Testing for sparse normal means: Is there a signal?

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- joint work with Leah Jager,
  U. S. Naval Academy

- Talk in the program on
  Statistical Theory and Methods for Complex,
  High-Dimensional Data

  Isaac Newton Institute for Mathematical Sciences,
  Cambridge 30 April, 2008

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Outline

- Testing problems for normal means

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- Detection boundaries and Tukey's "higher criticism" statistic

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- Application: the WMAP data

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- Initial setting: multiple testing of normal means
  For $i = 1, \ldots, n$ consider testing

$$H_{0,i} : X_i \sim N(0, 1)$$

versus

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- Main focus here: Q1; partial review of work on Q2.

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- Previous work: Q3: Where is the signal and how big is it?
  - Benjamini and Hochberg (1995)
  - Efron, Tibshirani, Storey, and Tusher (2001)
  - Storey, Dai, and Leek (2005)
  - Abramovich, Benjamini, Donoho, and Johnstone (2006).
  - Donoho and Jin (2006)

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  $H_1 : G = (1 - \epsilon)N(0,1) + \epsilon N(\mu, 1)$, and, in particular, against

$$H_1^{(n)} : G = (1 - \epsilon_n)N(0,1) + \epsilon_n N(\mu_n, 1).$$

for $\epsilon_n = n^{-\beta}, \quad \mu_n = \sqrt{2r \log n}$
$0 < \beta < 1, 0 < r < 1$.

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- Let $\Phi(z) \equiv P(Z \leq z) = \int_{-\infty}^{z} (2\pi)^{-1/2} \exp(-x^2/2)dx$,
  $Z \sim N(0,1)$.

- transform to $X_i \equiv 1 - \Phi(Y_i) \in [0, 1]$ i.i.d.

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- Then the testing problem becomes: test

$$H_0 : F = F_0 = U(0, 1) \qquad \text{versus}$$

$$H_1^{(n)} : F(u) = u + \epsilon_n\{(1 - u) - \Phi(\Phi^{-1}(1 - u) - \mu_n)\}$$
$$= (1 - \epsilon_n)u + \epsilon_n\{1 - \Phi(\Phi^{-1}(1 - u) - \mu_n)\}$$

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- Test statistics: Donoho-Jin

$$HC_n^* \equiv \sup_{X_{(1)} \leq u < X_{([n/2])}} \frac{\sqrt{n}(\mathbb{F}_n(u) - u)}{\sqrt{u(1 - u)}}$$

$$\equiv \text{Tukey's "higher criticism statistic"}$$

where $\mathbb{F}_n(u) \equiv n^{-1}\sum_{i=1}^{n} 1_{[0,u]}(X_i) =$ empirical distribution function of the X_i's.

- Optimal detection boundary $\rho^*(\beta)$ defined by:

$$\rho^*(\beta) = \begin{cases} \beta - 1/2, & 1/2 < \beta \leq 3/4 \\ (1 - \sqrt{1 - \beta})^2, & 3/4 < \beta < 1 \end{cases}$$

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- With $h_n(\alpha_n) = \sqrt{2 \log \log(n)}(1 + o(1))$

$$P_{H_0}(HC_n^* > h_n(\alpha_n)) = \alpha_n \to 0, \qquad \text{and}$$
$$P_{H_1^{(n)}}(HC_n^* > h_n(\alpha_n)) \to 1, \qquad \text{as } n \to \infty.$$
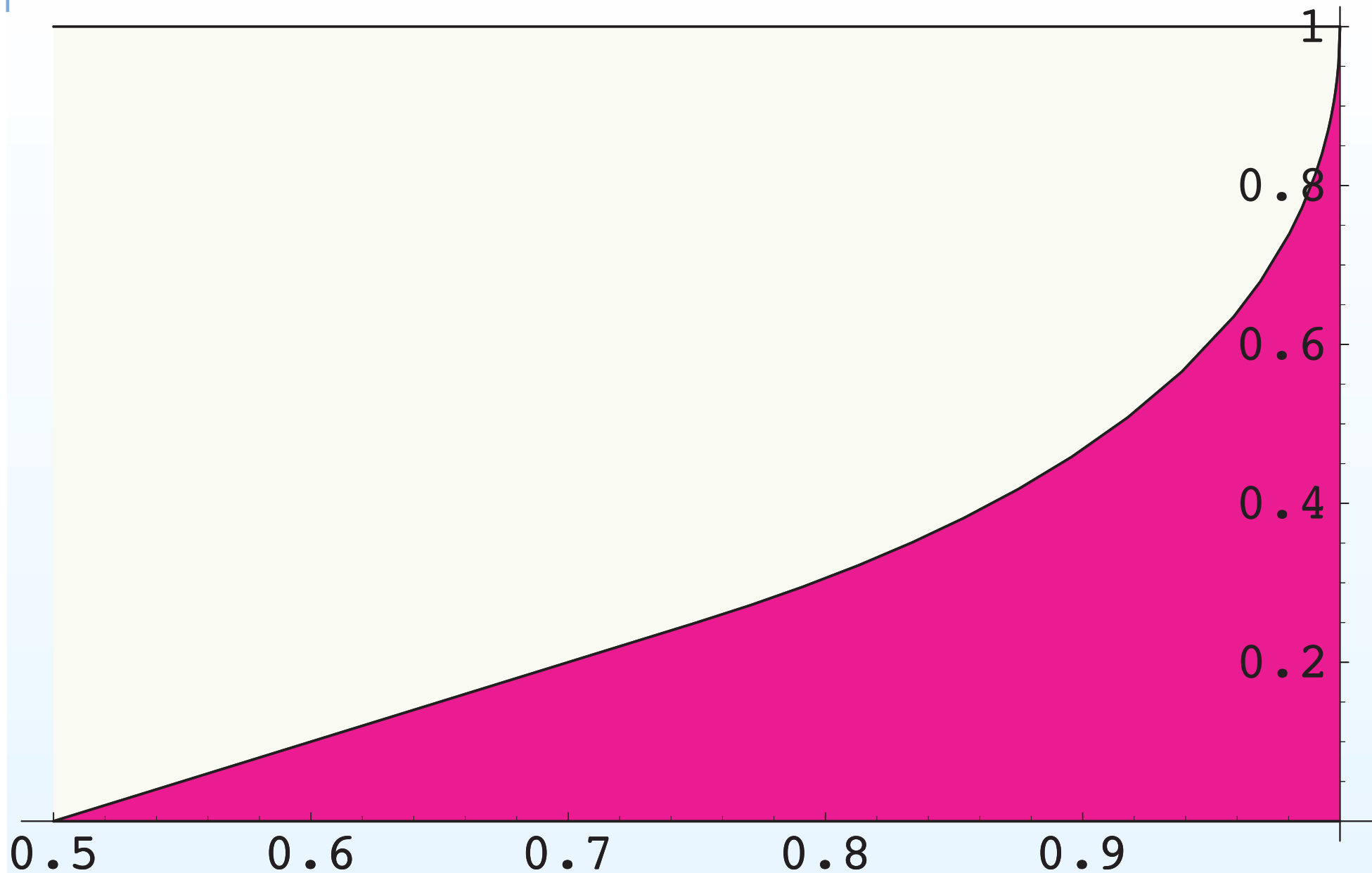
Figure 1. Detection boundary: $r > \rho^*(\beta)$ detectable

