Estimation and Testing with Current Status Data

Jon A. Wellner

University of Washington

- joint work with Moulinath Banerjee,
  University of Michigan

- Talk at Victoria University, Wellington, NZ,

  School of Mathematics, Statistics, and Computer Science,

  September 1, 2006

- *Email: jaw@stat.washington.edu*
  http: //www.stat.washington.edu/jaw/jaw.research.html

Moulinath Banerjee

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- Introduction: current status data

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- Further problems

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- Likelihood:

$$L_n(F) = \prod_{i=1}^n F(Y_i)^{\Delta_i} (1 - F(Y_i))^{1-\Delta_i}$$

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- The likelihood ratio statistic:

$$\lambda_n = \frac{\sup_F L_n(F)}{\sup_{F:F(t_0)=\theta_0} L_n(F)} = \frac{L_n(\hat{F}_n)}{L_n(\hat{F}_n^0)} \,.$$

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$$\mathbb{G}_n(t) = \frac{1}{n} \sum_{i=1}^{n} 1_{[Y_i \leq t]}, \qquad \mathbb{V}_n(t) = \frac{1}{n} \sum_{i=1}^{n} \Delta_i 1_{[Y_i \leq t]}.$$

Note that

$$\mathbb{G}_n(t) \to_{a.s.} G(t), \qquad \mathbb{V}_n(t) \to_{a.s.} \int_0^t F(y) dG(y) \equiv V(t).$$

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- Thus

$$\frac{dV}{dG}(t) = F(t)$$

- Partial sum diagram: Let $Y_{(1)} \leq Y_{(2)} \leq \cdots \leq Y_{(n)}$.
  The *partial sum diagram* $\mathcal{P} = \{P_i\}$ is given by

$$P_i = (\mathbb{G}_n(Y_{(i)}), \mathbb{V}_n(Y_{(i)})), \quad i = 1, \ldots, n\,.$$

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$$\hat{F}_n(Y_{(i)}) = \text{left derivative of the Greatest}$$
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- Greatest Convex Minorant $=$ GCM

Cumulative Sum Diagram: Example $n = 5$

i	$\Delta_{(i)}$	$Y_{(i)}$
1	1	0.4
2	0	0.7
3	1	1.3
4	1	1.5
5	0	2.0
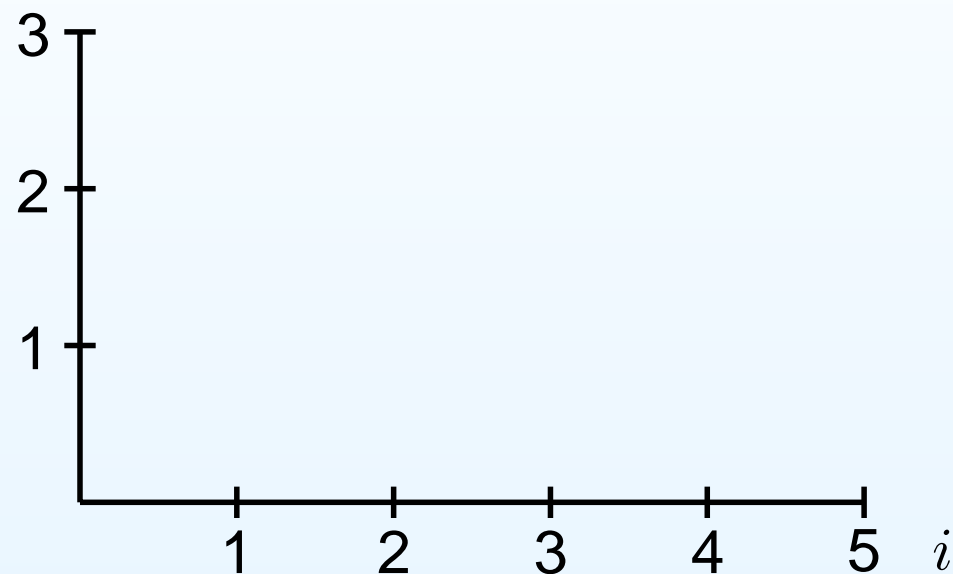
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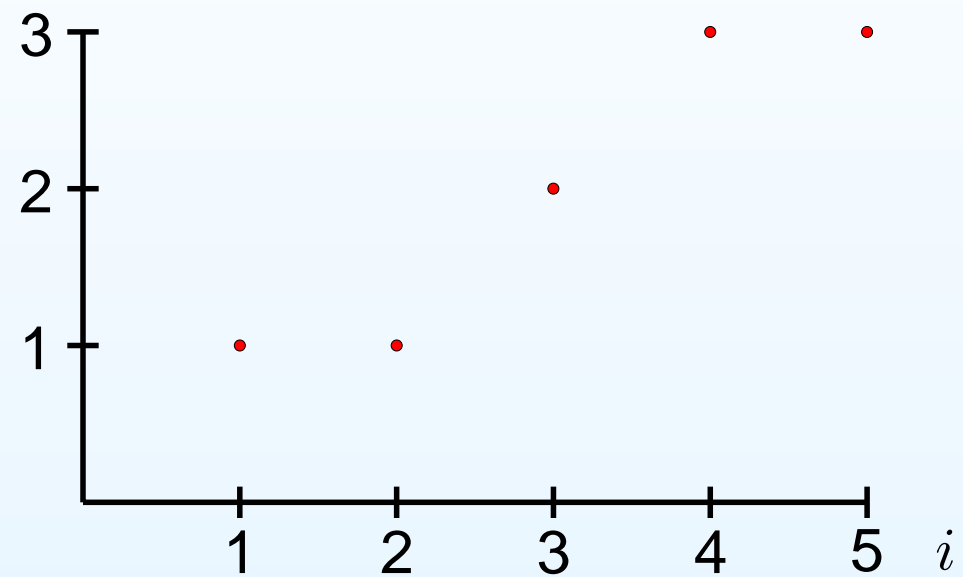
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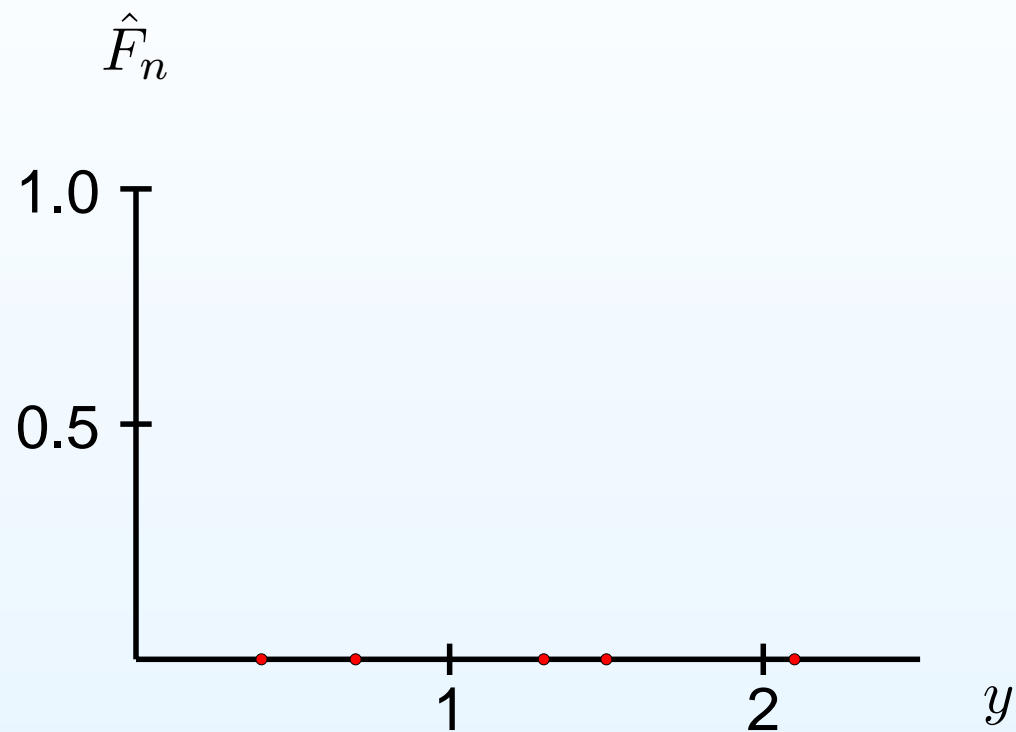
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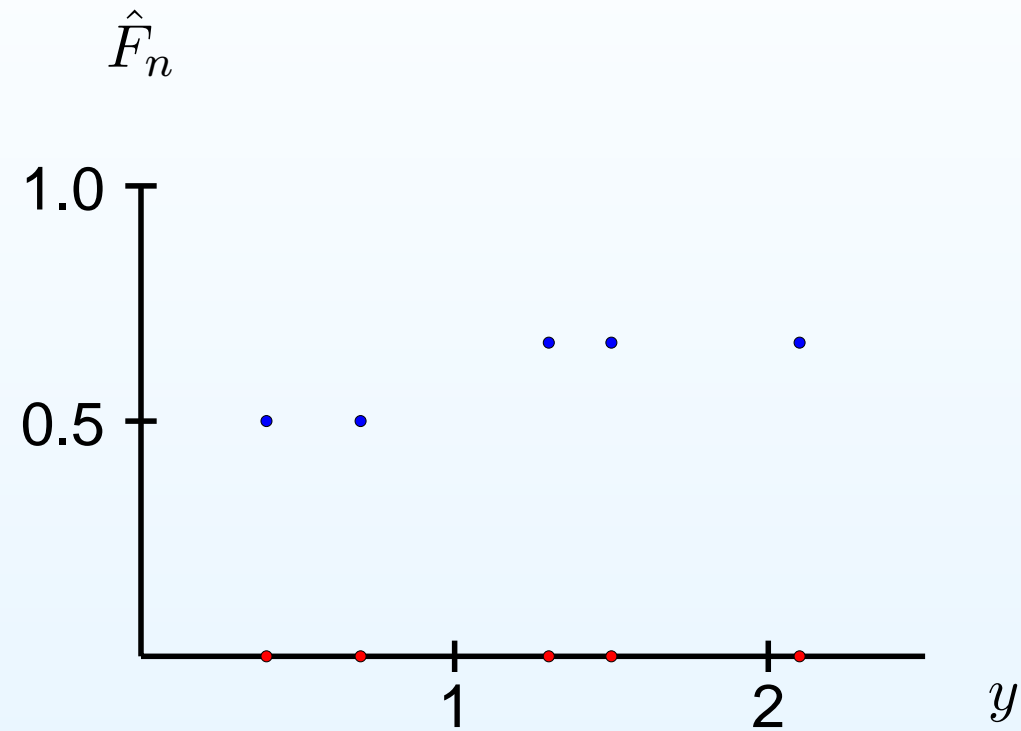
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$F =$ Exponential(1); $G =$ Uniform$(0, 3)$; $n = 30$
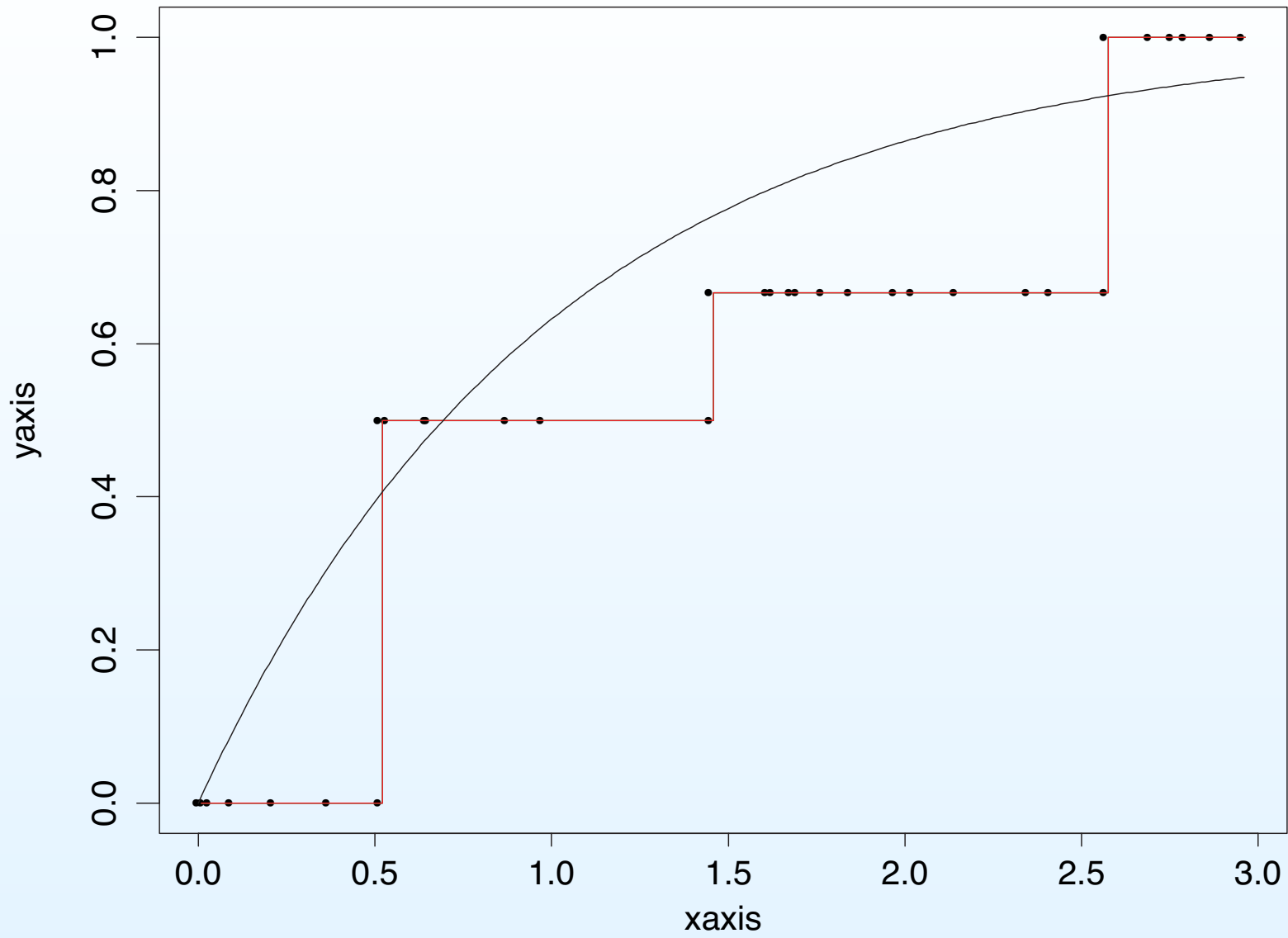
3. The constrained MLE $\hat{F}_n^0$. Recipe:

