Estimation and Testing with Shape Constraints

Ten Open Problems

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- Talk at University of Washington,
  Department of Statistics,
  Research Day, September 28, 2009
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Outline

- Introduction: shape constraints, nonparametric estimation and testing

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- Problems 1-4 from Gothenburg meeting: rearrangements versus maximum likelihood
- Problems 5-6 from Gothenburg meeting: how big is the Grenander estimator at zero
- Four more problems involving shape constraints ... very briefly

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- Confidence sets?
  - Assuming shape constraint?
  - Testing to see if a shape constraint is true?

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2. Problems 1-4 from Gothenburg meeting

Monotone rearrangements estimator

versus

maximum likelihood?

Continuous setting

$X_1, \ldots, X_n$ i.i.d. with density f on $[0, \infty)$ where $f \searrow 0$.

The Maximum Likelihood Estimator is

$$\begin{aligned} \widehat{f}_n &= \text{argmax}_{f \in \mathcal{M}_1} \left\{ \sum_{i=1}^{n} \log f(X_i) \right\} = \text{the MLE} \\ &= \text{Grenander estimator of } f. \end{aligned}$$

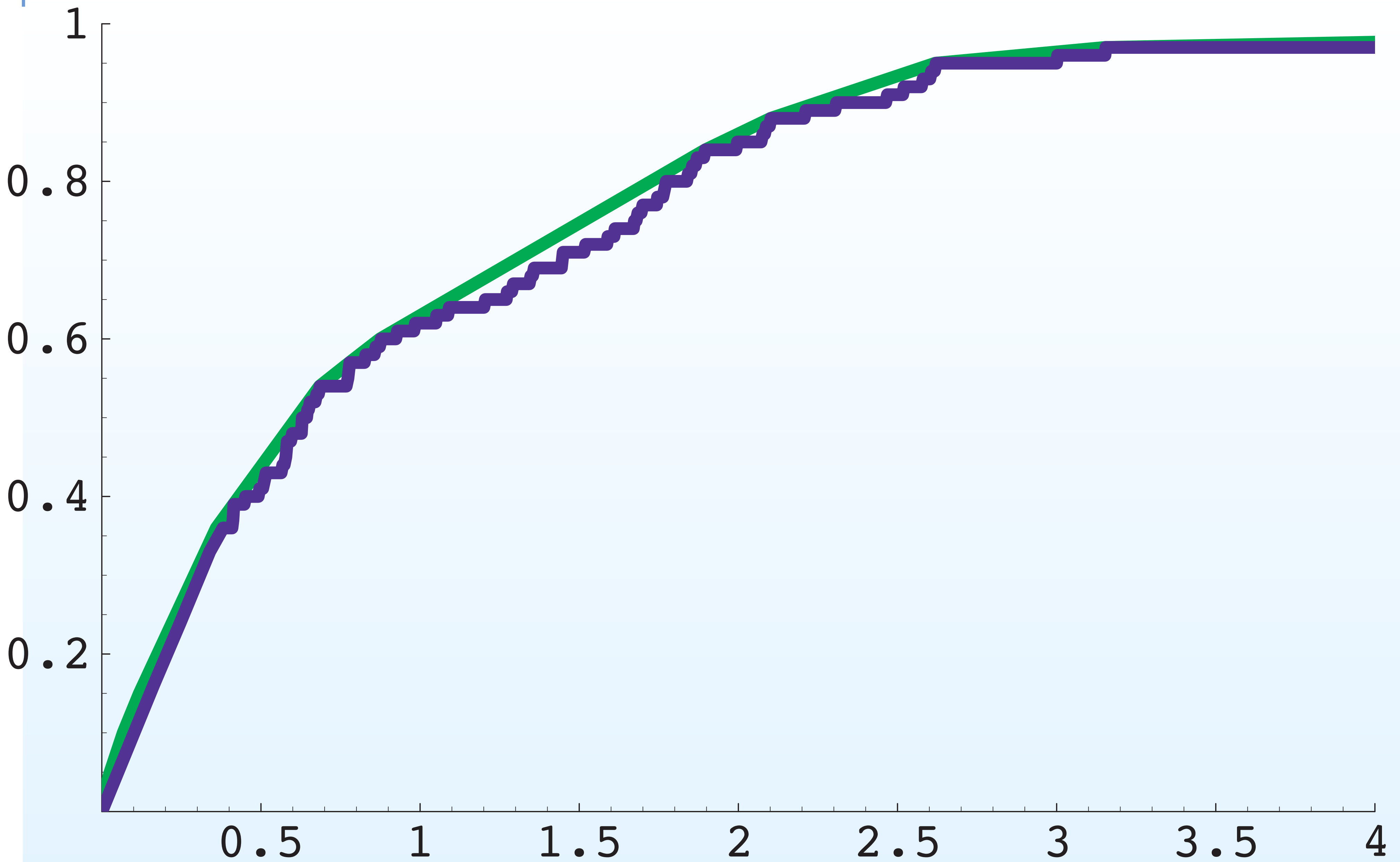
From Grenander (1956), the MLE is characterized by the **Fenchel conditions**:

$$\mathbb{F}_n(x) \le \widehat{F}_n(x) \equiv \int_0^x \widehat{f}_n(t)dt \qquad \text{for all } x \in [0,\infty), \quad \text{and}$$

$$\mathbb{F}_n(x) = \widehat{F}_n(x) \quad \text{if and only if } \widehat{f}_n(x-) > \widehat{f}_n(x+).$$