Some alternative statistics:

- Berk-Jones (1979) test statistic:

$$R_n \equiv \sup_x \log \lambda_n(x) = \sup_x K(\mathbb{F}_n(x), F_0(x)) \text{ with}$$

$$K(u, v) \equiv u \log \left(\frac{u}{v} \right) + (1 - u) \log \left(\frac{1 - u}{1 - v} \right)$$

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- Adaptation to one-sided $p-$value setting:

$$BJ_n^+ \equiv n \sup_{X_{(1)} \le u \le 1/2} K^+(\mathbb{F}_n(u), u)$$

where

$$K^+(u, v) \equiv \begin{cases} K(u, v), & \text{if } 0 < v < u < 1, \\ 0, & \text{if } 0 \le u \le v \le 1, \\ +\infty, & \text{otherwise}. \end{cases}$$

- **Theorem 2:** (Donoho - Jin, 2004). For $r > \rho^*(\beta)$ the test based on BJ_n^+ is size and power consistent for testing H_0 versus $H_1^{(n)}$; i.e. with $h_n(\alpha_n) = \sqrt{2 \log \log(n)}(1 + o(1))$

$$P_{H_0}(BJ_n^+ > h_n(\alpha_n)) = \alpha_n \to 0, \qquad \text{and}$$

$$P_{H_1^{(n)}}(BJ_n^+ > h_n(\alpha_n)) \to 1, \qquad \text{as } n \to \infty.$$

3. A new family of statistics via phi-divergences

A family of test statistics connecting "Higher criticism" and Berk-Jones:

- For $s \in \mathbb{R}$, $x \geq 0$ define

$$\phi_s(x) = \begin{cases} \frac{1-s+sx-x^s}{s(1-s)}, & s \neq 0, 1 \\ x \log x - x + 1, & s = 1 \\ -\log x + x - 1, & s = 0 \,. \end{cases}$$

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- Then define

$$K_s(u,v) = v\phi_s(u/v) + (1-v)\phi_s((1-u)/(1-v)).$$

- Special cases:

$$K_1(u,v) = K(u,v)$$
$$= u \log(u/v) + (1-u) \log((1-u)/(1-v))$$
$$K_0(u,v) = K(v,u)$$
$$K_2(u,v) = \frac{1}{2} \frac{(u-v)^2}{v(1-v)}$$
$$K_{-1}(u,v) = K_2(v,u) = \frac{1}{2} \frac{(u-v)^2}{u(1-u)}$$
$$K_{1/2}(u,v) = 2\{(\sqrt{u} - \sqrt{v})^2 + (\sqrt{1-u} - \sqrt{1-v})^2\}$$
$$= 4\{1 - \sqrt{uv} - \sqrt{(1-u)(1-v)}\}.$$

s=-1.0

s=-0.5

s= 0.0

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s= 1.5

s= 2.0
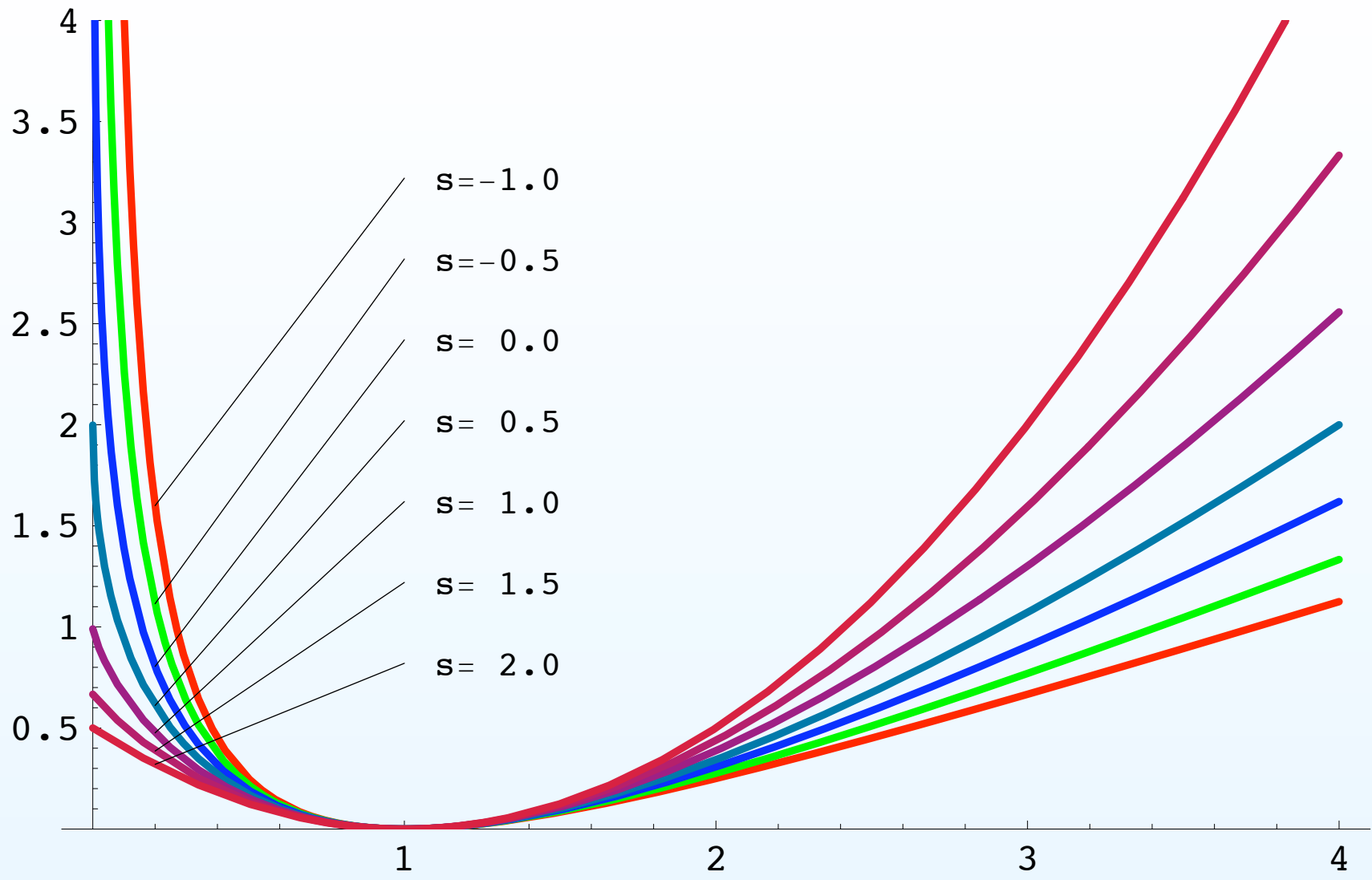
- The new family of statistics:

