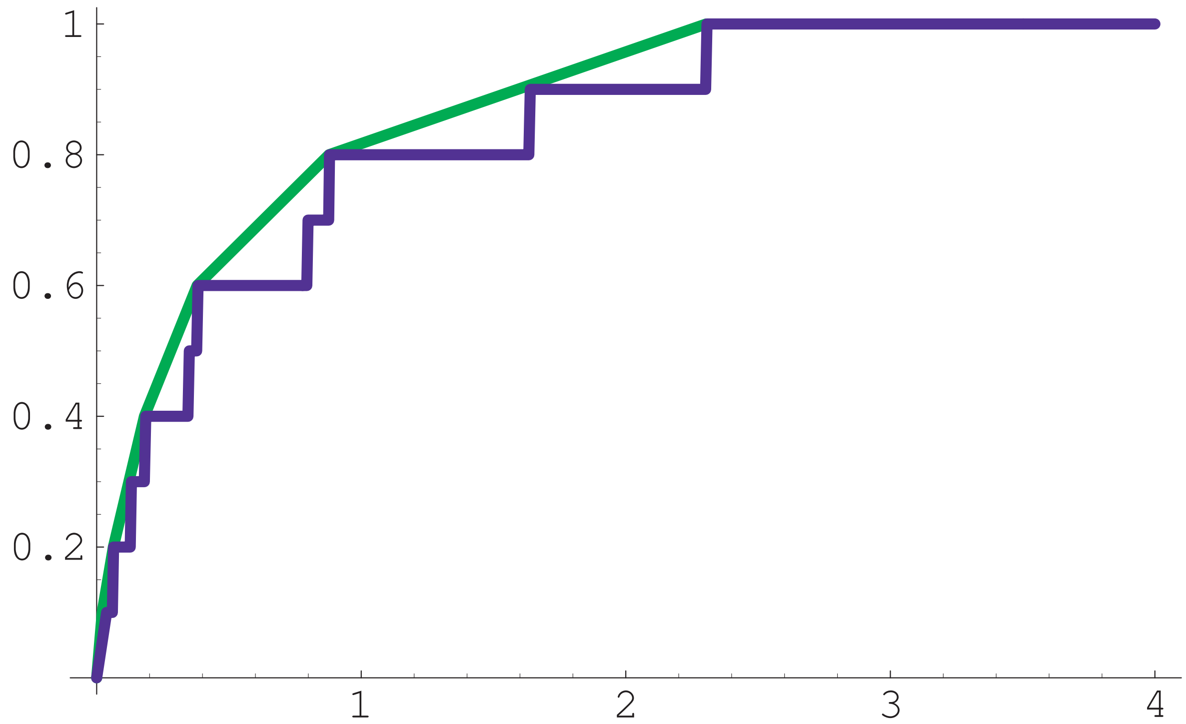
$$L_n(g) \equiv n^{-1} \sum_{i=1}^{n} \log\{g(X_i)\} = \mathbb{P}_n(\log g)$$

then $\widehat{f}_n$ satisfies

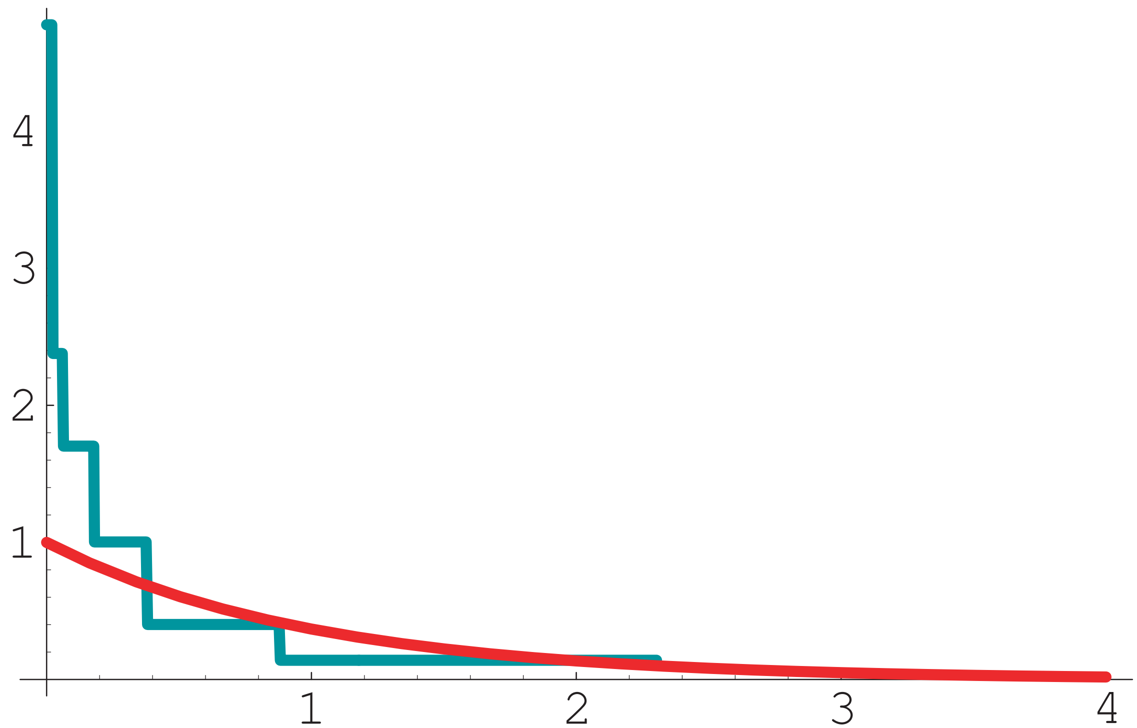
$$L_n(\widehat{f}_n) = \max\{L_n(g) : \ g \ \text{ a monotone decreasing density on } \mathbb{R}^+\}.$$