The geometric interpretation of these two conditions is

$$\begin{aligned} \widehat{f}_n(x) \quad &= \quad \left\{ \begin{array}{l} \text{the left-derivative of the slope at } x \text{ of the} \\ \text{least concave majorant } \widehat{F}_n \text{ of } \mathbb{F}_n \end{array} \right\} \\ &\equiv \quad \partial \mathcal{I}_1(\mathbb{F}_n) \end{aligned}$$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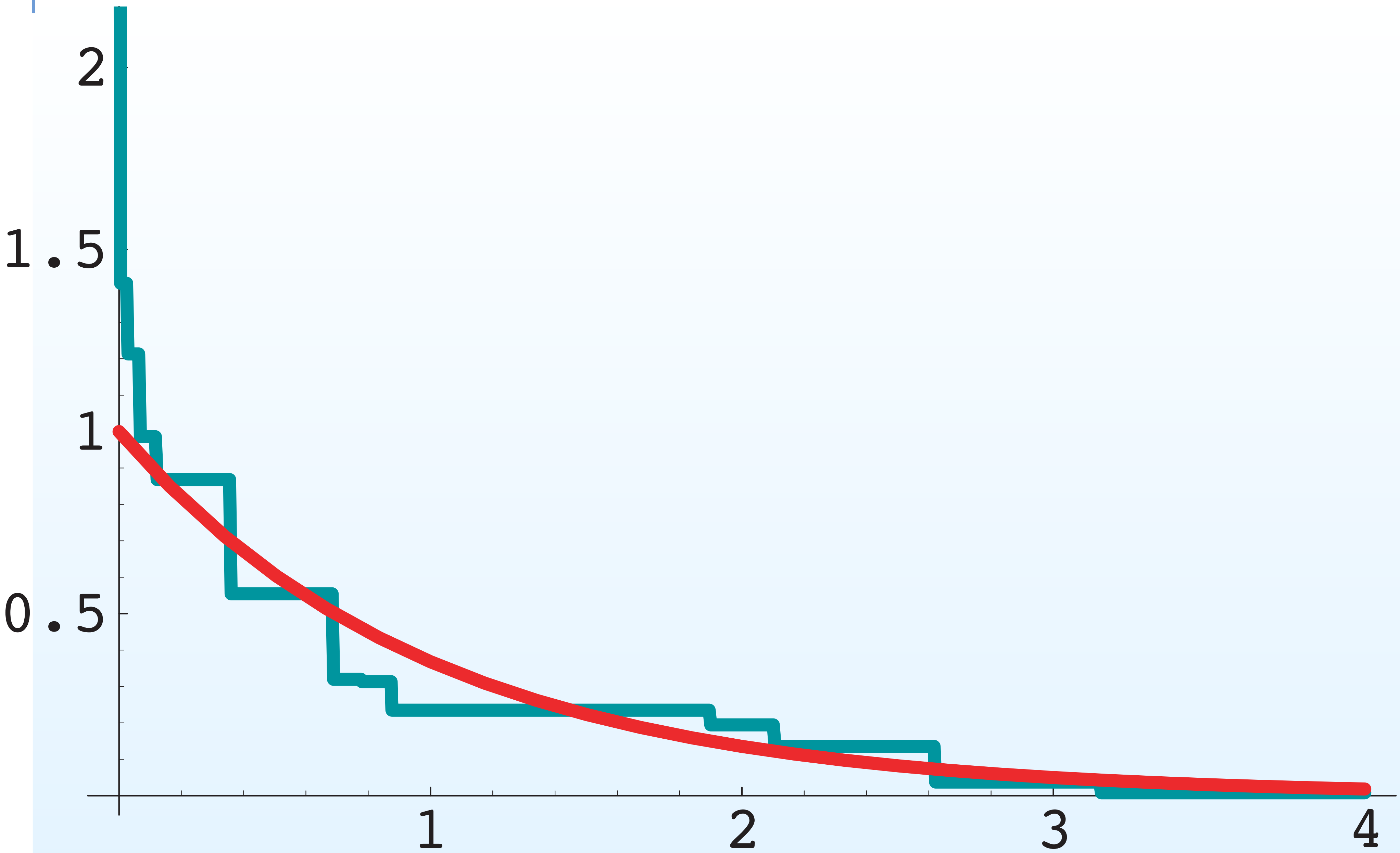
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Monotone rearrangement estimator

- Monotone rearrangement, continuous case: $\widehat{f}^{rearr} \equiv R(\tilde{f}_n)$ where

$$Z_f(s) = \lambda\{x : \ f(x) \geq s\}, \qquad R(f)(x) = Z_f^{-1}(x).$$

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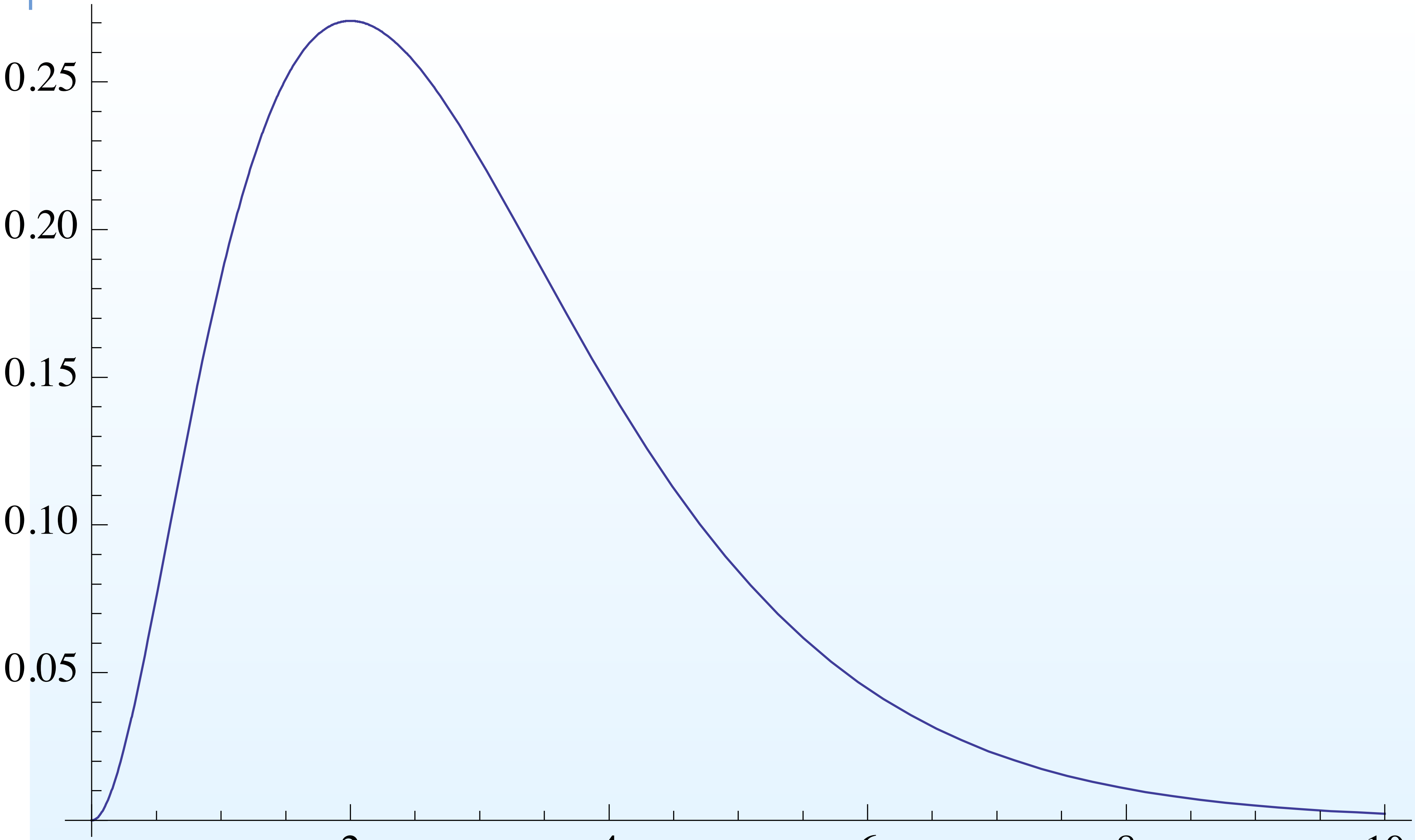
- Monotone rearrangement, discrete case: $\widehat{p}_n^{rearr} \equiv R(\widehat{p}_n)$ where