$$
S_n(s) = \begin{cases} \sup_{x \in \mathbb{R}} K_s(\mathbb{F}_n(x), F_0(x)), & s \geq 1 \\ \sup_{x \in [X_{(1)}, X_{(n)})} K_s(\mathbb{F}_n(x), F_0(x)), & s < 1, \end{cases}
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- Thus, with $F_0(x) = x$,

$$S_n(1) = R_n, \qquad S_n(0) = \text{``reversed'' Berk-Jones} \equiv \widetilde{R}_n$$

$$S_n(2) = \frac{1}{2} \sup_{x \in \mathbb{R}} \frac{(\mathbb{F}_n(x) - x)^2}{x(1 - x)},$$

$$S_n(-1) = \frac{1}{2} \sup_{x \in [X_{(1)}, X_{(n)})} \frac{(\mathbb{F}_n(x) - x)^2}{\mathbb{F}_n(x)(1 - \mathbb{F}_n(x))}$$

$$S_n(1/2)$$

$$= 4 \sup_{x \in [X_{(1)}, X_{(n)})} \{1 - \sqrt{\mathbb{F}_n(x)x} - \sqrt{(1 - \mathbb{F}_n(x))(1 - x)}\}$$

- Version of the statistics for one-sided $p-$value setting:

$$S_n^+ \equiv n \sup_{X_{(1)} \leq u \leq 1/2} K_s^+(\mathbb{F}_n(u), u)$$

where

$$K_s^+(u, v) \equiv \begin{cases} K_s(u, v), & \text{if } 0 < v < u < 1, \\ 0, & \text{if } 0 \leq u \leq v \leq 1, \\ +\infty, & \text{otherwise.} \end{cases}$$

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- **Theorem:** (Jager - Wellner, 2007). For $r > \rho^*(\beta)$ the tests based on $S_n^+(s)$ with $-1 \leq s \leq 2$ are size and power consistent for testing H_0 versus $H_1^{(n)}$; i.e. with $s_n(\alpha_n) = \log\log(n)(1 + o(1))$

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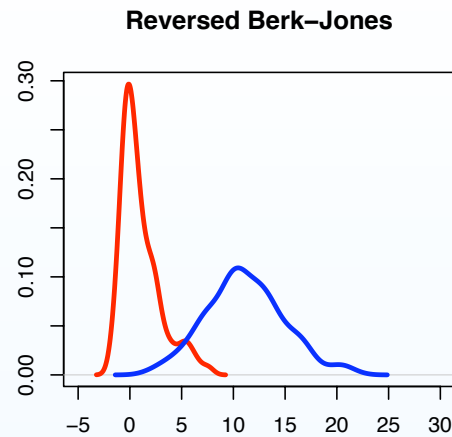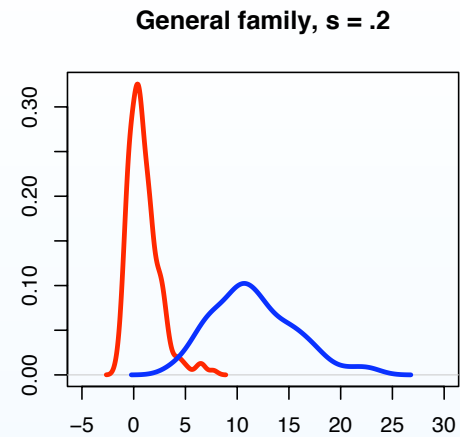
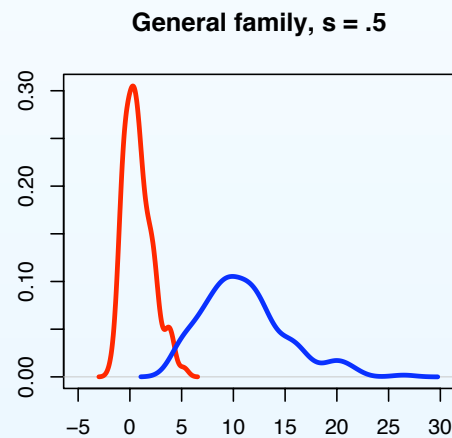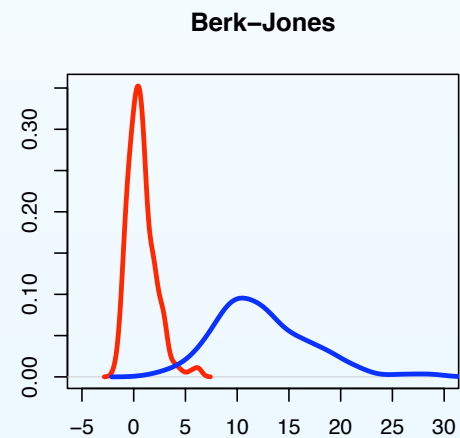Figure 2. Separation plots: $n = 5 \times 10^5$, $r = .15$, $\beta = 1/2$
Smoothed histograms of reps $= 200$ of the statistics under the
null hypothesis and the the alternative hypothesis

4. Beyond normality:

generalized Gaussian distributions, ...

- Donoho and Jin (2004) also computed detection boundaries for sparse mixtures of "Generalized Gaussian" or Subbotin distributions: $X \sim GN_\gamma(\mu)$ has density function

$$f_{\gamma,\mu}(x) = \frac{1}{C_\gamma} \exp\left(-\frac{|x - \mu|^\gamma}{\gamma}\right), \quad C_\gamma = 2\Gamma(1/\gamma)\gamma^{1/\gamma - 1}.$$

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- Suppose $Y_1, \ldots, Y_n$ i.i.d. G on $\mathbb{R}$.
- Test $H_0 : G = GN_\gamma(0)$ versus
$H_1^{(n)} : G = (1 - \epsilon_n)GN_\gamma(0) + \epsilon_n GN_\gamma(\mu_n)$ where

$$\epsilon_n = n^{-\beta}, \qquad \mu_{\gamma,n} = (\gamma r \log n)^{1/\gamma},$$

where $1/2 < \beta < 1, \ \ 0 < r < 1$.