Grenander (1956) showed that the maximum likelihood estimator $\widehat{f}_n$ of f is the (left-) derivative of the least concave majorant of the empirical distribution function $\mathbb{F}_n$

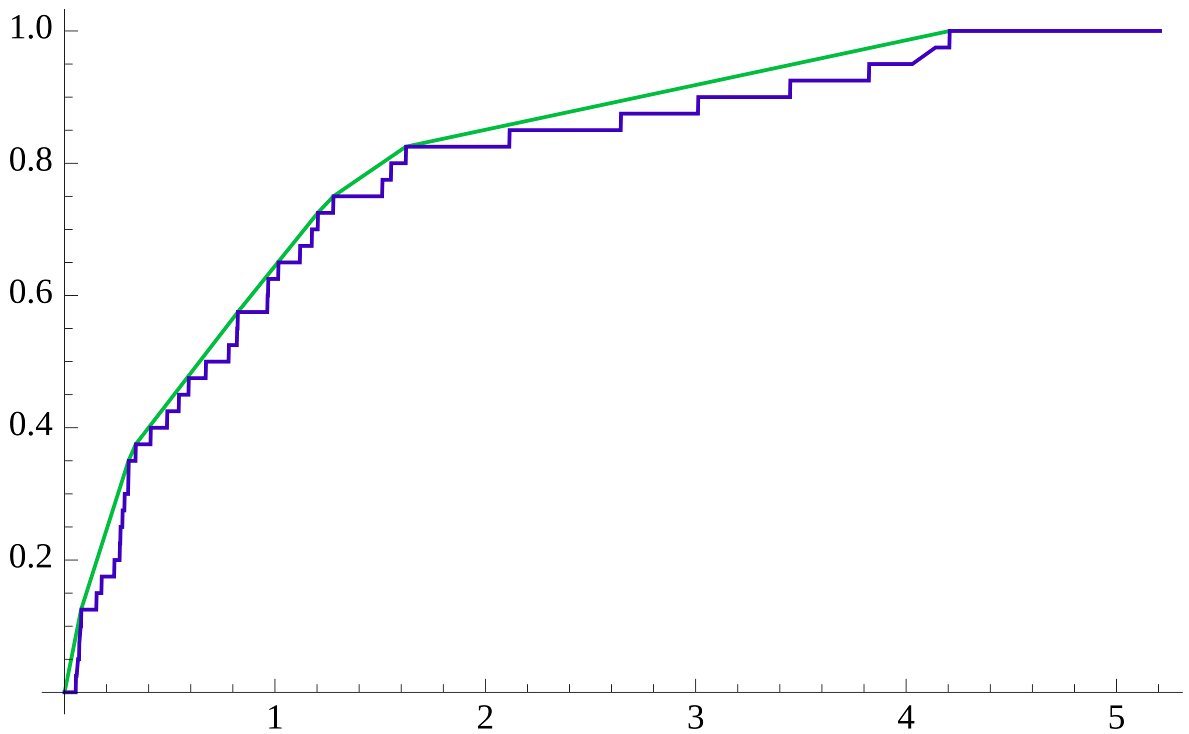
$$\widehat{f}_n \;=\; \text{left derivative of the least concave majorant of } \mathbb{F}_n, \quad \text{the empirical distribution of } \; X_1, \ldots, X_n \; \text{ i.i.d. } F$$



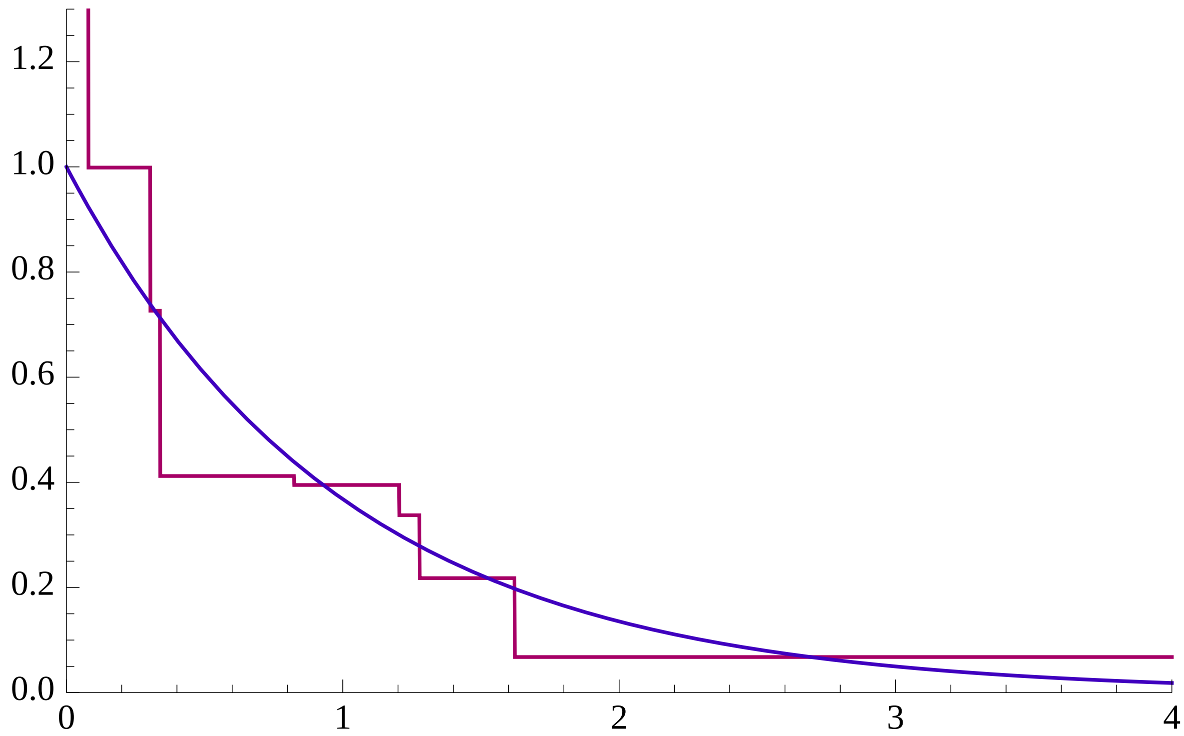
Least Concave Majorant and Empirical $n = 10$