$$Z_p(s) = \#\{i \in \mathbb{N}^+ : \ p(i) \geq s\}, \qquad R(p)(i) = Z_p^{-1}(i).$$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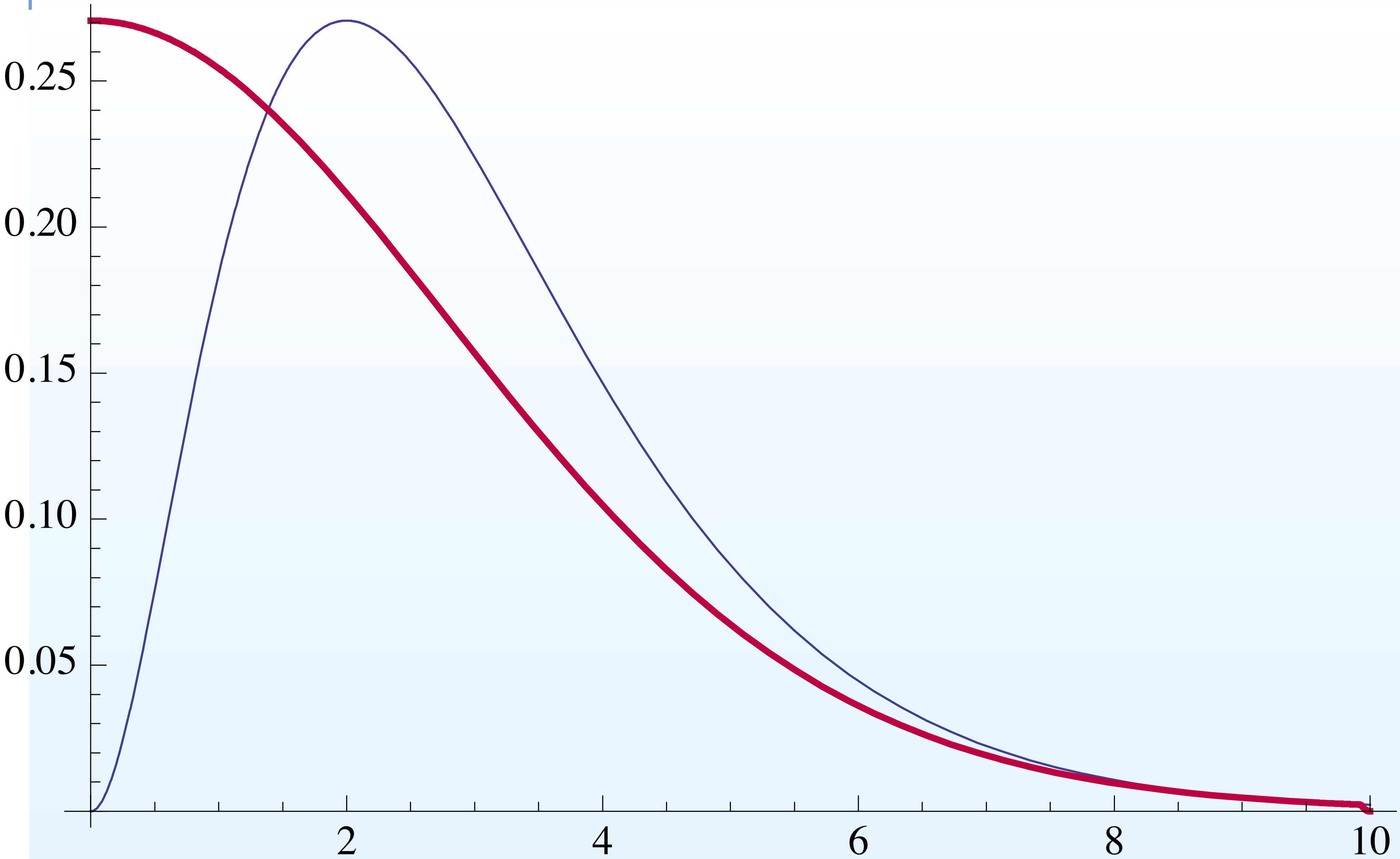