- Detection boundary for $1 < \gamma \leq 2$:

$$\rho_\gamma^*(\beta) = \begin{cases} (2^{1/(\gamma-1)} - 1)^{\gamma-1}(\beta - 1/2), & 1/2 < \beta \leq 1 - 2^{-\gamma/(\gamma-1)}, \\ (1 - (1 - \beta)^{1/\gamma})^\gamma, & 1 - 2^{-\gamma/(\gamma-1)} \leq \beta < 1. \end{cases}$$

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- Detection boundary for $0 < \gamma \leq 1$:

$$\rho_\gamma^*(\beta) = 2(\beta - 1/2), \qquad 1/2 < \beta < 1.$$

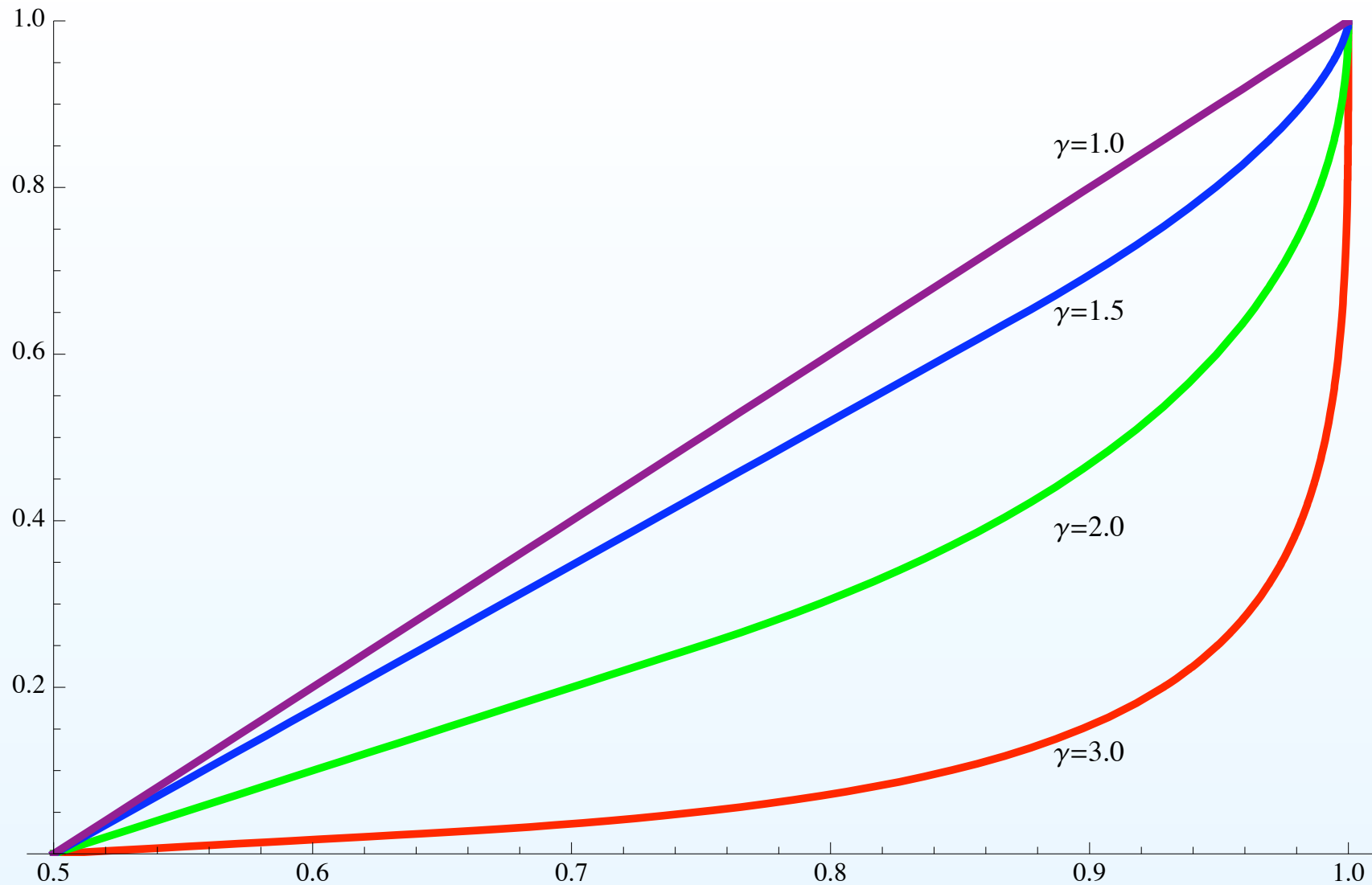Note: The detection boundary is the same for all for $0 < \gamma \leq 1$!

Figure 3. Detection boundaries for GN testing problem, $\gamma \in \{1, 1.5, 2, 3\}$.

- **Theorem:** (Donoho - Jin, 2004). For the higher criticism test statistic applied to the p-values $p_i \equiv P(GN_\gamma(0) > Y_i)$, $i = 1, \ldots, n$. Then the detection boundary $\rho_{HC,\gamma}$ for this procedure is the same as the efficient detection boundary:

$$\rho_{HC,\gamma}(\beta) = \rho_\gamma^*(\beta), \qquad 1/2 < \beta < 1.$$

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- Similar theorem for χ_ν^2 mixtures.

5. Estimating the proportion of false null hypotheses

- Meinshausen and Rice (2006): Assume
  $Y_i \sim (1 - \epsilon_n)N(0,1) + \epsilon_n F$, F arbitrary.
  $\epsilon_n = n^{-\beta}$, $1/2 < \beta < 1$.
  M & R construct $\hat{\epsilon}_n^{MR}$ such that