- Break $\mathcal{P}$ into $\mathcal{P}_L$ and $\mathcal{P}_R$ where

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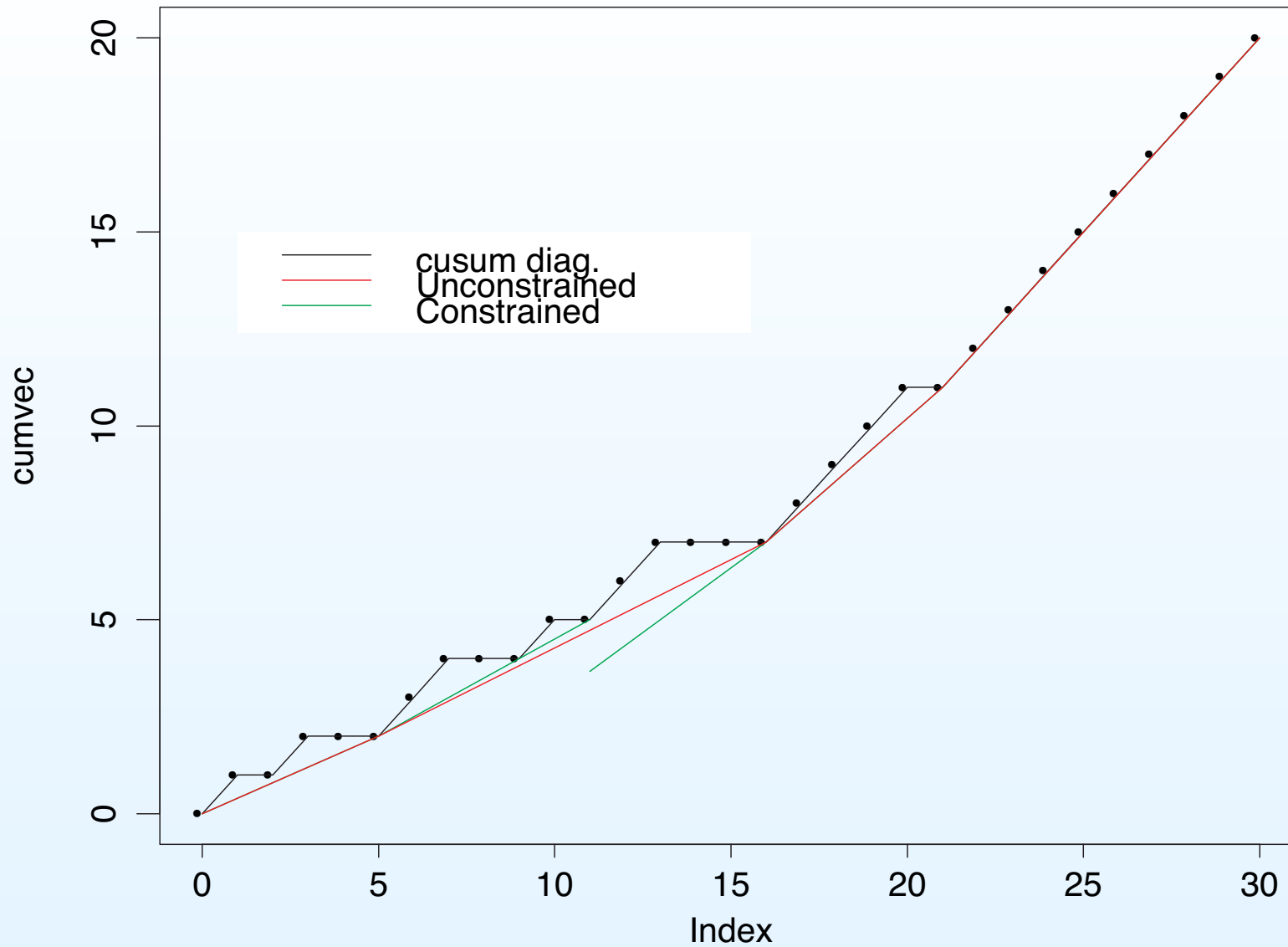
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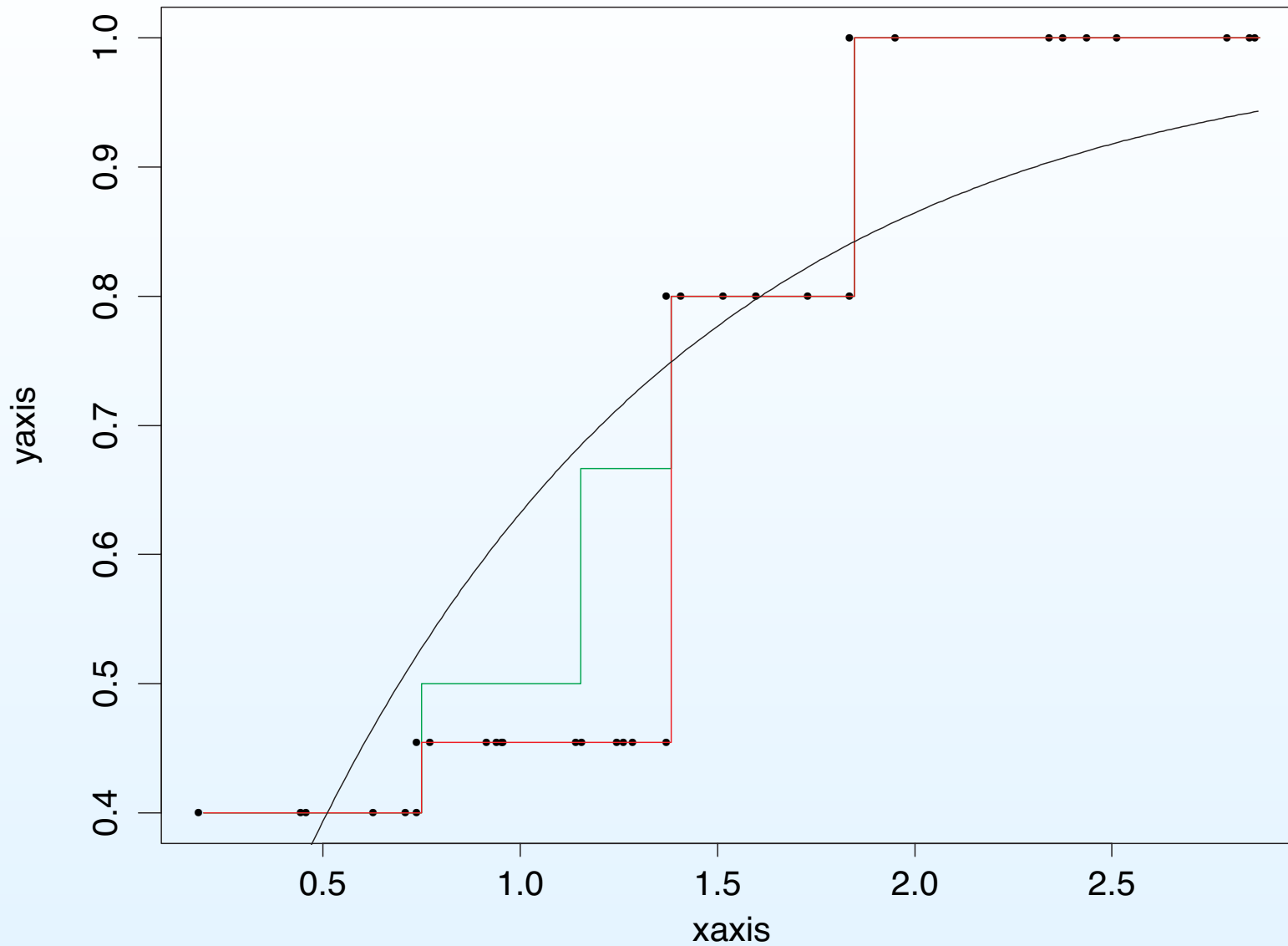## 3. The constrained MLE $\hat{F}_n^0$. Recipe:

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- The resulting (truncated or constrained) slope process yields the constrained MLE $\hat{F}_n^0$.

$F =$Exponential(1); $G =$Uniform$(0, 3)$; $n = 30$; $F(t_0) = 2/3$;
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- Answer: Yes! Banerjee and Wellner (2001)