Grenander Estimator and Exp(1) density, $n = 10$



Least Concave Majorant and Empirical $n = 40$



Grenander Estimator and Exp(1) density, $n = 40$

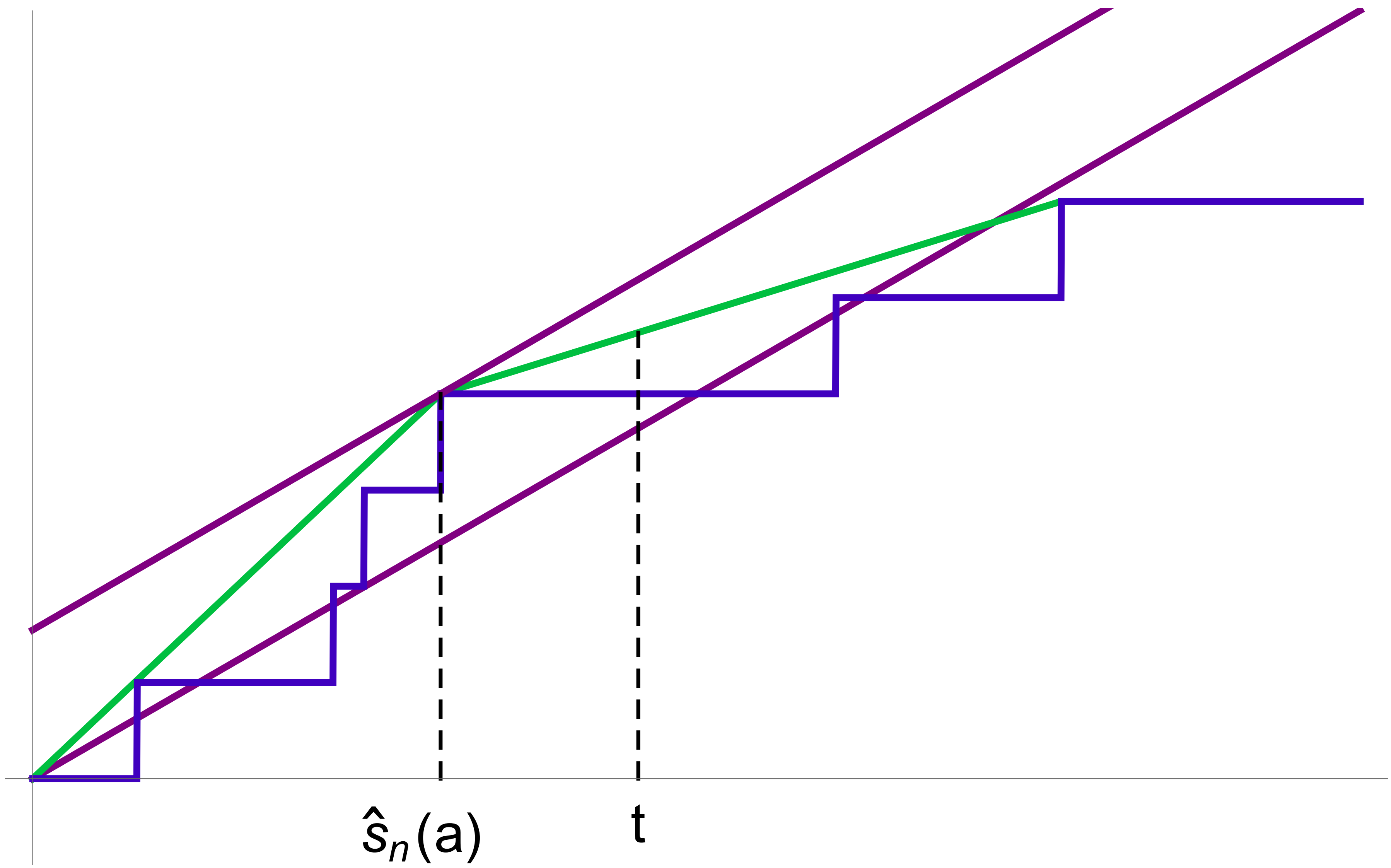
How to study $\widehat{f}_n$? Groeneboom's switching relation!