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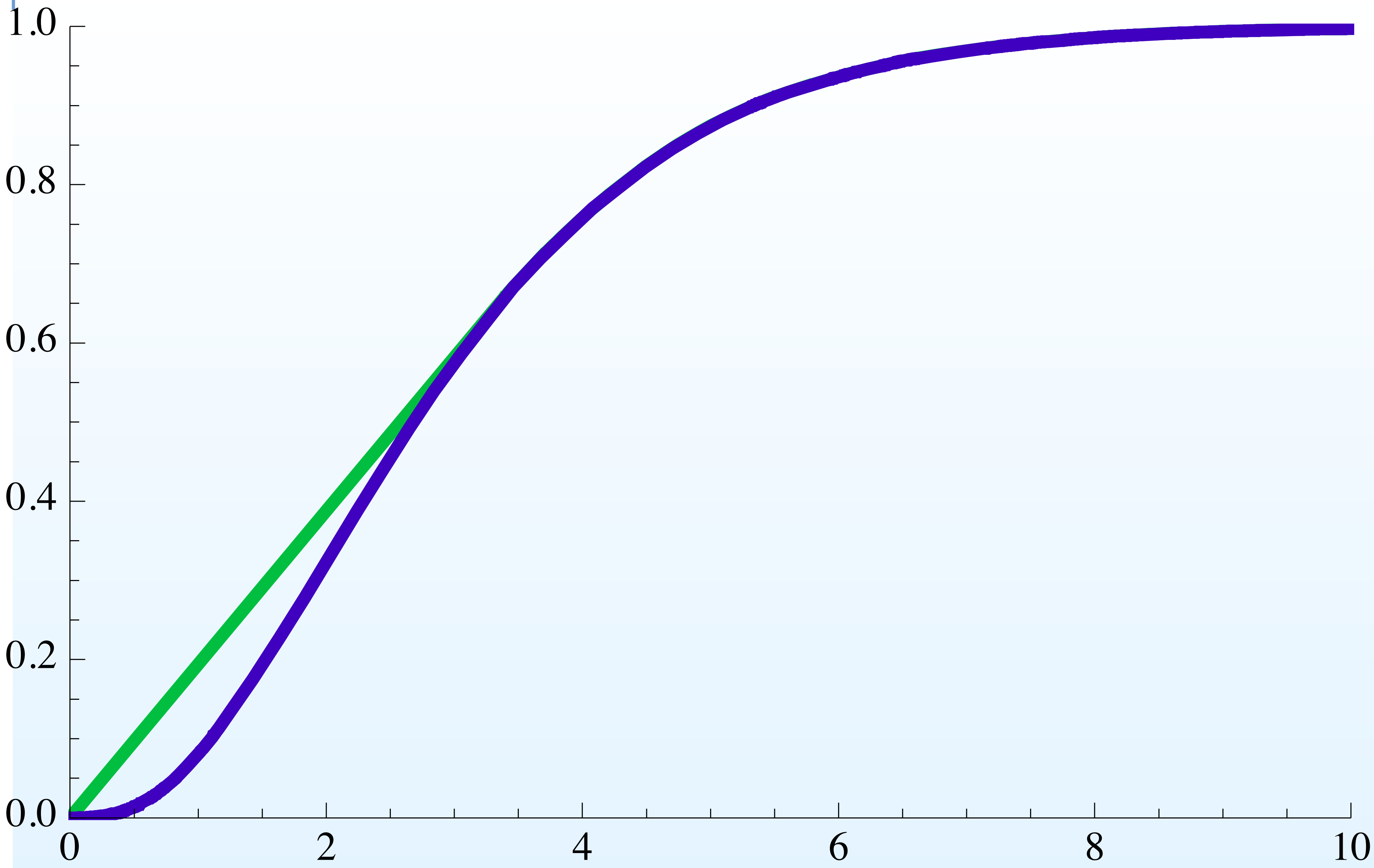
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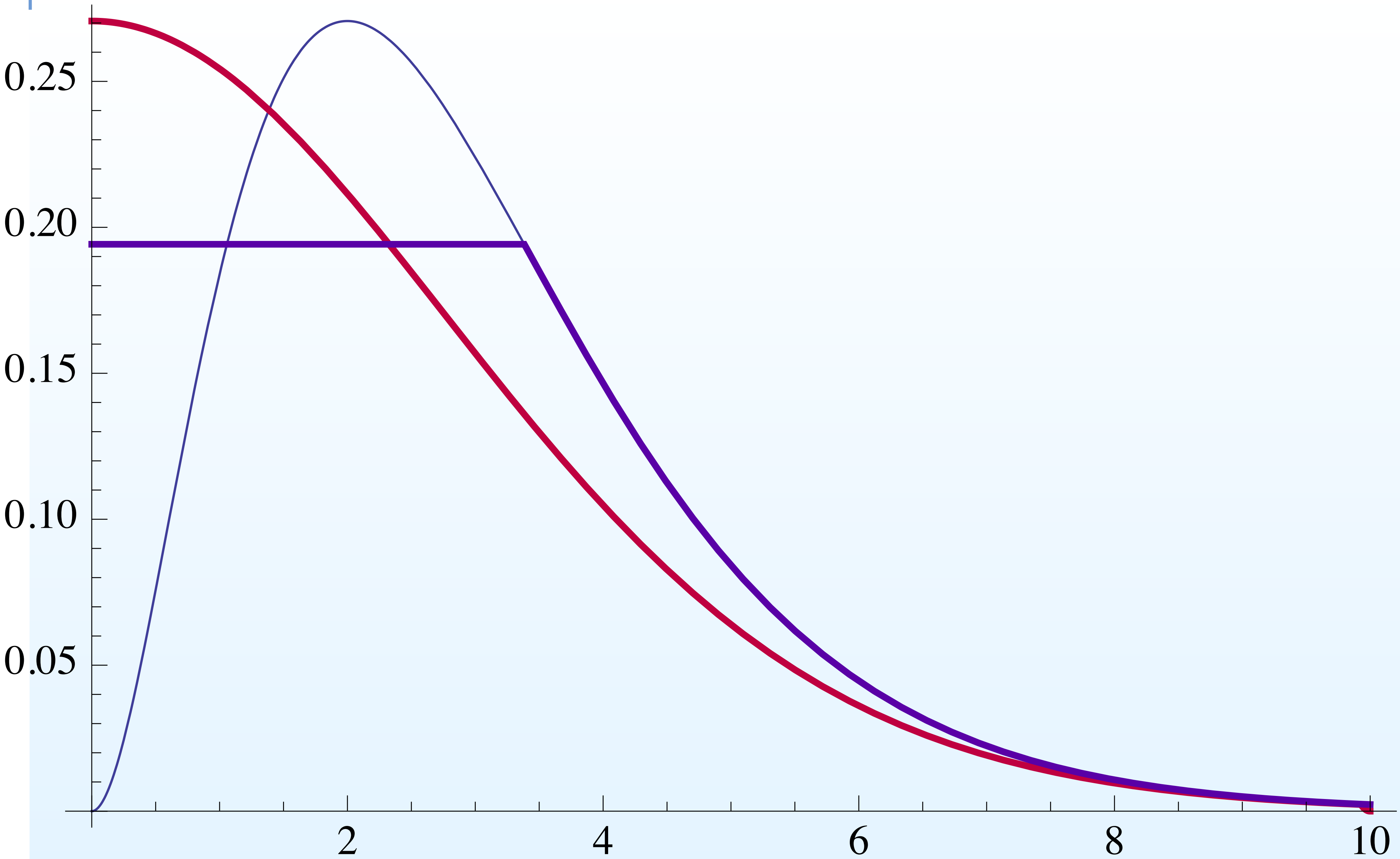