5. How big is "too big"? The limiting Gaussian problem

- Suppose that we observe $\{X(t) : t \in R\}$ where

$$X(t) = F(t) + \sigma W(t)$$

  - $F(t) = \int_{-\infty}^{t} f(s)ds$,
  - f monotone non-decreasing, and
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- The "canonical monotone function" is a linear one, and we can change σ to 1 by virtue of scaling arguments so the "canonical" version of the problem is as follows:

$$dX(t) = 2tdt + dW(t)\,,$$

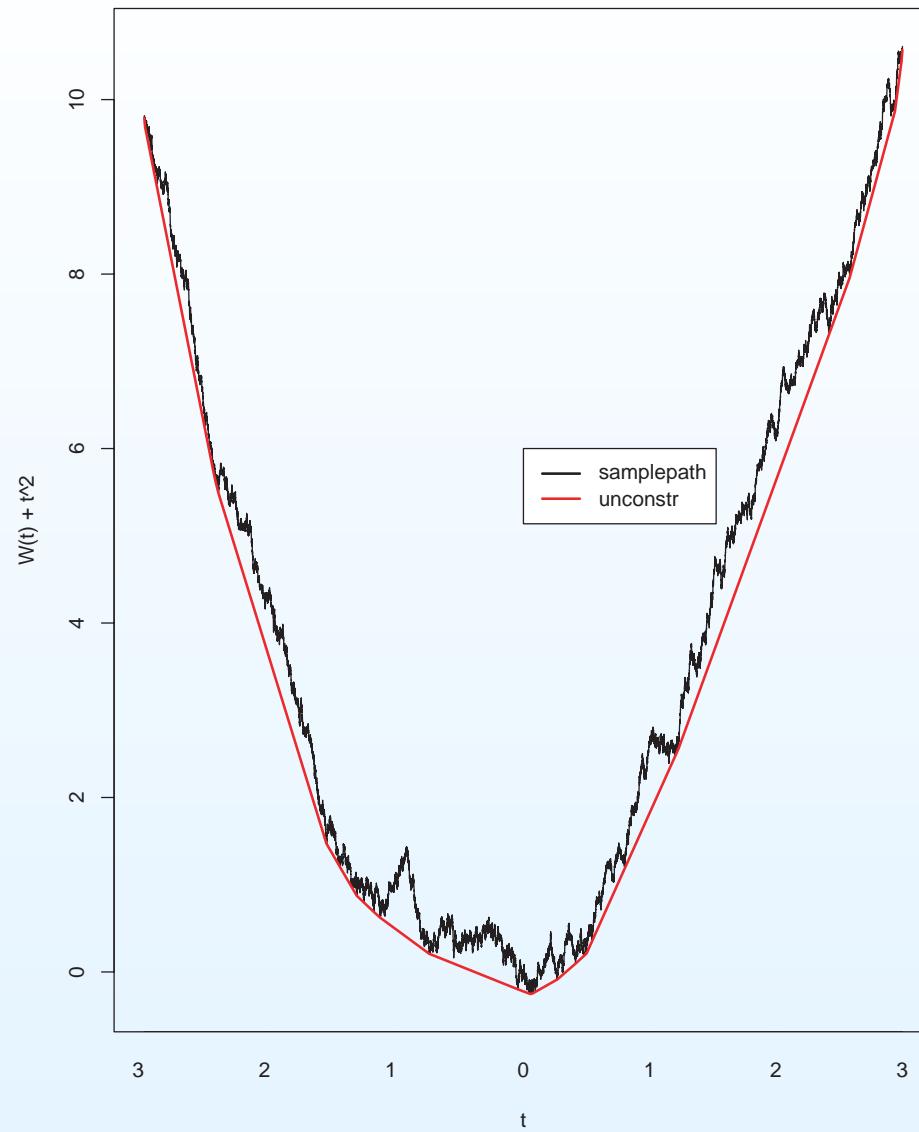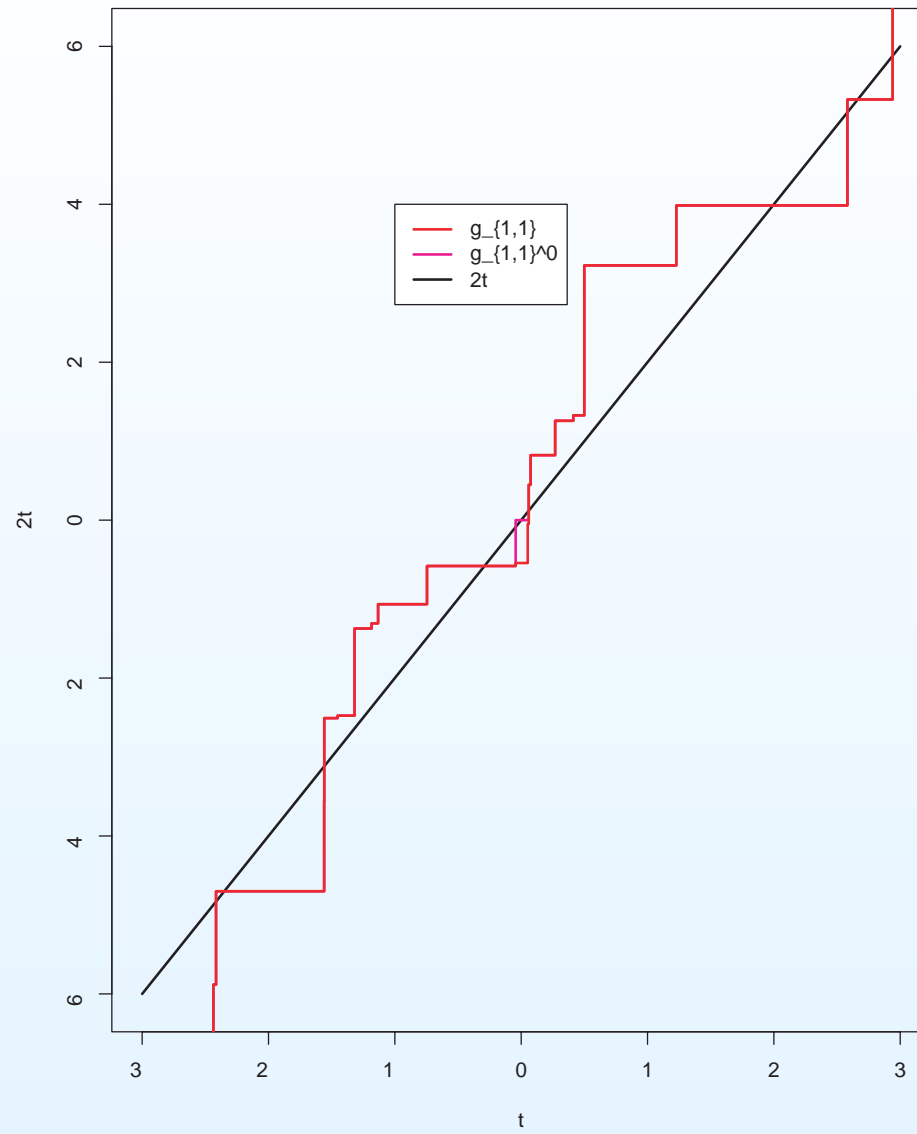- "estimate" $2t$ when $\{X(t) : \ t \in R\}$, is observed. Thus