$$P_{\epsilon_n, F}(\epsilon_n \geq \hat{\epsilon}_n^{MR}) \geq 1 - \alpha$$

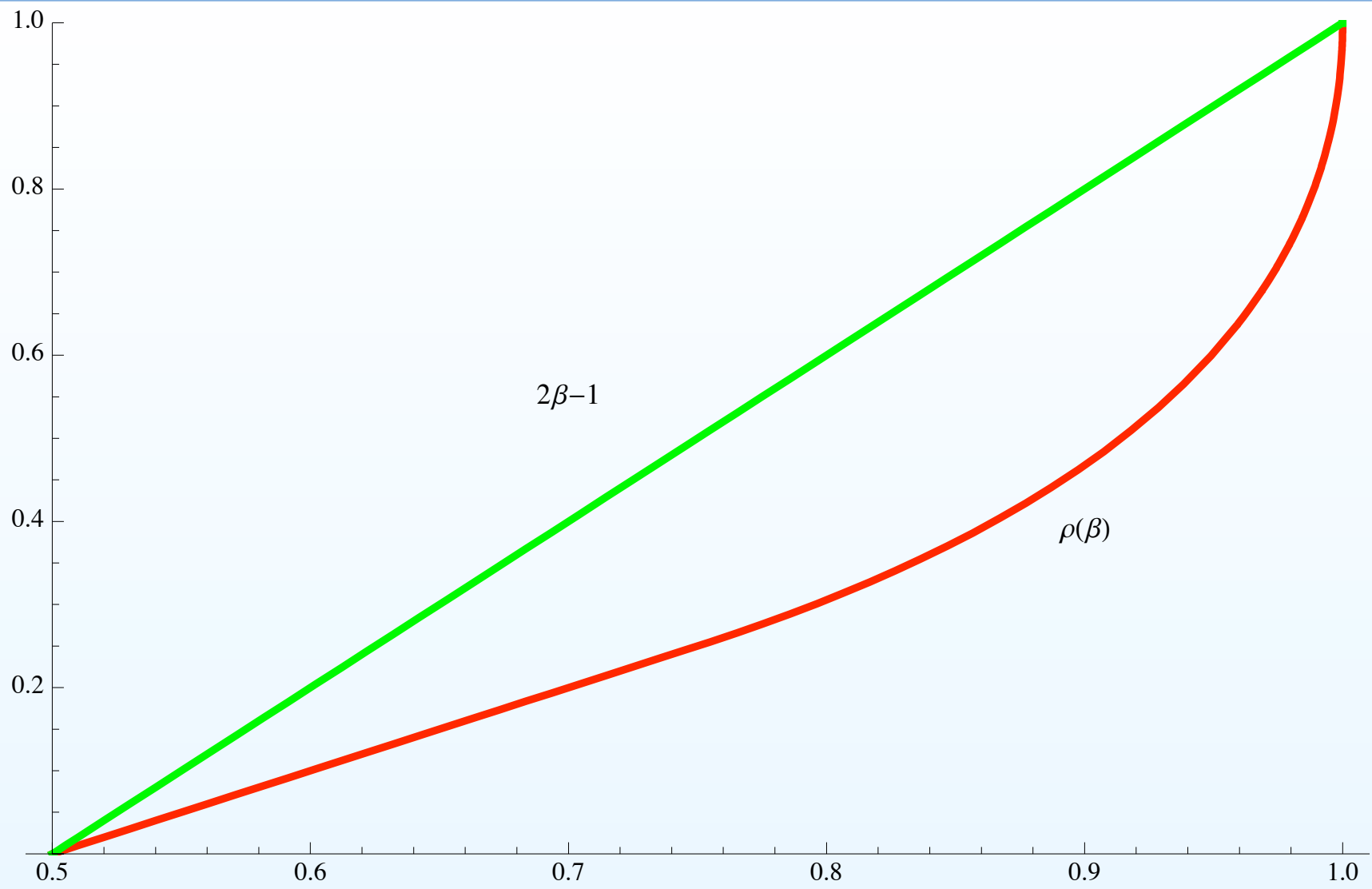# 5. Estimating the proportion of false null hypotheses

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- When $F = N(\mu_n, 1)$, $\mu_n = \sqrt{2r \log n}$, then if $r > 2\beta - 1$,

$$P_{\epsilon_n, \mu_n} \left(\left| \frac{\hat{\epsilon}_n^{MR}}{\epsilon_n} - 1 \right| > \delta \right) \to 0$$

  for every $\delta > 0$.

$2\beta-1$

$\rho(\beta)$

- Cai, Jin, and Low (2007): Assume
  $Y_i \sim (1 - \epsilon_n)N(0, 1) + \epsilon_n N(\mu_n, 1)$,
  $\epsilon_n = n^{-\beta}$, $\mu_n = \sqrt{2r \log n}$, with $1/2 < \beta < 1$, $0 < r < 1$.
  Cai, Jin, and Low construct $\hat{\epsilon}_n^{CJL}$ such that

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- For any closed set Ω in the interior of $\{(\beta, r): \ r > \rho^*(\beta)\}$,

$$\sup_{(\beta, r) \in \Omega} P_{\epsilon_n, \mu_n} \left(\left| \frac{\hat{\epsilon}_n^{CJL}}{\epsilon_n} - 1 \right| > \delta \right) \to 0$$

for every $\delta > 0$.

- Cai, Jin, and Low (2007): Assume
$Y_i \sim (1 - \epsilon_n)N(0,1) + \epsilon_n N(\mu_n, 1)$,
$\epsilon_n = n^{-\beta}$, $\mu_n = \sqrt{2r \log n}$, with $1/2 < \beta < 1$, $0 < r < 1$.
Cai, Jin, and Low construct $\hat{\epsilon}_n^{CJL}$ such that

$$P_{\epsilon_n, \mu_n}(\epsilon_n \geq \hat{\epsilon}_n^{CJL}) \geq 1 - \alpha$$

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- Moreover ...

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$$E_{\epsilon_n,\mu_n} \left(\frac{\hat{\epsilon}_n^{CJL}}{\epsilon_n} - 1 \right)^2 \leq C_n(\beta, r)$$

for $C_n(\beta, r) \to 0$ at the optimal rate up to powers of $\log n$.

6. Application: non-Gaussian-ness in the WMAP data?

The Cosmic Microwave Background radiation, CMB
also CMBR, or CBR, or MBR
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- 1989: COBE NASA Satellite launched: 7° angular resolution 1992: first anisotropies detected by COBE

