STAT 542 HW#1 SOLUTIONS

1. For <u>any</u> matrix $A \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ such that A_{22}^{-1} exists,

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} I & A_{12}A_{22}^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} A_{11\cdot 2} & 0 \\ 0 & A_{22} \end{pmatrix}\begin{pmatrix} I & 0 \\ A_{22}^{-1}A_{21} & I \end{pmatrix}$$

[A need not be symmetric or positive definite]

Thus $|A| = |A_{11\cdot 2}|\cdot|A_{22}| = |A_{22\cdot 1}|\cdot|A_{11}|$ if A_{11}^{-1} and A_{22}^{-1} exist.
Apply this with $A = \begin{pmatrix} S & -U \\ U' & I_q \end{pmatrix}$ to obtain $|S+UU'| = |S|\cdot|I_q + U'S^{-1}U|$.

2. Let $b = S^{-\frac{1}{2}}a$, where $S^{\frac{1}{2}}$ is the symmetric square root of S and $S^{-1/2} = (S^{1/2})^{-1}$. Then the desired identity becomes

$$b'(I+bb')^{-1}b = \frac{b'b}{1+b'b}.$$

But
$$(1+b'b)\,b'(I+bb')^{-1}b = b'(I+bb')^{-1}b + b'\left[bb'(I+bb')^{-1}\right]b$$
$$= \cancel{b'(I+bb')^{-1}b} + b'\left[I - \cancel{(I+bb')^{-1}}\right]b = b'b, \text{ as required.}$$

3. For any positive definite matrix Σ, $\underset{\sim}{ch}(\Sigma^{-1}) = \frac{1}{\underset{\sim}{ch}(\Sigma)}$. (That is, the characteristic roots of Σ^{-1} are the reciprocals of the ch. rts of Σ [why?].)
Thus for any positive semidefinite matrix A,

$$\underset{\sim}{ch}(I+A)^{-1} = \frac{1}{\underset{\sim}{ch}(I+A)} = \frac{1}{1+\underset{\sim}{ch}(A)},$$

so

$$\underset{\sim}{ch}\left[A(I+A)^{-1}\right] = \underset{\sim}{ch}\left[I-(I+A)^{-1}\right] = 1 - \underset{\sim}{ch}(I+A)^{-1}$$
$$= 1 - \frac{1}{1+\underset{\sim}{ch}(A)} = \frac{\underset{\sim}{ch}(A)}{1+\underset{\sim}{ch}(A)}.$$

Now apply this with $A = S^{-\frac{1}{2}}TS^{-\frac{1}{2}}$ and use the fact that $\underset{\sim}{ch}(BC) = \underset{\sim}{ch}(CB)$ [why?] to obtain the result.

4. We use the basic relations (see MDP notes on matrix algebra)

(1) $A^{11} = (A_{11\cdot 2})^{-1}$ [$\equiv A_{11\cdot 2}^{-1}$ for short]

(2) $(A^{11})^{-1}A^{12} = -A_{12}A_{22}^{-1}$

(i) $A_{11} - B_{11} = (I\ 0)(A-B)(I\ 0)^t \geq 0.$

(ii) $A \geq B \Rightarrow I \geq A^{-1/2}BA^{-1/2}$ [$A^{1/2}$ = symmetric square root]

$\Rightarrow \begin{pmatrix} I & A^{-1/2} \\ A^{-1/2} & B^{-1} \end{pmatrix} \geq 0$ [why?]

$\Rightarrow B^{-1} - A^{-1/2}A^{-1/2} = B^{-1} - A^{-1} \geq 0.$

(or, use char. roots)

(iii) $(B_{11\cdot 2})^{-1} \overset{(1)}{=} B^{11} \equiv (B^{-1})_{11} \overset{(i),(ii)}{\geq} (A^{-1})_{11} = A^{11} \overset{(1)}{=} (A_{11\cdot 2})^{-1}.$

$\therefore A_{11\cdot 2} \geq B_{11\cdot 2}$ by (ii).

5. By (1), $S^{11} = (S_{11\cdot 2})^{-1} \equiv (S_{11} - S_{12}S_{22}^{-1}S_{21})^{-1} \geq S_{11}^{-1}$, with "=" iff $S_{12} = 0$.

6. Let $C \equiv \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = S_{(12)\cdot 3} \overset{(1)}{=} [S^{(12)}]^{-1}$,

so $C^{-1} \equiv \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix} = S^{(12)} \equiv \begin{pmatrix} S^{11} & S^{12} \\ S^{21} & S^{22} \end{pmatrix}.$

(i) $[S_{11\cdot(23)}]^{-1} \overset{(1)}{=} S^{11} \equiv C^{11} \overset{(1)}{=} (C_{11\cdot 2})^{-1} \equiv [(S_{(12)\cdot 3})_{11\cdot 2}]^{-1}$, which yields the result.

(ii) $S_{11\cdot 3} = C_{11} \overset{(1)}{=} (C^{11\cdot 2})^{-1} \equiv (S^{11\cdot 2})^{-1}.$ Similarly, $S_{11\cdot 2} = (S^{11\cdot 3})^{-1}.$

(iii) By (2),
$$\begin{array}{ccc} (C^{11})^{-1}C^{12} & = & -C_{12}C_{22}^{-1} \\ \parallel & & \parallel \\ (S^{11})^{-1}S^{12} & = & -S_{12\cdot 3}(S_{22\cdot 3})^{-1}. \end{array}$$

(iv) $S_{11} \geq S_{11\cdot 2} \equiv S_{11} - S_{12}S_{22}^{-1}S_{21}$ is obvious, with "=" iff $\boxed{S_{12} = 0}$.

Similarly, $S^{11} \geq S^{11\cdot 3}$ with equality iff $\boxed{S^{13} = 0}$.

Thus $S_{11\cdot 2} \overset{(iii)}{=} (S^{11\cdot 3})^{-1} \geq (S^{11})^{-1} \overset{(i)}{=} S_{11\cdot(23)}$, with "=" iff $\boxed{S^{13} = 0}$, which by (iii) is equivalent to $\boxed{S_{13\cdot 2} = 0}$.