$$X(t) = t^2 + W(t) \,.$$

Unconstrained "Estimator": Slope of GCM of $X(t)$. Call this process of slopes of the GCM $\mathbb{S}$.

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- Estimate the monontone function $f(t) = 2t$ subject to the constraint that $f(0) = 0$ when $\{X(t) : t \in R\}$ is observed.

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- Recipe:
  - Break $\{X(t) : t \in R\}$ into $X^L \equiv \{X(t) : t < 0\}$ and $X^R \equiv \{X(t) : t \geq 0\}$.
  - Form the GCM's of X^L and X^R say Y^L and Y^R.
  - If the slope of Y^L exceeds 0, replace it by 0; if the slope of Y^R drops below 0, replace it by 0.
  - The resulting (truncated or constrained) slope process $\mathbb{S}^0$ is the constrained MLE of $f(t) = 2t$ in the Gaussian problem.
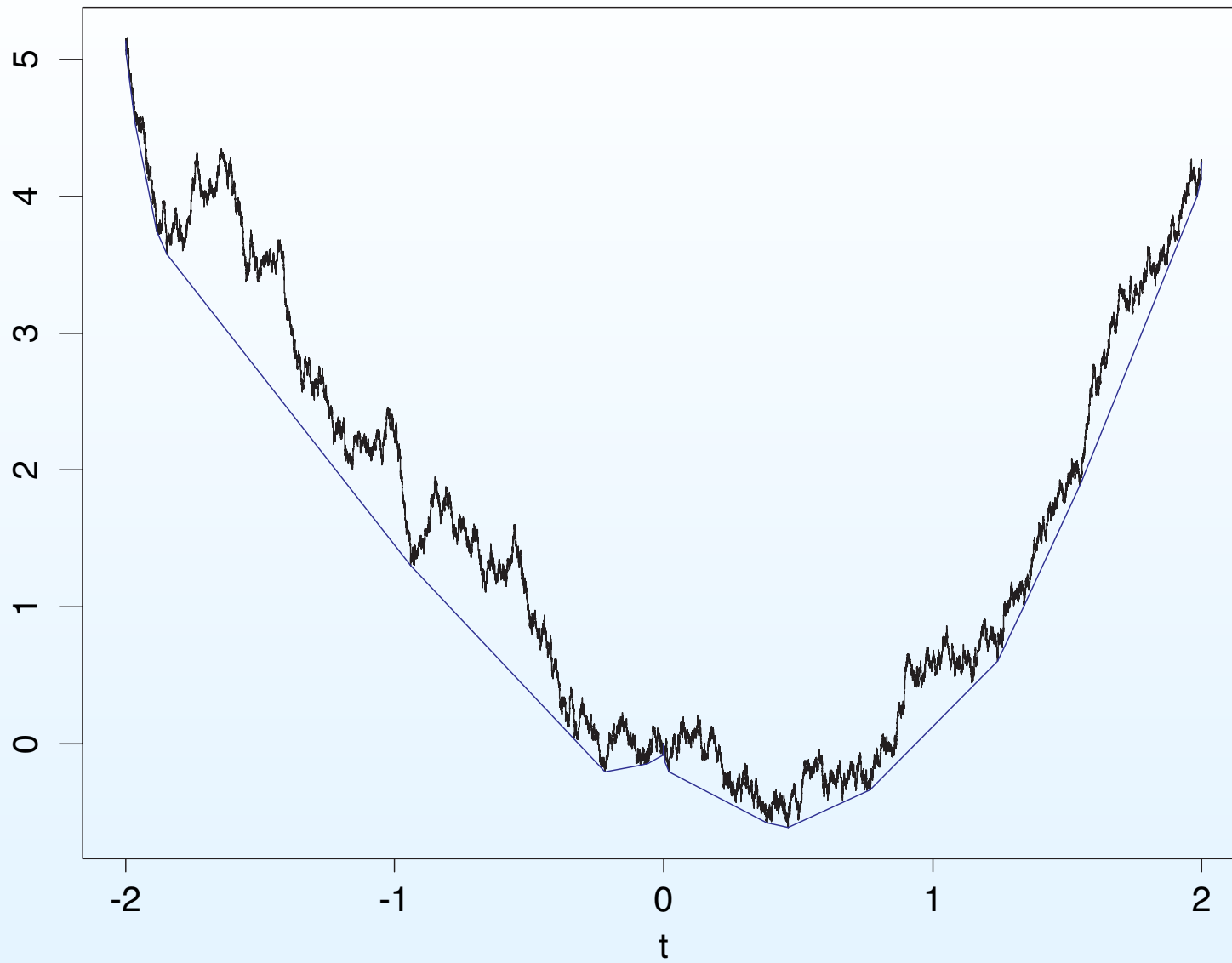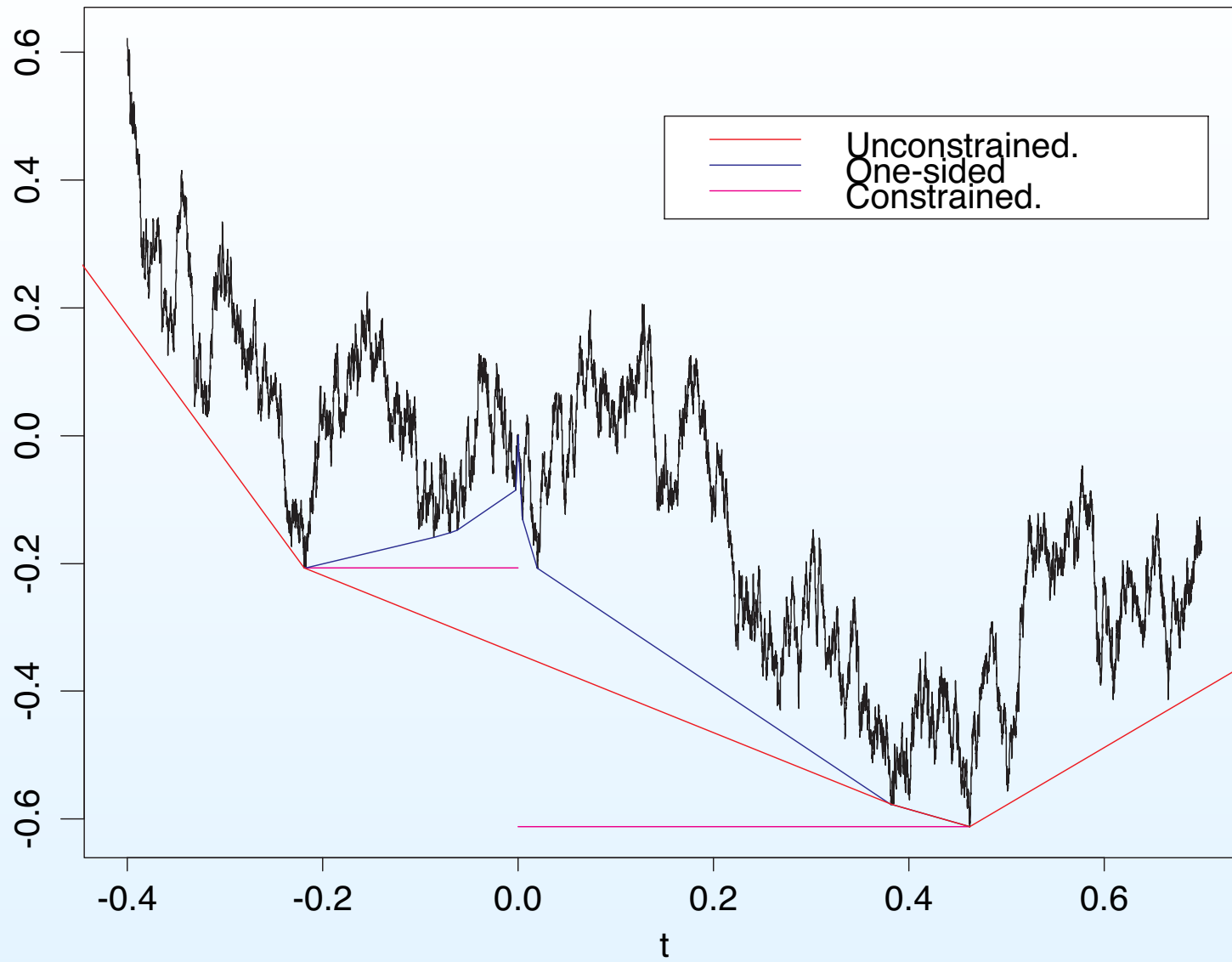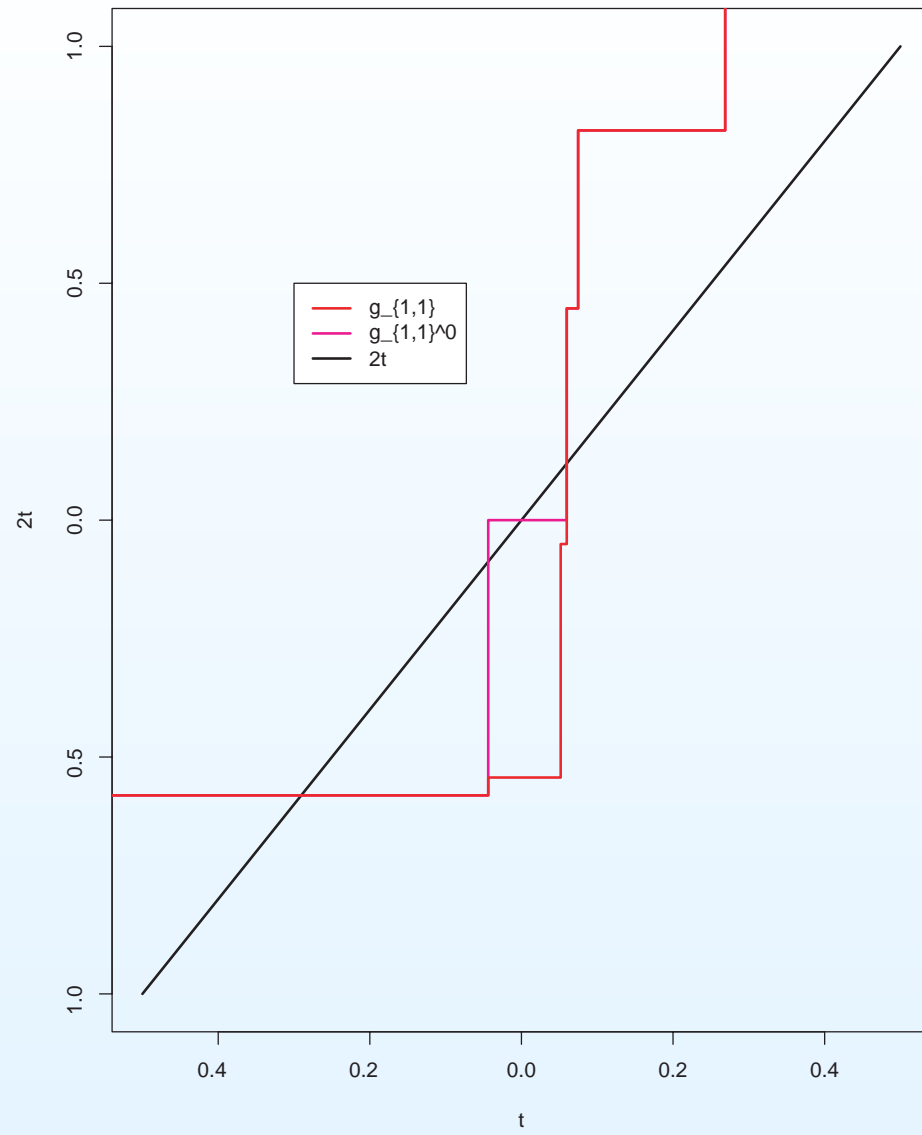
Likelihood ratio test statistic in the Gaussian problem?

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- Radon-Nikodym derivative (drifted process relative to zero drift):

$$\frac{dP_f}{dP_0} = \exp\left(\int_{-c}^{c} f\,dX - \frac{1}{2}\int_{-c}^{c} f^2(t)dt\right).$$

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- $\mathcal{F}(c, K) = \{\text{monotone functions } f : [-c, c] \to \mathbb{R}, \|f\|_c \leq K\}$
  $\mathcal{F}_0(c, K) = \{f \in \mathcal{F}(c, K) : f(0) = 0\}$

- Then

$$
\begin{aligned}
2\log \lambda_c &= 2\log\left(\frac{\sup_{f\in\mathcal{F}(c,K)} dP_f/dP_0}{\sup_{f\in\mathcal{F}_0(c,K)} dP_f/dP_0}\right) = 2\log\left(\frac{dP_{\hat f}/dP_0}{dP_{\hat f_0}/dP_0}\right) \\
&= 2\left\{\int_c^c \hat f_c\, dX - \frac{1}{2}\int_{-c}^c \hat f_c^2(t)\, dt \right. \\