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- 2008: European Space Agency Planck CMB probe to be launched on 31 October: $5'$ angular resolution.
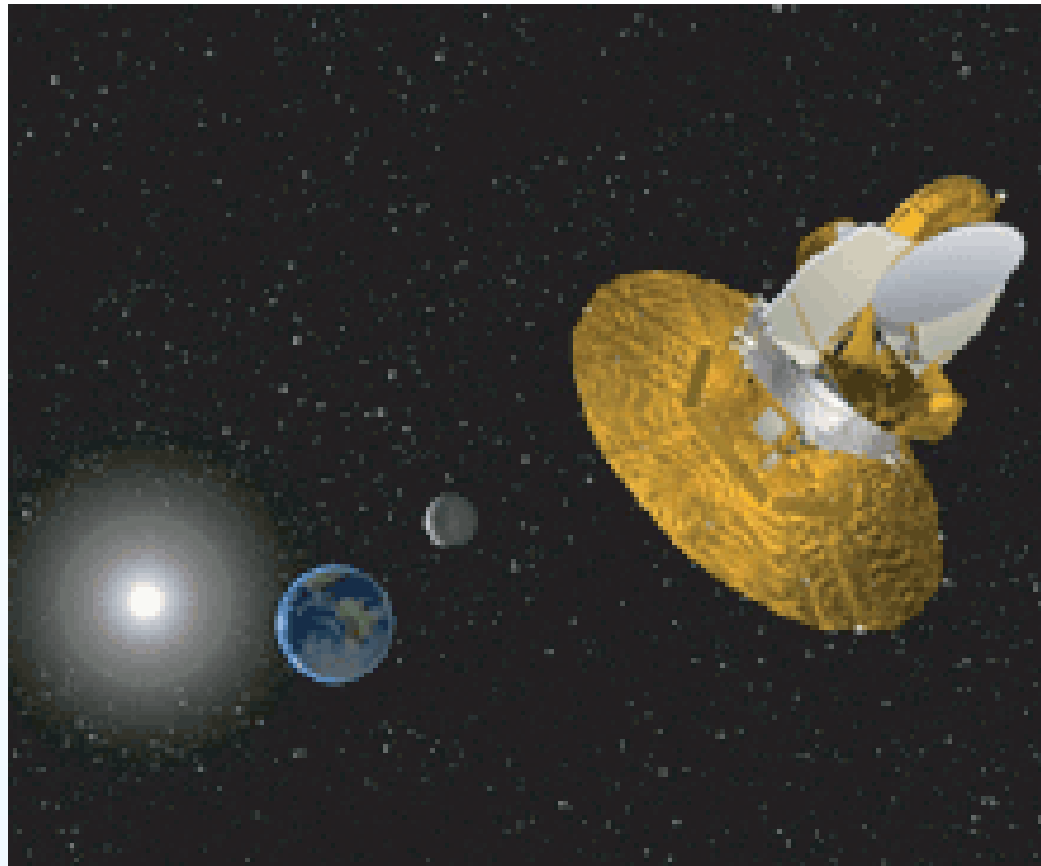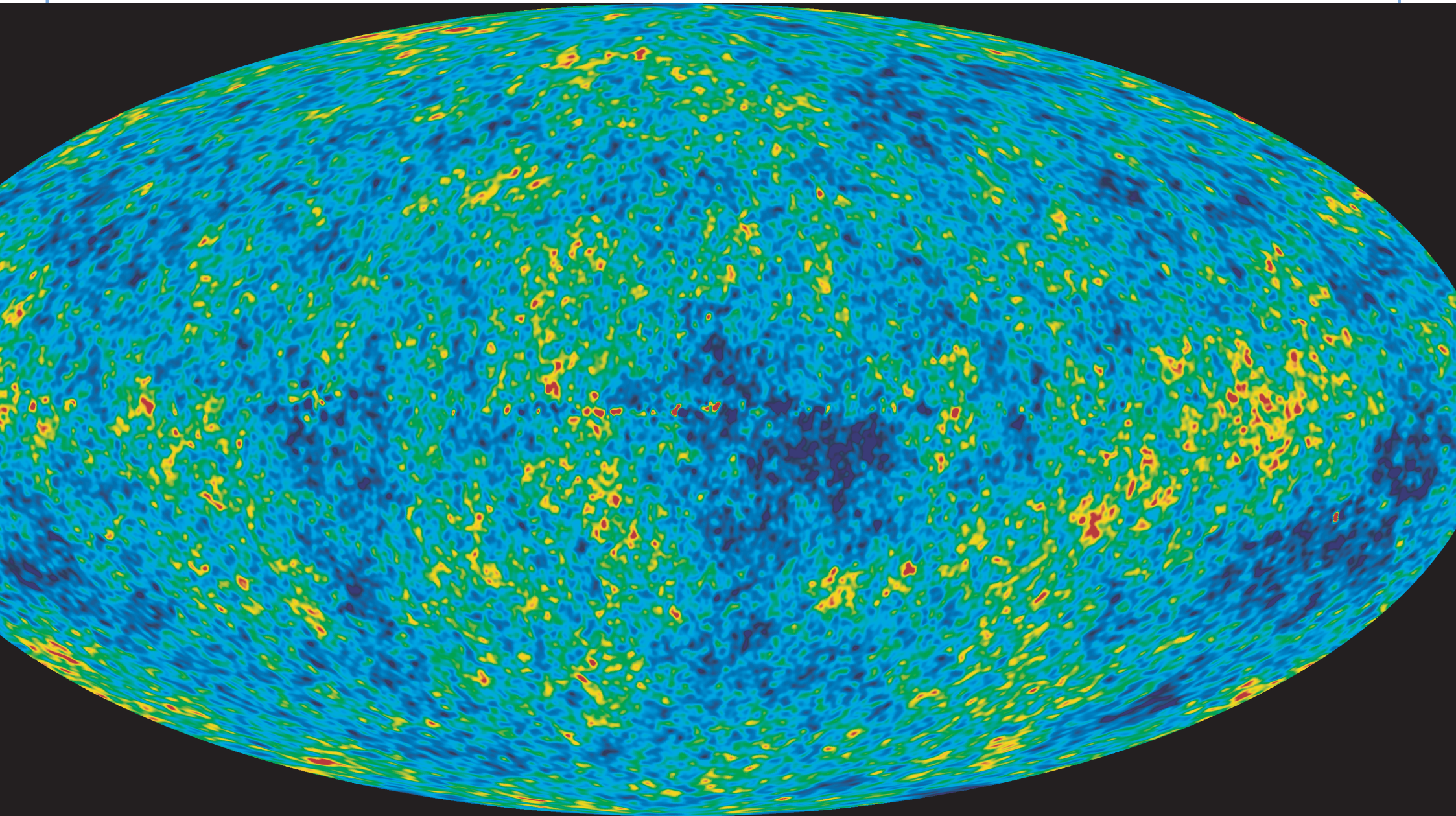
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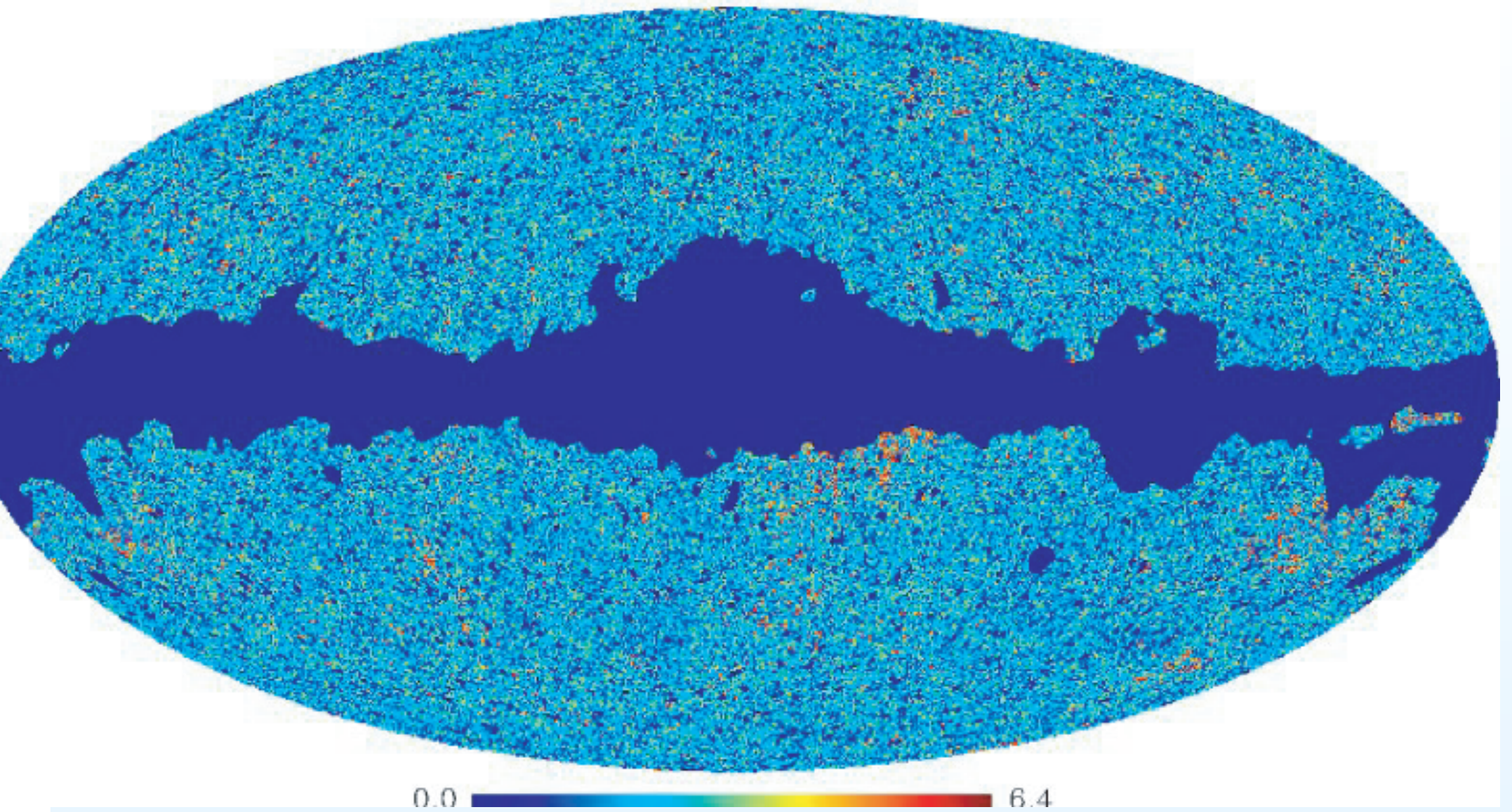 - $< 1\%$ neutrinos.