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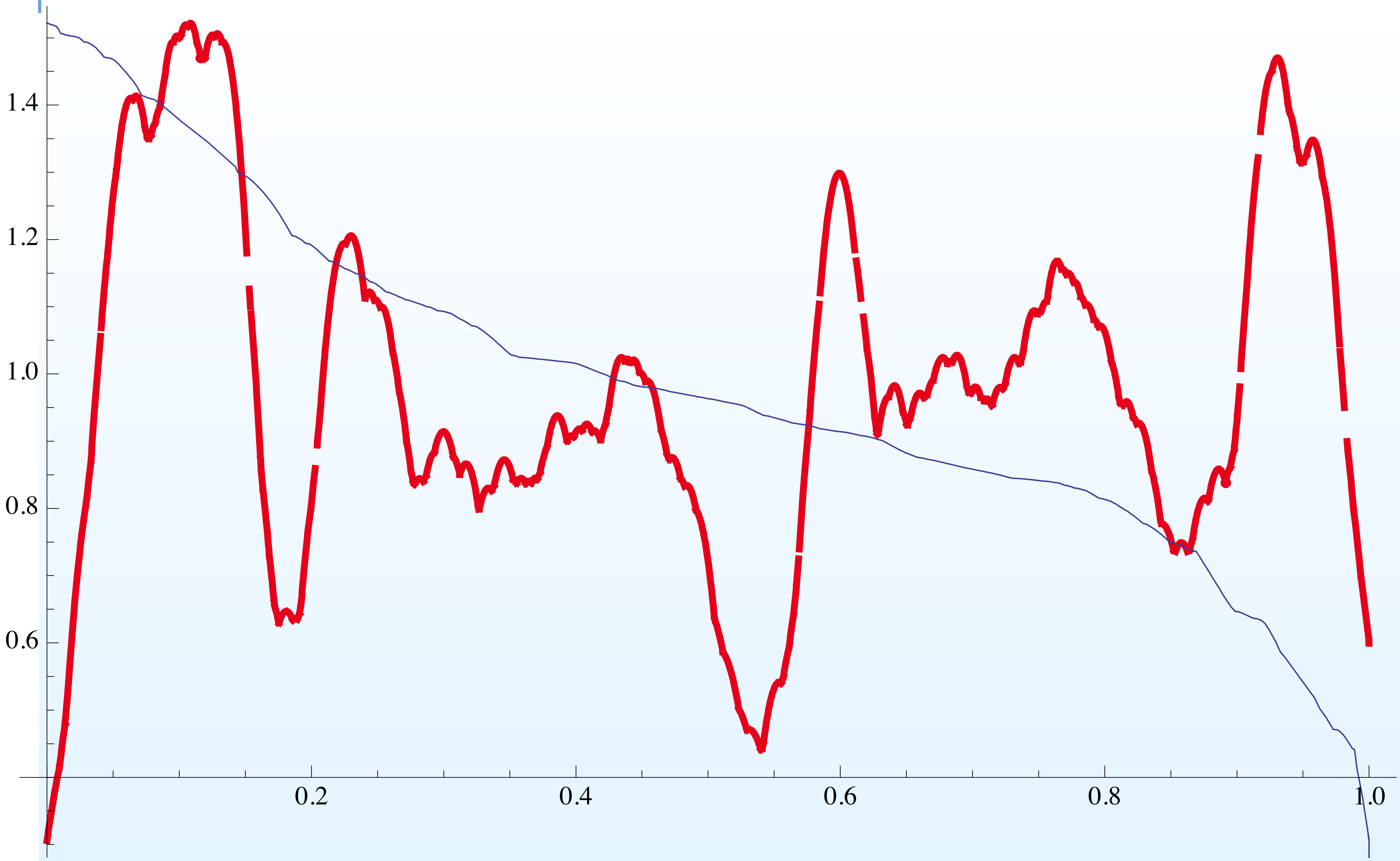
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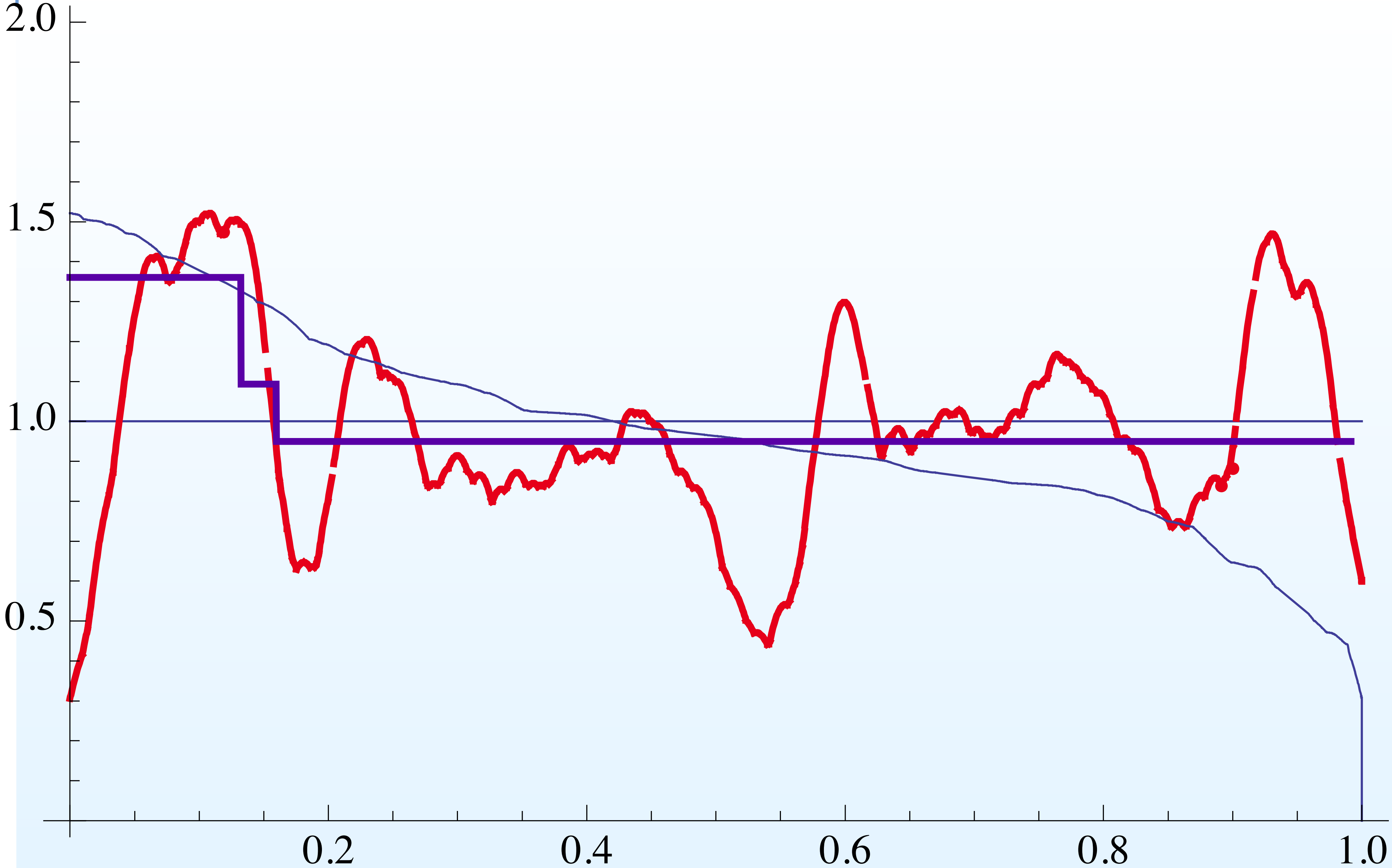