&\qquad\qquad \left. - \int_c^c \hat f_{c,0}\, dX + \frac{1}{2}\int_{-c}^c \hat f_{c,0}^2(t)\, dt\right\} \\
&= 2\int_{-c}^c (\hat f_c - \hat f_{c,0})\, dX - \int_{-c}^c \{\hat f_c^2(t) - \hat f_{c,0}^2(t)\}\, dt\,.
\end{aligned}
$$

- Taking the limit as $c \to \infty$ with $K = K_c = 5c$, this yields

$$2 \log \lambda = 2 \int_D (\hat{f} - \hat{f}_0) dX - \int_D \{\hat{f}^2(t) - \hat{f}_0^2(t)\} dt$$

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- From the characterizations of $\hat{f}$ and $\hat{f}_0$:

$$\int_{\mathbb{R}} (X - \hat{F}) d\hat{f} = 0, \qquad \int_{\mathbb{R}} (X - \hat{F}_0) d\hat{f}_0 = 0 \,.$$

- Integration by parts:

$$
\begin{aligned}
\int_{\mathbb{R}} (\hat{f} - \hat{f}_0) dX &= \int_D (\hat{f} - \hat{f}_0) dX = -\int_D X d(\hat{f} - \hat{f}_0) \\
&= -\int_D \hat{F} d\hat{f} + \int_D \hat{F}_0 d\hat{f}_0 \\
&= \int_D \hat{f} d\hat{F} - \int_D \hat{f}_0 d\hat{F}_0 \\
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- Likelihood ratio statistic becomes:

$$2 \log \lambda = \int_D \{\hat{f}^2(t) - \hat{f}_0^2(t)\} dt.$$

6. Limit distribution, LR statistic under H

- Limit distributions for $\hat{F}_n$ and $\hat{F}_n^0$. Set

$$\mathbb{G}_n^{loc}(t, h) = n^{1/3}(\mathbb{G}_n(t + n^{-1/3}h) - \mathbb{G}_n(t))$$

$$\mathbb{V}_n^{loc}(t, h)$$
$$= n^{1/3}\left\{n^{1/3}(\mathbb{V}_n(t + n^{-1/3}h) - \mathbb{V}_n(t)) - \mathbb{G}_n^{loc}(t, h)F(t)\right\}.$$

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- **Theorem 1.** If $g(t_0) = G'(t_0)$ and $f(t_0) = F'(t_0)$ exist, then:
  A. $\mathbb{G}_n^{loc}(t_0, h) \to_p g(t_0)h$.
  B. $\mathbb{V}_n^{loc}(t_0, h) \Rightarrow aW(h) + bh^2$ where
  $a = \sqrt{F(t_0)(1 - F(t_0))g(t_0)}$, $b = f(t_0)g(t_0)/2$, and W is a
  two-sided Brownian motion starting from 0.