Switching for $\widehat{f}_n$: Define

$$\begin{aligned}\widehat{s}_n(a) &\equiv \text{argmax}_{s\ge 0}\{\mathbb{F}_n(s) - as\}, \quad a>0 \\ &\equiv \sup\{s \ge 0 : \ \mathbb{F}_n(s) - as = \sup_{z\ge 0}(\mathbb{F}_n(z) - az)\}.\end{aligned}$$

Then for each fixed $t \in (0,\infty)$ and $a > 0$

$$\left\{\widehat{f}_n(t) < a\right\} = \{\widehat{s}_n(a) < t\}\,.$$


$\hat{s}_n(a)$
t

Theorem. (Prakasa Rao (1969), Groeneboom (1985)): If $f(x) > 0$, $f'(x) < 0$, and f' is continuous at x, then with $c \equiv (4f(x)/(f'(x))^2)^{1/3}$ and $C \equiv (2^{-1}f(x)|f'(x)|)^{1/3}$,

$$\begin{aligned} \mathbb{S}_n(x,t) &\equiv n^{1/3}\left(\widehat{f}_n(x+n^{-1/3}ct) - f(x)\right) \\ &\to_d C \cdot \mathbb{S}(t) \end{aligned}$$

where

W is a standard two-sided Brownian motion starting at 0,

and

$\mathbb{S} \equiv$ the left - derivative of the least concave majorant $\mathbb{C}$ of $W(t) - t^2$.

In particular, with $t = 0$,

$$n^{1/3}\left(\widehat{f}_n(x) - f(x)\right) \to_d C\mathbb{S}(0) \stackrel{d}{=} C2Z$$

where the distribution of $Z \equiv \text{argmax}_h\{W(h) - h^2\}$ is

Chernoff's distribution .

Proof: By the switching relation $\{\widehat{f}_n(t) < a\} = \{\widehat{s}_n(a) < t\}$:

$$\begin{aligned}
& P(n^{1/3}(\widehat{f}_n(x_0 + n^{-1/3}t) - f(x_0)) < y) \\
&= P(\widehat{f}_n(x_0 + n^{-1/3}t) < f(x_0) + yn^{-1/3}), \\
&= P(\widehat{s}_n(f(x_0) + yn^{-1/3}) < x_0 + n^{-1/3}t) \\
&= P(\text{argmax}_v\{\mathbb{F}_n(v) - (f(x_0) + n^{-1/3}y)v\} < x_0 + n^{-1/3}t)
\end{aligned}$$

Now we change variables $v = x_0 + n^{-1/3}h$ in the argument of $\mathbb{F}_n$, center and scale to find: the right side in the last display equals

$$
\begin{aligned}
& P(\text{argmax}_h\{\mathbb{F}_n(x_0 + n^{-1/3}h) - (f(x_0) + n^{-1/3}y)(x_0 + n^{-1/3}h)\} < t) \\
& \quad = \; P\Big(\text{argmax}_h\{\mathbb{F}_n(x_0 + n^{-1/3}h) - \mathbb{F}_n(x_0) \\
& \qquad\qquad - \; (F(x_0 + n^{-1/3}h) - F(x_0)) \qquad (1) \\
& \qquad + \; F(x_0 + n^{-1/3}h) - F(x_0) - f(x_0)n^{-1/3}h - n^{-2/3}yh\} < t\Big).
\end{aligned}
$$

Let $\mathbb{G}_n(t) \equiv n^{-1}\sum_{i=1}^n 1\{\xi_i \le t\}$, $\mathbb{U}_n(t) \equiv \sqrt{n}(\mathbb{G}_n(t) - t)$ with ξ_i i.i.d. Uniform$[0,1]$. Thus $\mathbb{F}_n \stackrel{d}{=} \mathbb{G}_n(F)$. The the stochastic term in (1) satisfies, with $W \equiv$ two-sided standard Brownian motion,

$$
\begin{aligned}
n^{2/3} & \left\{\mathbb{F}_n(x_0+n^{-1/3}h) - \mathbb{F}_n(x_0) - (F(x_0+n^{-1/3}h) - F(x_0))\right\} \\
& \stackrel{d}{=} \ n^{2/3-1/2}\left\{\mathbb{U}_n(F(x_0+n^{-1/3}h)) - \mathbb{U}_n(F(x_0))\right\} \\
& = \ n^{1/6}\left\{\mathbb{U}(F(x_0+n^{-1/3}h)) - \mathbb{U}(F(x_0))\right\} + o_p(1) \qquad \text{by KMT} \\
& \stackrel{d}{=} \ n^{1/6} W(f(x_0)n^{-1/3}h) + o_p(1) \\
& \stackrel{d}{=} \ \sqrt{f(x_0)}W(h) + o_p(1)
\end{aligned}
$$

where W is a standard two-sided Brownian motion process starting from 0. On the other hand, with $\delta_n \equiv n^{-1/3}$,

$$
\begin{aligned}
n^{2/3} & \left(F(x_0+n^{-1/3}) - F(x_0) - f(x_0)n^{-1/3}h\right) \\
& = \ \delta_n^{-2}(F(x_0+\delta_n h) - F(x_0) - f(x_0)\delta_n h) \\
& \to \ -b|h|^2 \qquad \text{with} \ \ b = |f'(x_0)|/2
\end{aligned}
$$

by our hypotheses, while $n^{2/3}n^{-1/3}n^{-1/3}h = n^0 h = h$.