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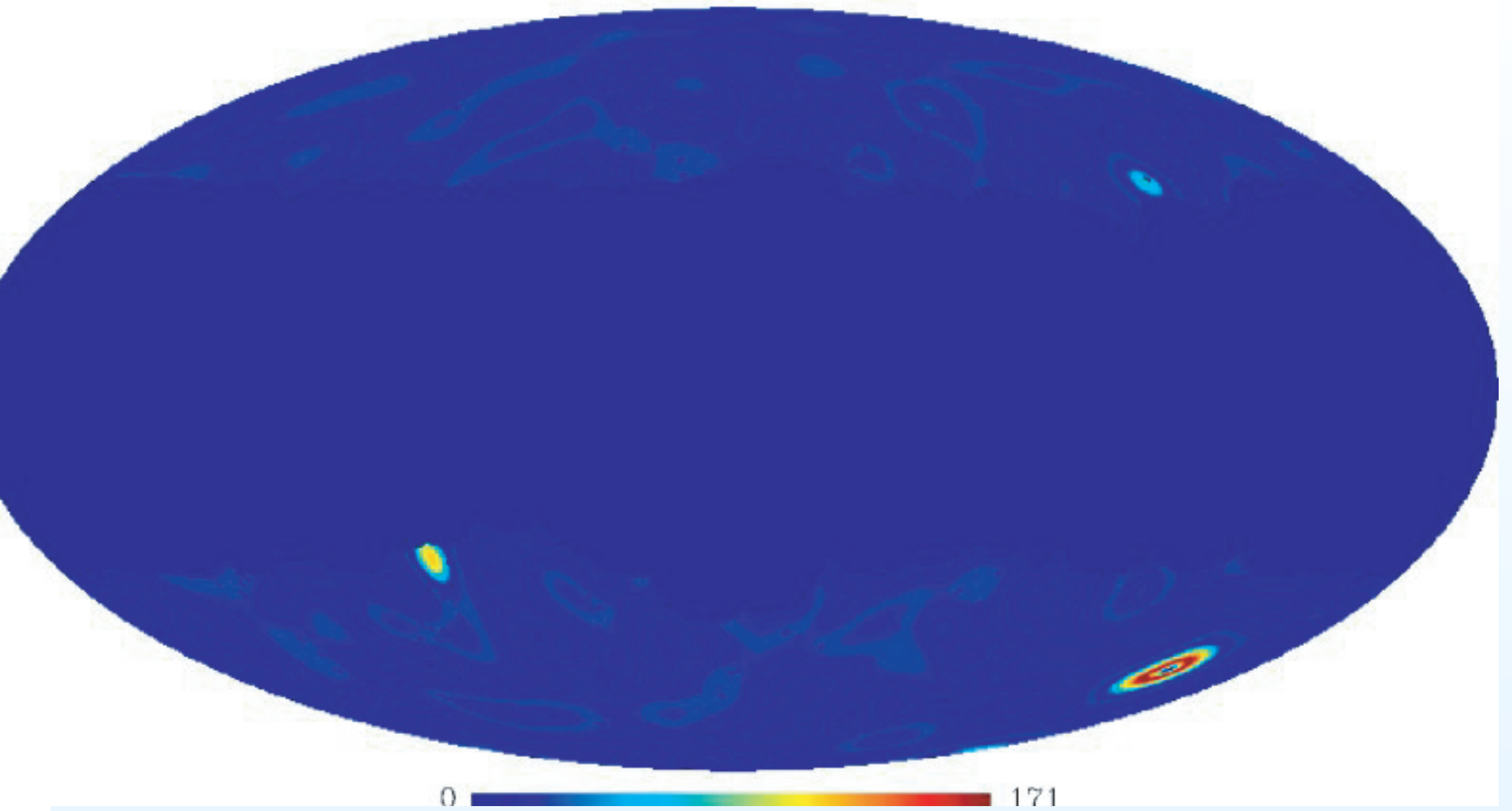
- Results support the "Lambda -Cosmic Dark Matter" model