- Now define

$$\mathbb{Z}_n(h) = n^{1/3}(\hat{F}_n(t_0 + hn^{-1/3}) - F(t_0)),$$

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- **Theorem 2.** If the hypotheses of Theorem 1 hold with $f(t_0) > 0$, $g(t_0) > 0$, and $F(t_0) = \theta_0$, then

$$(\mathbb{Z}_n(h), \mathbb{Z}_n^0(h)) \Rightarrow (\mathbb{S}_{a,b}(h), \mathbb{S}_{a,b}^0(h))/g(t_0)$$

where $\mathbb{S}_{a,b}$ and $\mathbb{S}_{a,b}^0$ are the constrained and unconstrained slope processes corresponding to $X_{a,b}(h) = aW(h) + bh^2$.

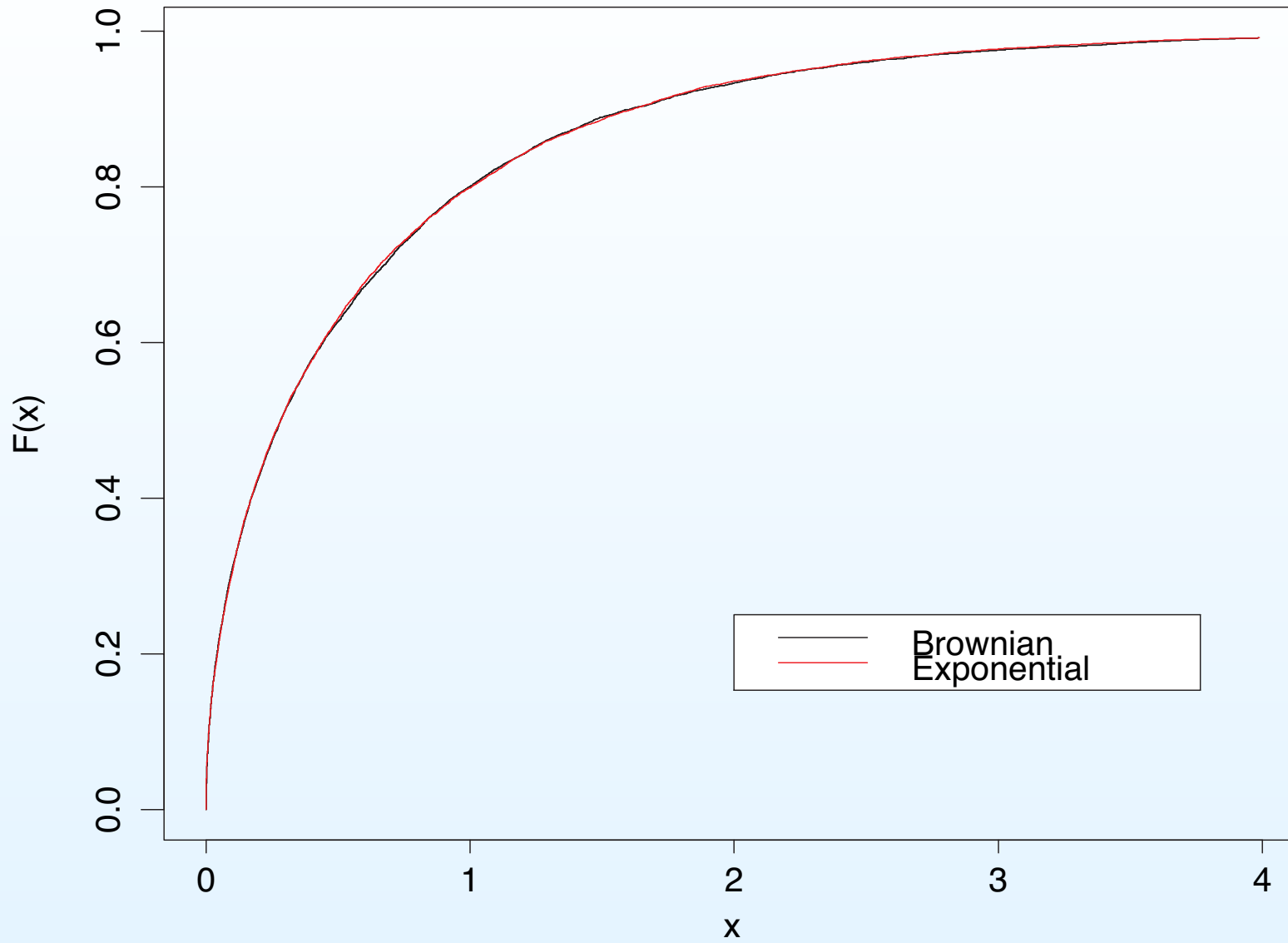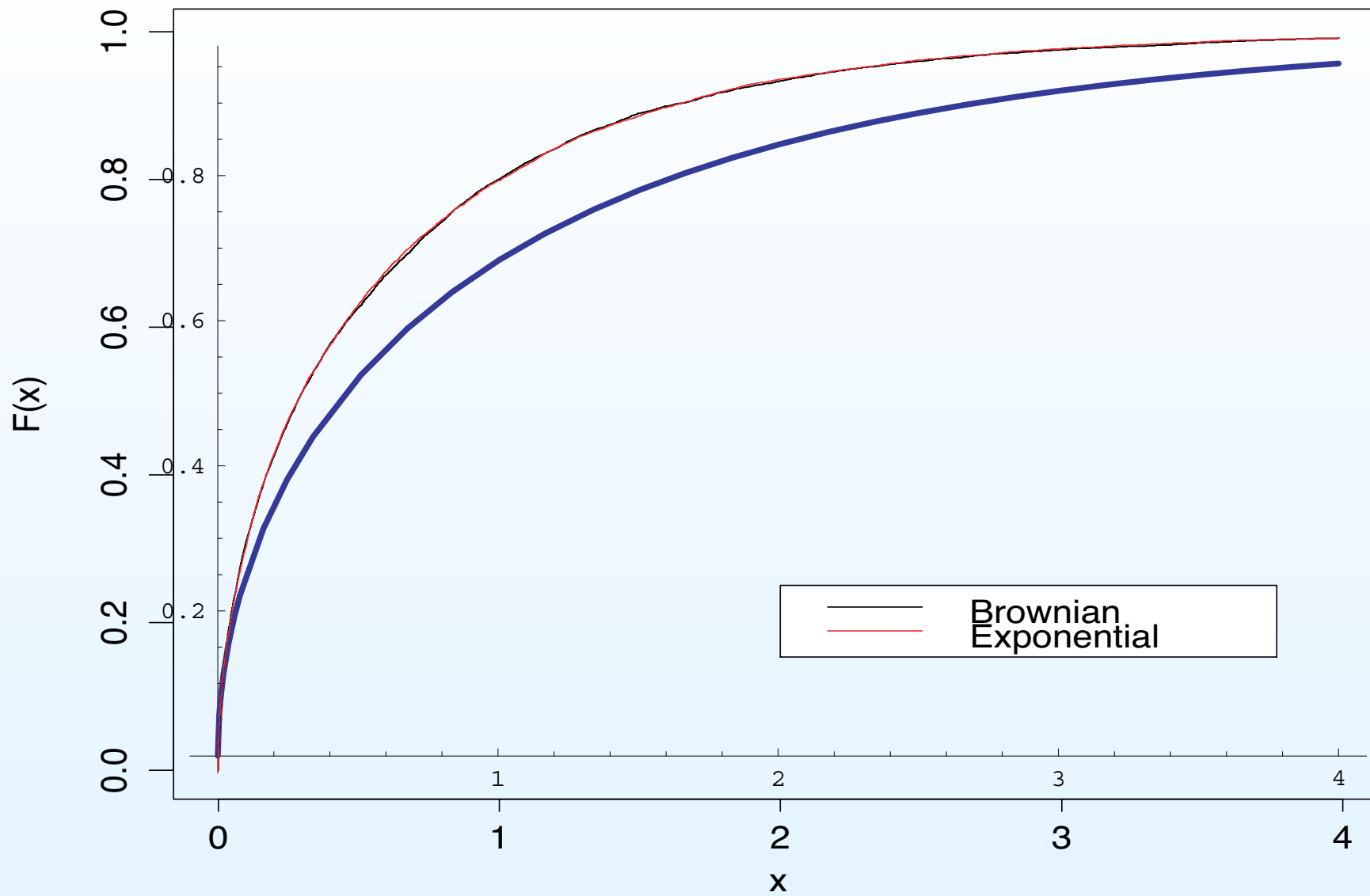- Limit distribution for $2 \log \lambda_n$

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- **Theorem 3.** (Banerjee and Wellner, 2001). Suppose that F and G have densities f and g which are strictly positive and continuous in a neighborhood in a neighborhood of t_0. Suppose that $F(t_0) = \theta_0$. Then

$$2 \log \lambda_n \quad \to_d \quad \frac{1}{g(t_0)a^2} \int \left((\mathbb{S}_{a,b}(z))^2 - (\mathbb{S}^0_{a,b}(z))^2 \right) dz$$

$$\stackrel{d}{=} \quad \int \{ (\mathbb{S}(z))^2 - (\mathbb{S}^0(z))^2 \} dz \equiv \mathbb{D} \,,$$

and the distribution of $\mathbb{D}$ is universal (free of parameters).