Thus it follows that the last probability above converges to

$$P\left(\text{argmax}_h\left\{aW(h)-b|h|^2-yh\right\}<t\right)$$
$$=\begin{cases} P(\mathbb{S}_{a,b}(t)<y) & \text{by switching again} \\ P\left((a/b)^{(2/3)}\text{argmax}_h\{W(h)-h^2\}-(2b)^{-1}y<t\right), & \text{by (2) below} \end{cases}$$

where

$$\begin{aligned} \mathbb{S}_{a,b}(t) &= \text{slope at } t \text{ of the least concave majorant of} \\ & \qquad aW(h)-bh^2 \equiv \sqrt{f_0(x_0)}W(h)-|f_0'(x_0)||h|^2/2 \\ &\stackrel{d}{=} |2^{-1}f_0(x_0)f_0'(x_0)|\mathbb{S}(t/c_0). \end{aligned}$$

and where we used

$$\text{argmax}\{aW(h)-bh^2\} \stackrel{d}{=} \left(\frac{a}{b}\right)^{2/3}\text{argmax}\{W(h)-h^2\}-\frac{1}{2b}y. \quad (2)$$

First appearance of Z:
Chernoff (1964), *Estimation of the mode*:

- $X_1, \ldots, X_n$ i.i.d. with density f and distribution function F.
- Fix $a > 0$; $\widehat{x}_a \equiv$ center of the interval of length $2a$ containing the most observations.
- $x_a \equiv$ center of the interval of length $2a$ maximizing $F(x+a) - F(x-a)$.
- Chernoff shows:
  - ▷ $n^{1/3}(\widehat{x}_a - x_a) \rightarrow_d \left(\frac{8f(x_a+a)}{c}\right)^{1/3} Z$ where $c \equiv f'(x_a - a) - f'(x_a + a)$.
  - ▷ $f_Z(z) = \frac{1}{2} g(z) g(-z)$ where

$$g(t) \equiv \lim_{x \nearrow t^2} \frac{\partial}{\partial x} u(t,x) = \lim_{x \nearrow t^2} u_x(t,x),$$

▷ $u(t,x) \equiv P^{(t,x)}(W(z) > z^2, \quad \text{for some } z \geq t)$ is a solution to the backward heat equation

$$\frac{\partial}{\partial t}u(t,x) = -\frac{1}{2}\frac{\partial^2}{\partial x^2}u(t,x)$$

under the boundary conditions

$$u(t,t^2) = \lim_{x \nearrow t^2} u(t,x) = 1, \quad \lim_{x \to -\infty} u(t,x) = 0.$$

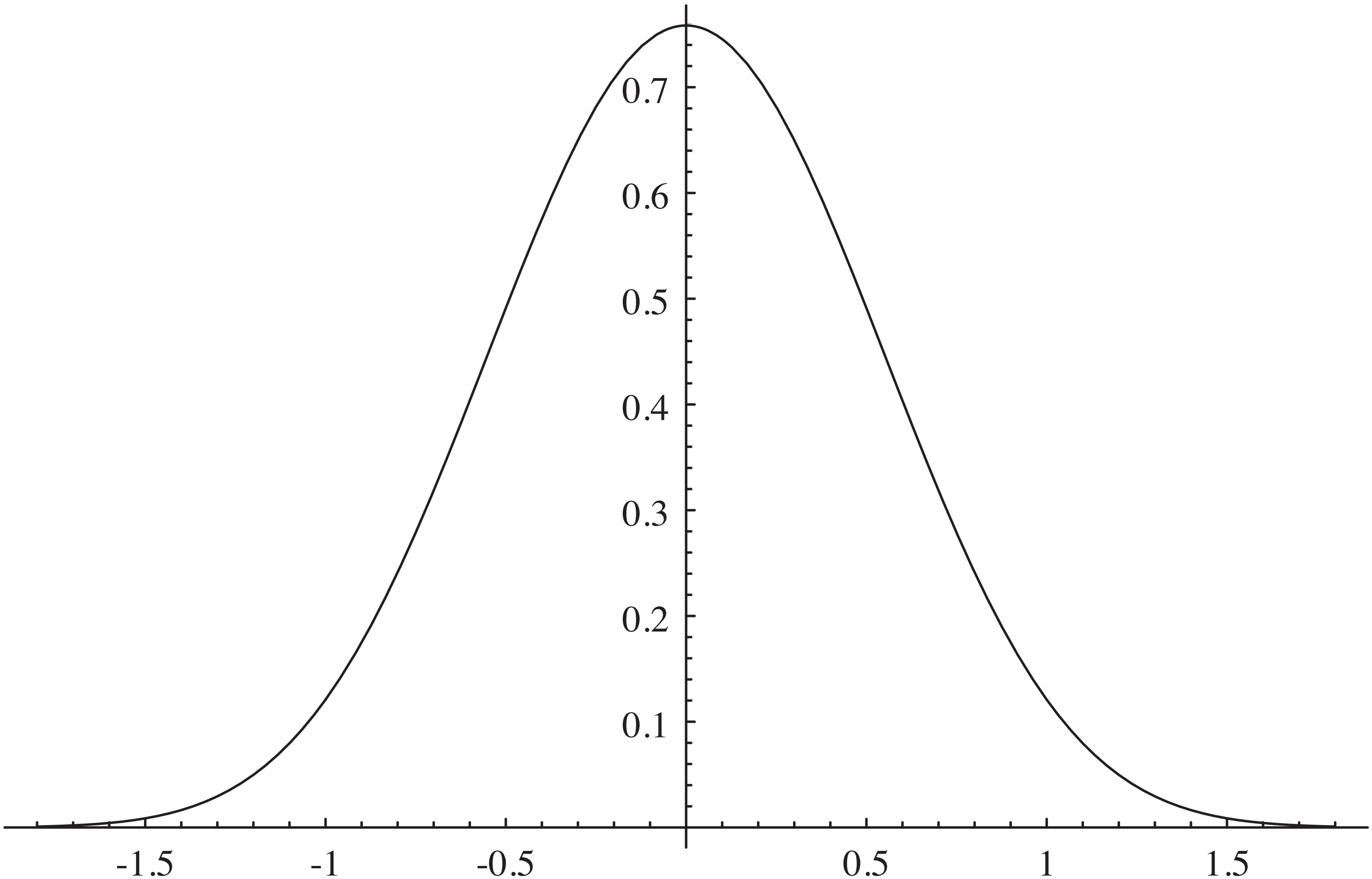
Groeneboom (1989) showed that g has Fourier transform given by