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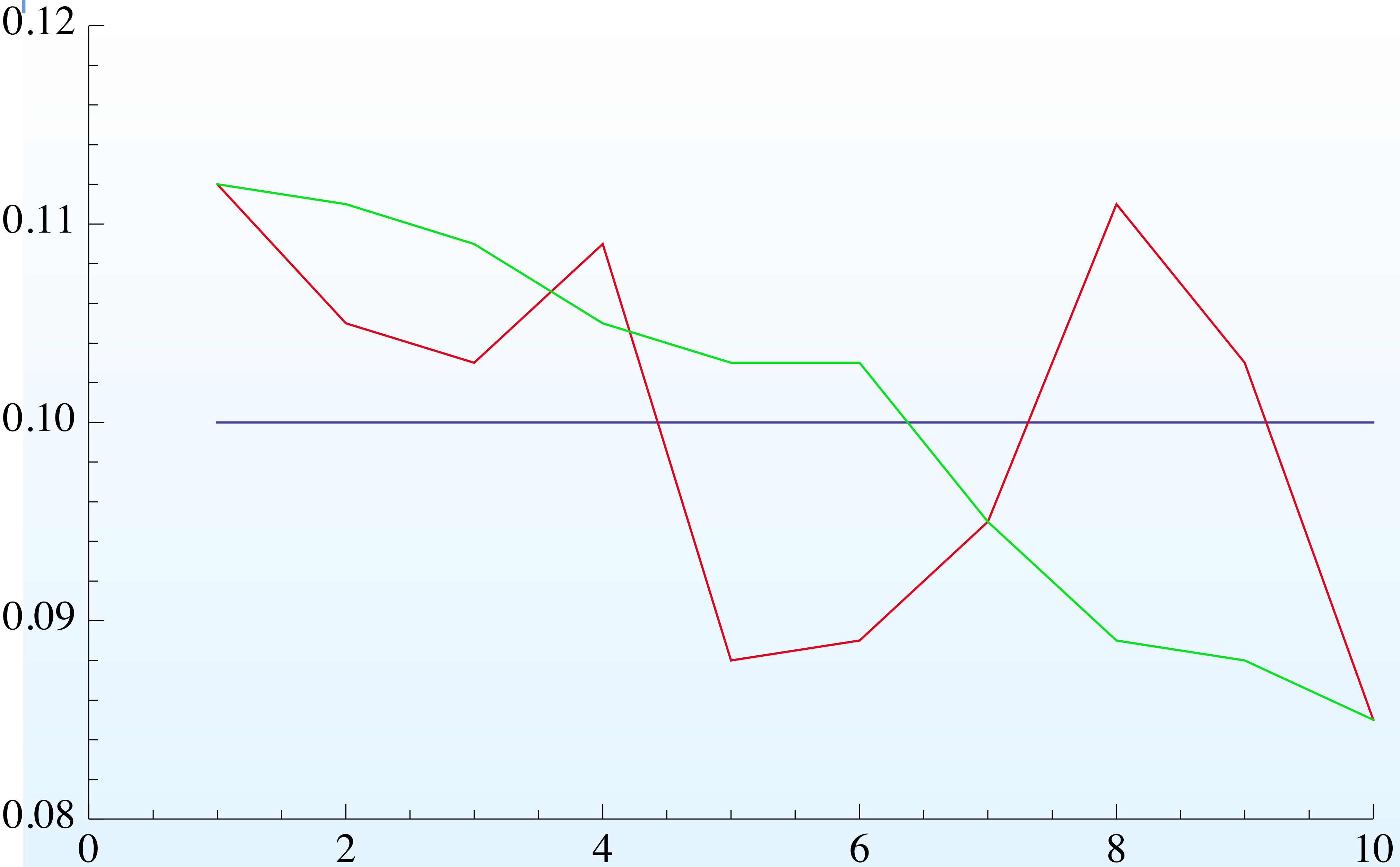
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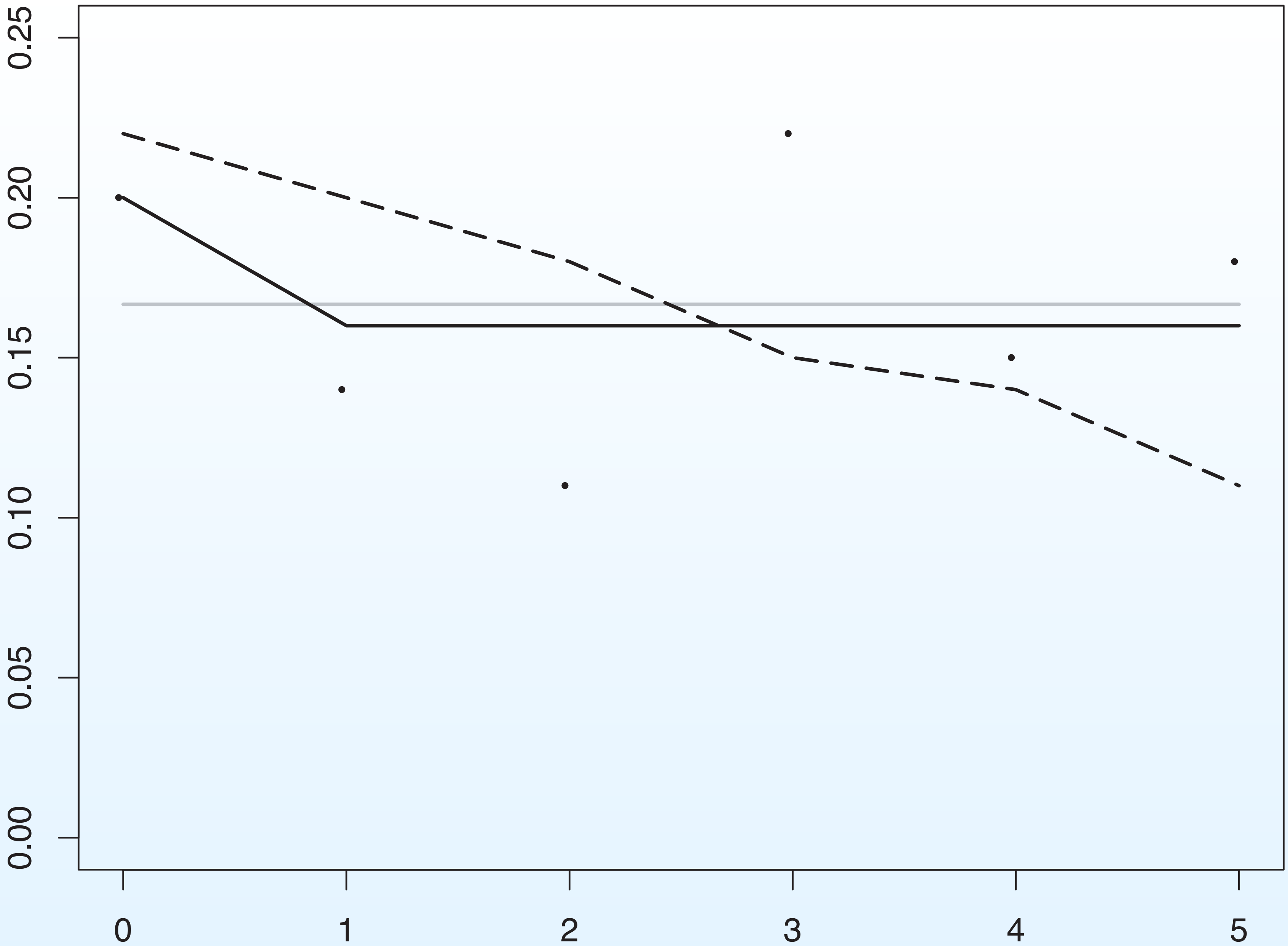
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