HC WMAP

0.0 ▭ 6.4
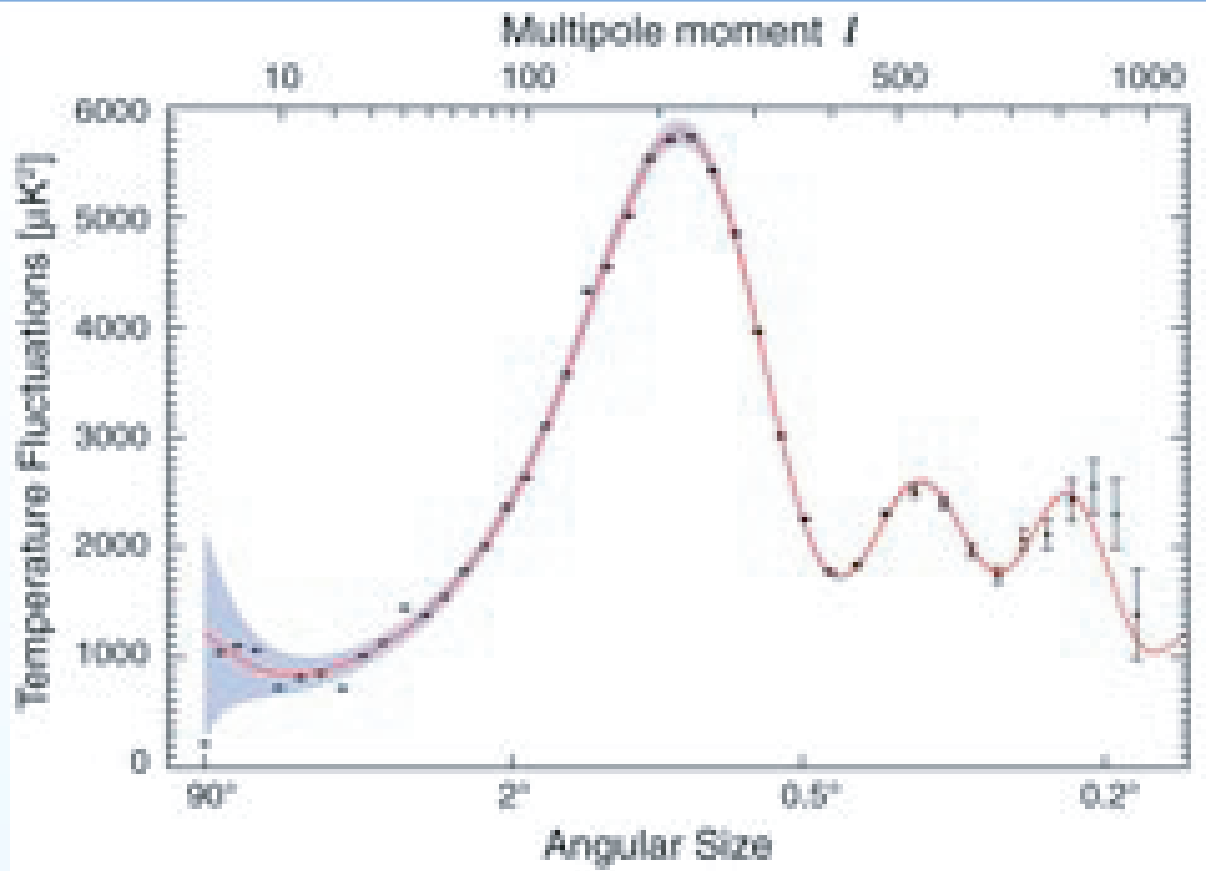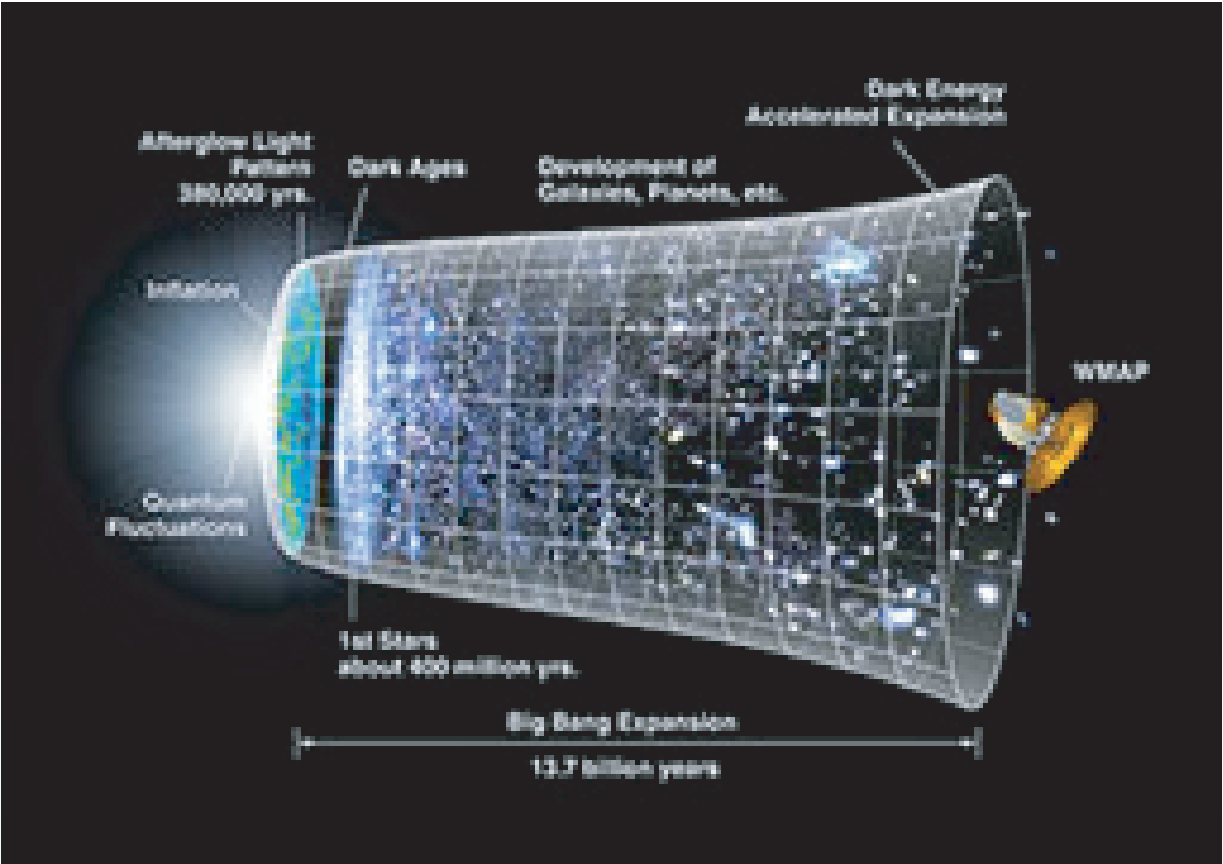
HC WMAP sc=300 arcmin

0 ▬▬▬▬▬▬▬▬▬▬▬▬▬▬ 171

From Spergel et al., 2007:

"The detection of primordial non-Gaussian fluctuations in the CMB would have a profound impact of the physics of the early universe. While the simplest inflationary models predict only mild non-Gaussianities that should be undetectable in the WMAP data, there are a wide range of plausible mechanisms for generating significant and detectable non-Gaussian fluctuations … "

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8. Some references for WMAP

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