$$\hat{g}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda s} g(s)\, ds = \frac{2^{1/3}}{Ai(i(2)^{-1/3}\lambda)}; \tag{3}$$

Groeneboom (1989) also showed that

$$f_Z(z) \sim \frac{1}{2Ai'(a_1)} 4^{4/3}\, z \exp\left(-\frac{2}{3}z^3 + 3^{1/3} a_1 z\right),$$
$$P(Z > z) \sim \frac{1}{2Ai'(a_1)} 4^{4/3} \frac{1}{z} \exp\left(-\frac{2}{3}z^3\right) \quad \text{as} \quad z \to \infty$$

where $a_1 \approx -2.3381...$ is the largest zero of the Airy function Ai on the negative real axis and $Ai'(a_1) \approx 0.7022$.


0.7
0.6
0.5
0.4
0.3
0.2
0.1
-1.5
-1
-0.5
0.5
1
1.5

4. A LIL for Grenander's estimator

The tail behavior of Z given in the last display leads naturally to the following conjecture concerning a LIL for the Grenander estimator:

$$\limsup_{n\to\infty} \frac{n^{1/3}(\widehat{f}_n(x_0) - f(x_0))}{((3/2)\mathrm{loglog}n)^{1/3}} =_{a.s.} \left|\frac{1}{2} f(x_0) f'(x_0)\right|^{1/3} \cdot 2;$$

note that if $\delta > 0$ and $n_k \equiv \lfloor \alpha^k \rfloor$ with $\alpha > 1$, then

$$\begin{aligned} & \exp\left(-\frac{2}{3}[(1+\delta)((3/2)\mathrm{loglog}n_k)^{1/3}]^3\right) \\ &= \exp\left(-(1+\delta)^3 \mathrm{loglog}n_k\right) \\ &= (\mathrm{log}n_k)^{-(1+\delta)^3} \sim \left(k\mathrm{log}\alpha)^{-(1+\delta)^3}\right). \end{aligned}$$

Equivalently

$$\begin{aligned} \limsup_{n\to\infty} \frac{n^{1/3}(\widehat{f}_n(x_0) - f(x_0))}{(2\mathrm{loglog}n)^{1/3}} &=_{a.s.} \left|\frac{1}{2} f(x_0) f'(x_0)\right|^{1/3} \cdot 2 \cdot \frac{1}{2^{1/3}} \cdot \left(\frac{3}{2}\right)^{1/3} \\ &=_{a.s.} \left|\frac{1}{2} f(x_0) f'(x_0)\right|^{1/3} \cdot 2 \cdot \left(\frac{3}{4}\right)^{1/3}. \end{aligned}$$

How to prove this conjecture?

- Step 1: Switching!
- Step 2: localize; use a functional LIL for the local empirical process: Deheuvels and Mason (1994); Mason (2004).
- Step 3: Find the set of limit points via 1 and 2; study these via properties of the natural Strassen limit sets analogous to the distributional equivalents for Brownian motion.
- Step 4: Solve the resulting variational problem over the set of limit points expressed in terms of the Strassen limit points.

Step 1: Switching. Let $r_n \equiv (n^{-1} 2\text{loglog} n)^{1/3}$.
We want to find a number y_0 such that

$$P(r_n^{-1}(\widehat{f}_n(x_0) - f(x_0)) > y \ \ \text{i.o.}) = \begin{cases} 0, & \text{if } y > y_0, \\ 1, & \text{if } y < y_0. \end{cases}$$

Now $\{\widehat{f}_n(x_0) > a\} = \{\widehat{s}_n(a) > x_0\}$, so

$$\left\{\widehat{f}_n(x_0) > f(x_0) + r_n y \text{ i.o.}\right\} = \{\widehat{s}_n(f(x_0) + r_n y) > x_0 \text{ i.o.}\} . \quad (4)$$

By letting $s = x_0 + r_n h$ in the definition of $\widehat{s}_n$ we see that

$$\begin{aligned} &\widehat{s}_n(f(x_0) + r_n y) - x_0 \\ &\quad = r_n \text{argmax}_h \{\mathbb{F}_n(x_0 + r_n h) - f(x_0 + r_n y)(x_0 + r_n h)\} \end{aligned}$$

and hence the right side of (4) can be rewritten as $\{\widehat{h}_n > 0 \ \ \text{i.o.}\}$ where

$$\begin{aligned} \widehat{h}_n &= \text{argmax}_h \{\mathbb{F}_n(x_0 + r_n h) - (f(x_0) + r_n y)(x_0 + r_n h)\} \\ &= \text{argmax}_h \Big\{ r_n^{-2} [\mathbb{F}_n(x_0 + r_n h) - \mathbb{F}_n(x_0) - (F(x_0 + r_n h) - F(x_0))] \\ &\qquad + r_n^{-2} [F(x_0 + r_n h) - F(x_0) - f(x_0) r_n h] - y h \Big\} . \quad (5) \end{aligned}$$

The deterministic (drift) term on the right side converges to $f'(x_0)h^2/2$ as $n \to \infty$.