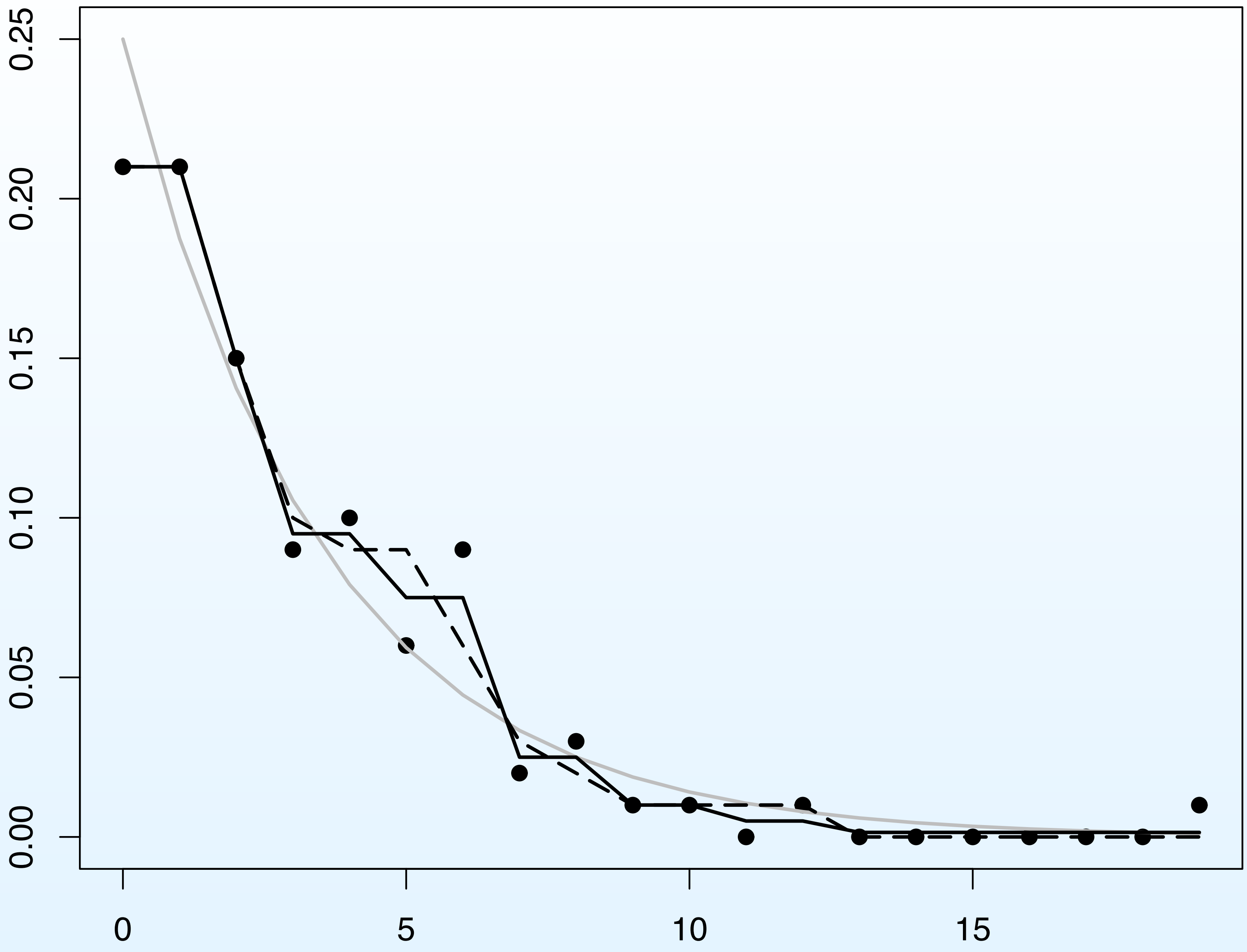
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  - $Y_x^R = \text{rear}(Y^{(r,s)})_x$ and $Y_x^G = \text{Gren}(Y^{(r,s)})_x$.