7. Confidence intervals for $F(t_0)$

- Wald-type intervals:

$$
\begin{aligned}
\mathbb{Z}_n(0) &= n^{1/3}(\hat{F}_n(t_0) - F(t_0)) \to_d \mathbb{S}_{a,b}(0)/g(t_0) \\
&\stackrel{d}{=} \left\{ \frac{F(t_0)(1 - F(t_0))f(t_0)}{2g(t_0)} \right\}^{1/3} \mathbb{S}(0) \\
&\equiv C(F, f, g)\, \mathbb{S}(0)
\end{aligned}
$$

where $\mathbb{S}(0) \stackrel{d}{=} 2\,\mathbb{Z} \equiv 2\mathrm{argmin}(W(h) + h^2), \quad \mathbb{S}(0) \equiv \mathbb{S}_{1,1}(0).$

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- Wald - interval:

$$
\hat{F}_n(t_0) \pm n^{-1/3}C(\hat{F}_n, \hat{f}_n, \hat{g}_n)\, t_\alpha
$$

where $\hat{f}_n$ and $\hat{g}_n$ are estimates of f and g (at t_0), and $t_{\alpha/2}$ satisfies

$$
P(2\mathbb{Z} > t_{\alpha/2}) = \alpha/2\,.
$$

- Problem: this involves smoothing to get estimators $\hat{f}_n$ and $\hat{g}_n$!

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- Tradeoff: need to compute constrained estimator(s) $\hat{F}_n^0$ of F and $\lambda_n(\theta)$ for many different values of the constraint θ.

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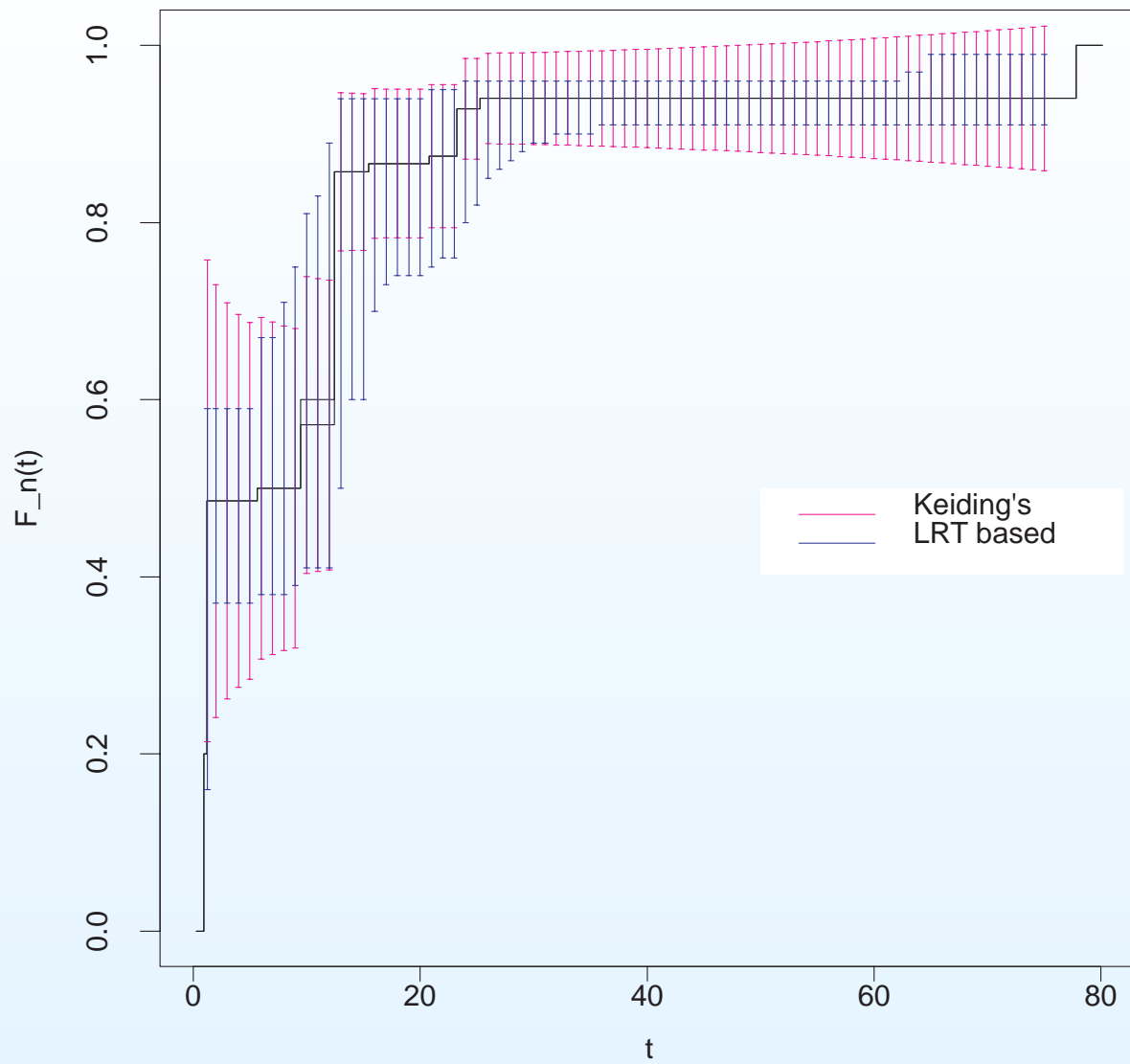 - no vaccination of males