Step 2: LIL for the local empirical process. By a theorem of Mason (1988), the sequence of functions

$$\left\{ r_n^{-2} [\mathbb{F}_n(x_0 + r_n h) - \mathbb{F}_n(x_0) - (F(x_0 + r_n h) - F(x_0)) : \ h \in R \right\}$$

is almost surely relatively compact with limit set

$$\{ g(f(x_0) \cdot) : \ g \in \mathcal{G} \}$$

where the two-sided Strassen limit set $\mathcal{G}$ is given by

$$\mathcal{G} = \left\{ g : \mathbb{R} \to \mathbb{R} \middle| \ g(t) = \int_0^t \dot{g}(s) ds, \ t \in \mathbb{R}, \ \int_{-\infty}^{\infty} \dot{g}^2(s) ds \le 1 \right\}.$$

Proof: Reduce to the local empirical process of $\xi_1, \ldots, \xi_n$ i.i.d. Uniform$(0,1)$. As in Mason (1988), take $k_n \equiv n r_n = n^{2/3}(2\text{loglog}n)^{1/3} \nearrow \infty$ and $n^{-1} k_n = r_n \searrow 0$.

Thus the processes involved in the argmax in (5) are almost surely relatively compact with limit set

$$\{g(f(x_0)h) + 2^{-1}f'(x_0)h^2 - yh : \quad g \in \mathcal{G}\} \tag{6}$$

Step 3: Strassen set equivalences.

Lemma 1. Let $c > 0$ and $d \in R$. then

$$\{t \mapsto g(ct+d) - g(d) : \quad g \in \mathcal{G}\} = \sqrt{c}\mathcal{G}.$$

[This is analogous to $W(ct+d) - W(d) \stackrel{d}{=} \sqrt{c}W(t)$.]

Lemma 2. Let α, β be positive constants and $\gamma \in R$. Then

$$\begin{aligned} &\Big\{\text{argmax}\{\alpha g(h) - \beta h^2 - \gamma h\} : \quad g \in \mathcal{G}\Big\} \\ &\quad = \Big\{(\alpha/\beta)^{2/3}\text{argmax}_h\{g(h) - h^2\} - \gamma/(2\beta) : \quad g \in \mathcal{G}\Big\}. \end{aligned}$$

[This is analogous to

$$\text{argmax}\Big(\alpha W(h) - \beta h^2 - \gamma h\Big) \stackrel{d}{=} (\alpha/\beta)^{2/3}\{W(h) - h^2)\} - \gamma/(2\beta).]$$

By Lemmas 1 and 2, with $a = \sqrt{f(x_0)}$, $b = |f'(x_0)|/2$,

$$\begin{aligned} &\{g(f(x_0)h) + 2^{-1}f'(x_0)h^2 - yh : \quad g \in \mathcal{G}\} \\ &\quad = \{ag(h) - bh^2 - yh : \quad g \in \mathcal{G}\} \\ &\quad = \{(a/b)^{2/3}\text{argmax}\{g(h) - h^2\} - y/(2b) : \quad g \in \mathcal{G}\} \end{aligned}$$

Hence with $T_g \equiv \text{argmax}_h\{g(h) - h^2\}$,

$$\begin{aligned}
\left\{\widehat{h}_n > 0 \text{ i.o.}\right\} &\stackrel{a.s.}{=} \left\{\left(\frac{a}{b}\right)^{2/3} \sup_{g\in\mathcal{G}} T_g > \frac{y}{2b}\right\} \\
&= \left\{2b\left(\frac{a}{b}\right)^{2/3} \sup_{g\in\mathcal{G}} T_g > y\right\} \\
&= \emptyset
\end{aligned}$$

if

$$y > y_0 \equiv 2b(a/b)^{2/3} \sup_{g\in\mathcal{G}} T_g = |2^{-1} f(x_0) f'(x_0)|^{1/3} 2 \sup_{g\in\mathcal{G}} T_g.$$