Theorem. (Jankowski and Wellner, 2009)

$$\left(\mathbb{Y}_n, \mathbb{Y}_n^R, \mathbb{Y}_n^G\right) \Rightarrow \left(\mathbb{Y}, \mathbb{Y}^R, \mathbb{Y}^G\right)$$

in $\ell_2 \times \ell_2 \times \ell_2$ where $\ell_2 \equiv \{\{y_x\} : \sum_{x\geq 0} y_x^2 < \infty\}$.

Corollary 1. If $p_{x+1} < p_x$ for all $x \geq 0$, then

$$\left(\mathbb{Y}_n, \mathbb{Y}_n^R, \mathbb{Y}_n^G\right) \Rightarrow (\mathbb{Y}, \mathbb{Y}, \mathbb{Y})$$

in $\ell_2 \times \ell_2 \times \ell_2$. In this case the three estimators are asymptotically equivalent.

Corollary 2. If $p_x = (y+1)^{-1} 1_{\{0,\ldots,y\}}(x)$, then

$$\left(\mathbb{Y}_n, \mathbb{Y}_n^R, \mathbb{Y}_n^G\right) \Rightarrow (\mathbb{Y}, \text{rear}(\mathbb{Y}), \text{Gren}(\mathbb{Y}))\ ,$$

and ...

$$E\|Y_n\|_2^2 = nE\left\{\sum_{x=0}^{y}(\widehat{p}_{n,x} - p_x)^2\right\} \to E\|Y_x\|_2^2 = 1 - \frac{1}{y+1},$$

$$E\|Y_n^R\|_2^2 = nE\left\{\sum_{x=0}^{y}(\widehat{p}_{n,x}^{\text{rear}} - p_x)^2\right\} \to E\|\text{rear}(Y)\|_2^2 = 1 - \frac{1}{y+1},$$

$$E\|Y_n^G\|_2^2 = nE\left\{\sum_{x=0}^{y}(\widehat{p}_{n,x}^{\text{Gren}} - p_x)^2\right\} \to E\|\text{Gren}(Y)\|_2^2$$

$$= \frac{1}{y+1}\sum_{x=1}^{y+1}\frac{1}{x} \sim \frac{\log(y+1)}{y}.$$

Hence $\widehat{p}_n^{\text{rear}}$ is (asymptotically) inadmissible!

What is the problem?

Proposition. $\{p_x\}$ is monotone decreasing if and only if it is a mixture of uniform mass functions $(y+1)^{-1}1_{\{0,\ldots,y\}}(x)$:

$$p_x = \sum_{y=0}^{\infty} (y+1)^{-1} 1_{\{0,\ldots,y\}}(x) q_y$$

for some probability mass function $\{q_y\}$. The inversion formula is given by

$$q_y = -(y+1)\Delta p_y \equiv -(y+1)(p_{y+1} - p_y).$$

Thus we can define two estimators of q:

$$\widehat{q}_{n,y}^{\text{rear}} \equiv -(y+1)(\widehat{p}_{n,y+1}^{\text{rear}} - \widehat{p}_{n,y}^{\text{rear}}),$$

$$\widehat{q}_{n,y}^{\text{Gren}} \equiv -(y+1)(\widehat{p}_{n,y+1}^{\text{Gren}} - \widehat{p}_{n,y}^{\text{Gren}}).$$

Define processes Z_n, Z_n^R, Z_n^G by

$$Z_{n,x} \equiv \sqrt{n}(\widehat{q}_{n,x} - q_x),$$
$$Z_{n,x}^R \equiv \sqrt{n}(\widehat{q}_{n,x}^{\text{rearr}} - q_x),$$
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We know that if $\sum_{x \geq 0} x^2 p_x = E(X^2) < \infty$, then

$$\mathbb{Z}_n \Rightarrow Z \equiv \{-(x+1)\Delta Y_x\} \qquad \text{in } \ell_2.$$

- Problem 1. If $\sum_{x \geq 0} x^2 p_x < \infty$, does it hold that

$$\mathbb{Z}_n^R \Rightarrow Z^R \equiv \{-(x+1)\Delta Y_x^R\} \qquad \text{in } \ell_2;$$
$$\mathbb{Z}_n^G \Rightarrow Z^G \equiv \{-(x+1)\Delta Y_x^G\} \qquad \text{in } \ell_2?$$

- Problem 2. If $\{p_x\}$ is strictly decreasing, for what sequences $a_n,\ b_n$ (with $a_n/\sqrt{n} \to \infty$, $b_n/\sqrt{n} \to \infty$) does it hold that

$$a_n \|\widehat{p}_n^{\text{rearr}} - \widehat{p}_n\|_2 \to_{p,a.s.} 0,$$

$$b_n \|\widehat{p}_n^{\text{Gren}} - \widehat{p}_n\|_2 \to_{p,a.s.} 0?$$

- **Problem 2.** If $\{p_x\}$ is strictly decreasing, for what sequences $a_n,\ b_n$ (with $a_n/\sqrt{n} \to \infty$, $b_n/\sqrt{n} \to \infty$) does it hold that

$$a_n \|\widehat{p}_n^{\text{rearr}} - \widehat{p}_n\|_2 \to_{p,a.s.} 0,$$
$$b_n \|\widehat{p}_n^{\text{Gren}} - \widehat{p}_n\|_2 \to_{p,a.s.} 0?$$

- **Problem 3.** When (or in exactly what senses) does $\widehat{q}_n^{\text{Gren}}$ beat $\widehat{q}_n^{\text{rearr}}$?

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- Problem 3. When (or in exactly what senses) does $\widehat{q}_n^{\text{Gren}}$ beat $\widehat{q}_n^{\text{rearr}}$?
- Problem 4. What are the analogues of these results when $\{p_s\}$ is $k-$monotone; i.e. when

$$p_x = \sum_{y=0}^{\infty} \frac{(y-x)_+^{k-1}}{\sum_{x'=0}^{y}(y-x')^{k-1}} q_y$$

for some probability mass function $\{q_y\}$?

3. Problems 5-6 from Gothenburg meeting

Known from Woodroofe and Sun (1993): in the continuous case, the Grenander estimator $\widehat{f}_n$ of a decreasing density is not consistent at zero:

$$\widehat{f}_n(0) \rightarrow_d f_0(0) Y_1 \equiv f_0(0) \sup_{t>0} \frac{\mathbb{N}(t)}{t} \stackrel{d}{=} f_0(0) U^{-1}$$

where $U \sim$ Uniform$(0,1)$.

Question: If f_0 is not bounded at zero, what is the behavior of $\widehat{f}_n(0)$?

Theorem. (Balabdaoui, Jankowski, Pavlides, Seregin and W, 2009): Suppose that F_0 is regularly varying at 0 with exponent $\gamma \in (0, 1]$. Then with a_n satisfying $nF_0(a_n) \to 1$ as $n \to \infty$,

$$n a_n \widehat{f}_n(t a_n) \Rightarrow \widehat{h}_\gamma(t) \qquad \text{in } D[0, \infty)$$

where $\widehat{h}_\gamma$ is the right derivative of the least concave majorant of $\mathbb{N}(t^\gamma)$ and $\mathbb{N}$ is a standard Poisson process.

Now suppose that f_0 is $k-$monotone on $(0, \infty)$ with $k \geq 2$; i.e.

$$f(x) = \int_0^\infty \frac{1}{y^k}(y - x)_+^{k-1} dG(y)$$

for some probability distribution G.

Problem 5. If f_0 is $k-$monotone, what is the behavior of $\widehat{f}_n(0)$?

Problem 6. If f_0 is completely monotone (i.e. representable as a scale mixture of exponentials), what is the behavior of $\widehat{f}_n(0)$?

4. Four more problems involving shape constraints ...

very briefly