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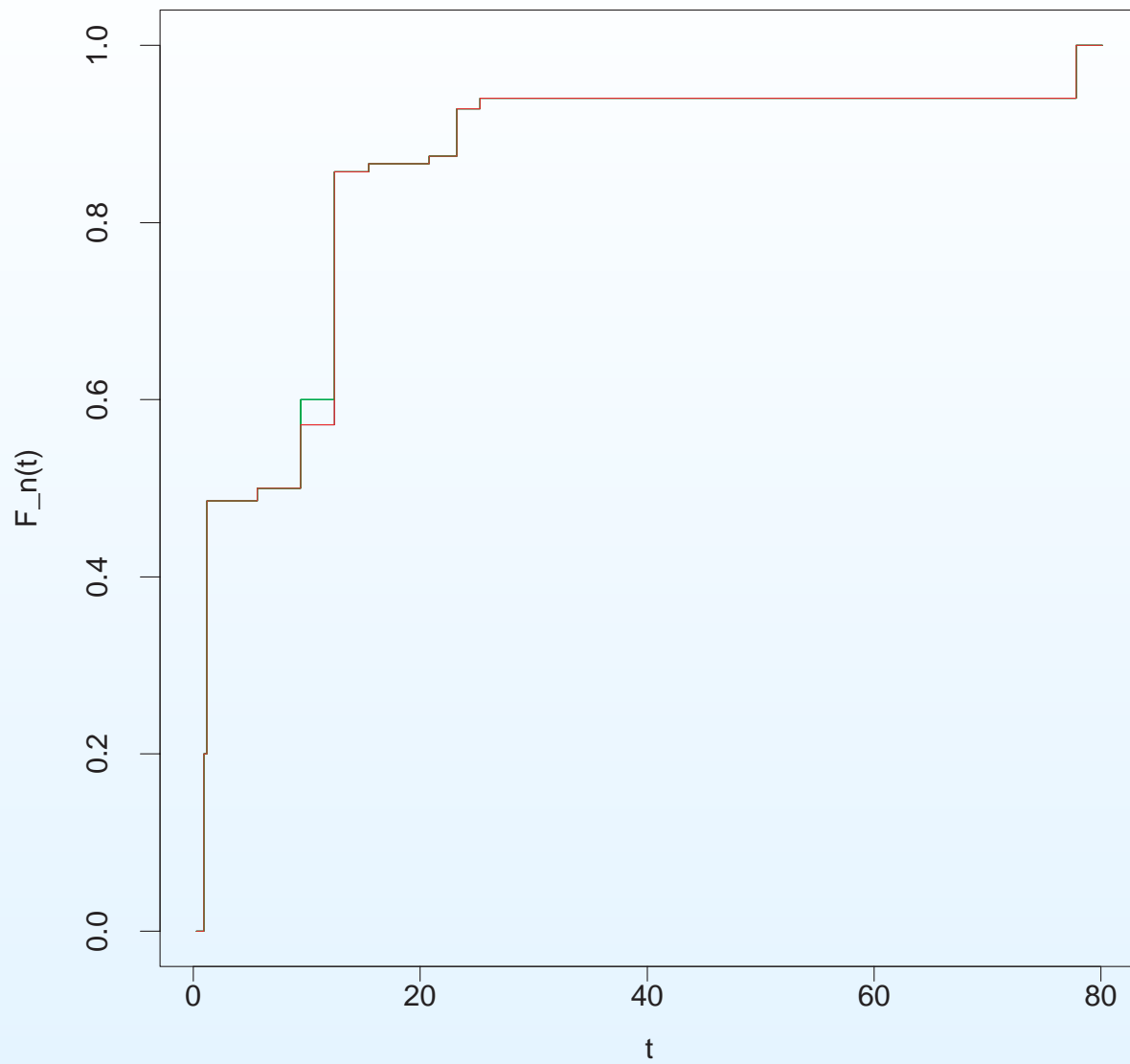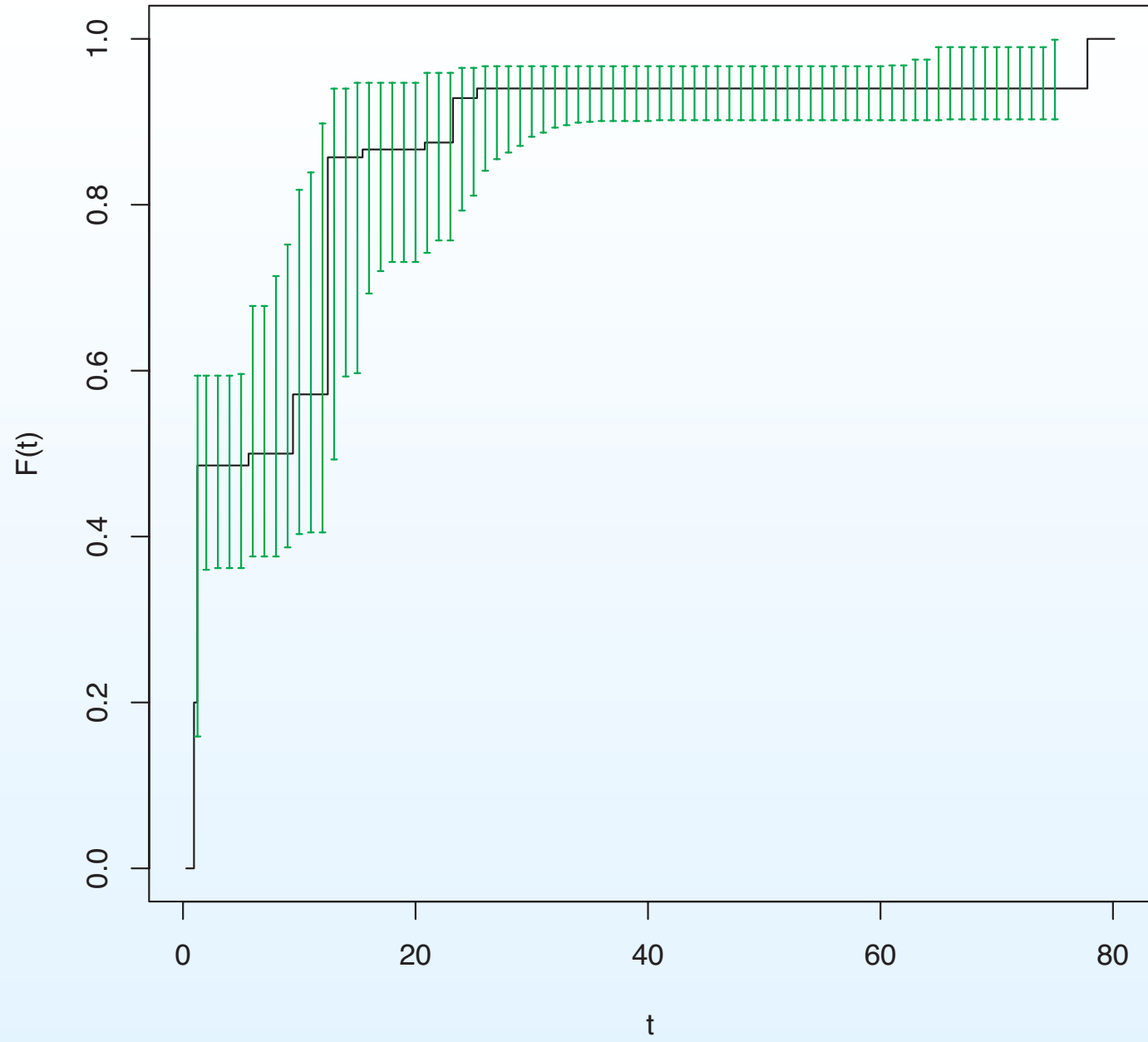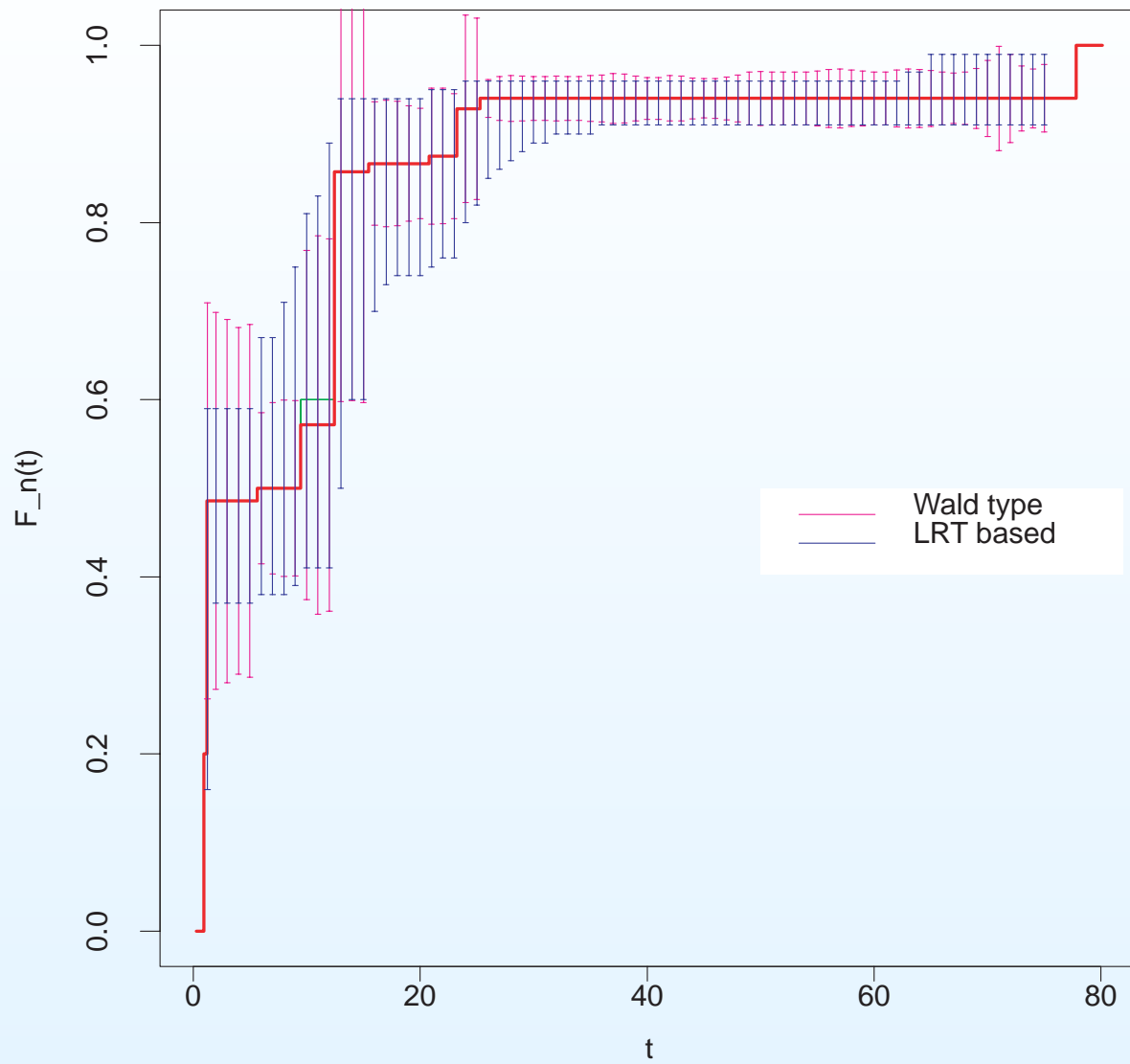
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- Confidence bands for the whole monotone function F? (No LR bands yet. Bands based on multi-scale methods: Duembgen (1999))

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- Confidence intervals (and bands?) for estimating a concave distribution function F?