It remains only to show that

$$\sup_{g\in\mathcal{G}} T_g = (3/4)^{1/3} \approx 0.90856\ldots.$$

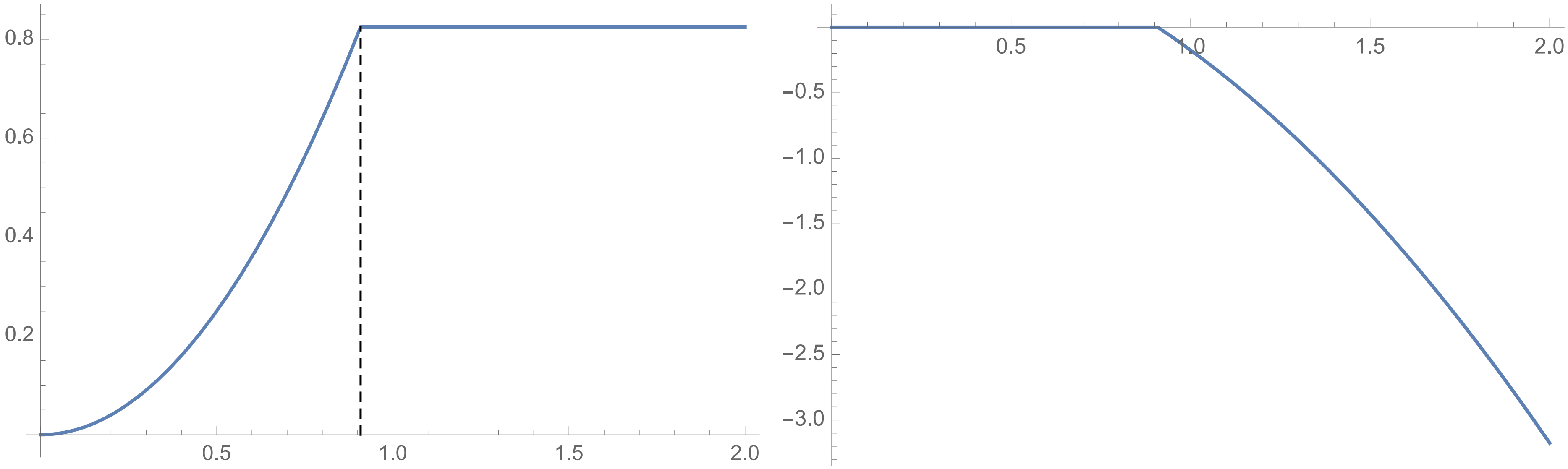
This follows from Lemma 3:

Lemma 3. Let t_0 be an arbitrary positive number and let $\dot{g} \in L_1([0, t_0])$ be an arbitrary function satisfying

$$\int_0^{t_0} \dot{g}(s)ds - t_0^2 \geq \int_0^t \dot{g}(s)ds - t^2 \quad \text{for} \quad 0 \leq t \leq t_0.$$

Then, with $\dot{g}_0(u) \equiv 2u$, $0 \leq t \leq u$,

$$\int_0^{t_0} \dot{g}(u)^2 du \geq \int_0^{t_0} \dot{g}_0(u)^2 du = \int_0^{t_0} (2u)^2 du = \frac{4t_0^3}{3}.$$



$g_0(t)$ and $g_0(t) - t^2$

Intuition: A few pictures and trial functions $\dot{g}$ quickly lead to

$$\dot{g}_0(s) \equiv \begin{cases} 2s, & 0 \le s \le t_0, \\ 0, & t_0 < s < \infty, \\ 0, & t < 0. \end{cases}$$

For this $\dot{g}_0$ we have

$$g_0(t) = t^2 1_{[0,t_0]}(t) + t_0^2 1_{(t_0,\infty)}(t) + 0 \cdot 1_{(-\infty,0)}(t).$$

Note that $\int_{-\infty}^{\infty} \dot{g}_0^2(s)ds = \int_0^{t_0} (2s)^2 ds = 4t_0^3/3.$ and that

$$g_0(t) - t^2 = \begin{cases} 0, & t \le t_0, \\ t_0^2 - t^2, & t > t_0. \end{cases}$$

Thus $\text{argmax}(g_0(t) - t^2) = t_0$ while $\sup_{t \ge 0}(g_0(t) - t^2) = 0$ is achieved for all $0 \le t \le t_0$. To force $g_0 \in \mathcal{G}$ we simply require that $\int_0^{t_0} \dot{g}_0^2(s)ds = 1$ and this yields $4t_0^3/3 = 1$ or $t_0 = (3/4)^{1/3}$.

Proof.

6. Open questions and further problems:

- For the mode estimator $M(\widehat{f}_n)$ of a log-concave density with mode $m \equiv M(f)$ and $f''(m) < 0$, we know that

$$n^{1/5}(M(\widehat{f}_n) - M(f)) \to_d C_2 M(H^{(2)})$$

where $H^{(2)}$ is the convex function given by the second derivative of the "invelope process"

$$H \quad \text{of} \quad Y(t) \equiv t^4 + \int_0^t W(s)ds.$$

Do we have a LIL for $M(\widehat{f}_n)$:

$$\limsup_{n\to\infty} \frac{n^{1/5}(M(\widehat{f}_n) - M(f))}{(2\log\log n)^{1/5}} =_{a.s.} \tilde{C}_2 < \infty?$$

- With definitions as in the last problem, do we have

$$P(M(H^{(2)}) \geq t) \leq K_1 \exp(-K_2 t^5) \quad \text{for all} \quad t \geq 0$$

for some $K_1, K_2 < \infty$? An answer to the previous problem would start to give information about K_2!

- Are the touch points of H and Y isolated? (See Groeneboom, Jongbloed, & W (2001), Groeneboom and Jongbloed (2015), page 320.)
- Balabdaoui and W (2014) show that Chernoff's distribution is log-concave. Is it strongly log-concave?
- Is there a Berry-Esseen type result for Grenander's estimator? That is, for what sequences $r_n \to \infty$ do we have

$$\sup_{t \in \mathbb{R}} \left| P\left(n^{1/3}(\widehat{f}_n(x) - f(x))/C \le t \right) - P(2Z \le t) \right| \le \frac{K}{r_n}$$

for some K finite.

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