PROBABILITY and MATHEMATICAL STATISTICS I.

CLASS NOTES FOR MATH/STAT 394-5, AMATH/STAT 506, and STAT 512

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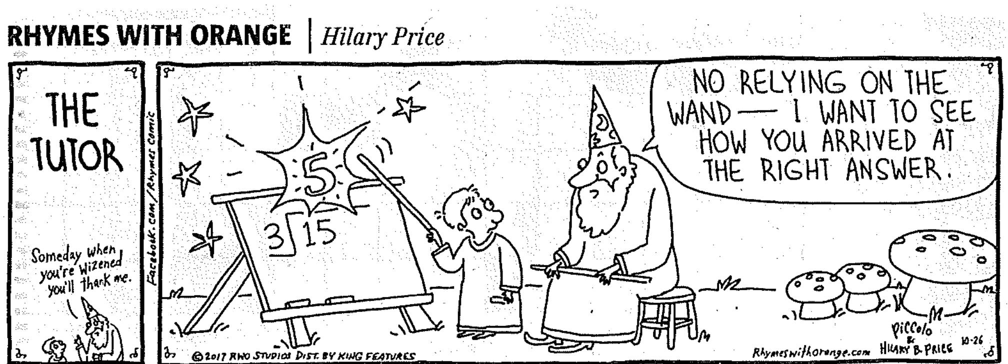

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PROBABILITY AND STATISTICAL INFERENCE

Probability vs. Statistical Inference – Standard Viewpoint:

"Probability" postulates a probability model and uses this to predict the behavior of observed data.

"Statistical inference" uses observed data to infer the probability model (= distribution) from which the data was generated.

1. Probability Distributions and Random Variables.

1.1. The components (Ω, P) of a probability model ($\equiv$ random experiment):

$\Omega :=$ *sample space* = set of all possible outcomes of the random experiment.

Ω either *discrete* (finite or countable) or *continuous* ($\approx$ open subset of $\mathbf{R}^n$).

Example 1.1. Toss a coin n times: $\Omega =$ all sequences $HHTH \ldots TH$ of length n ($H =$ Heads, $T =$ Tails). Thus $|\Omega| = 2^n$ (finite), so Ω is discrete.

Example 1.2. Toss a coin repeatedly until Heads appears and record the number of tosses needed: $\Omega = \{1, 2, \ldots\}$ (countable) so Ω is again discrete.

Example 1.3. Spin a pointer and record the angle where the pointer comes to rest: $\Omega = [0, 2\pi) \subset \mathbf{R}^1$, an entire interval, so Ω is continuous.

Example 1.4. Toss a dart at a circular board of radius d and record the impact point: $\Omega = \{(x, y) \mid x^2 + y^2 \leq d^2\} \subset \mathbf{R}^2$, a solid disk; Ω is continuous.

Example 1.5. Toss a coin *infinitely* many times: $\Omega =$ all *infinite* sequences $HHTH \ldots$. Here $\Omega \overset{1-1}{\longleftrightarrow} [0, 1] \subset \mathbf{R}^1$ [why?], so Ω is continuous.

Example 1.6. (Brownian motion) Observe the path of a particle suspended in a liquid or gaseous medium: Ω is the set of all continuous paths (functions), so Ω is continuous but *not* finite-dimensional.

Note: Examples 1.5 and 1.6 are examples of *discrete-time* and *continuous-time stochastic processes*, respectively.

$P :=$ a *probability measure*: $P(A) =$ probability that A occurs. Require:

(a) $0 \le P(A) \le 1$ for all $A \subseteq \Omega$;

(b) $P(\emptyset) = 0$, $P(\Omega) = 1$.

(c) $\{A_i\}$ *disjoint*, $\Rightarrow P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$. (countable additivity)

Three consequences of (c) [verify!]:

Inclusion: $A \subseteq B \Rightarrow P(A) \le P(B)$.

Complementation: $P(A^c) = 1 - P(A)$.

Monotone continuity:

$$\{A_n\}\uparrow \;\Rightarrow P(\cup_{n=1}^{\infty} A_n) =\uparrow \lim_{n\to\infty} P(A_n),$$
$$\{A_n\}\downarrow \;\Rightarrow P(\cap_{n=1}^{\infty} A_n) =\downarrow \lim_{n\to\infty} P(A_n).$$

In the **discrete case** where $\Omega = \{\omega_1, \omega_2, \ldots\}$, P is completely determined by the elementary probabilities $p_k \equiv P(\{\omega_k\})$, $k = 1, 2, \ldots$. This is because countable additivity implies that

$$P(A) = \sum_{\omega \in A} P(\{\omega\}) \qquad \forall\, A \in 2^{\Omega}. \tag{1.1}$$

Conversely, given any set of numbers $p_1, p_2, \ldots$ that satisfy

(a) $p_k \ge 0$,

(b) $\sum_{k=1}^{\infty} p_k = 1$,

we can *define* a probability measure P on 2^{Ω} via (1.1). Here, $\{p_1, p_2, \ldots\}$ is called a *probability mass function (pmf)*.

Example 1.7. The following $\{p_k\}$ are pmfs [verify (a) and (b)]:

(1.2) $p_k = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \ldots, n$ $(0 < p < 1)$ [Binomial(n, p)];

(1.3) $p_k = (1-p)^{k-1} p$, $k = 1, 2, \ldots$ $(0 < p < 1)$ [Geometric(p)];

(1.4) $p_k = e^{-\lambda} \lambda^k / k!$, $k = 0, 1, \ldots$ $(\lambda > 0)$ [Poisson(λ)].

The binomial distribution occurs in Example 1.1; the geometric distribution occurs in Example 1.2; the Poisson distribution arises as the limit of the binomial distributions $\mathrm{Bin}(n,p)$ when $n \to \infty$ and $p \to 0$ such that $np \to \lambda$ (Law of Rare Events). $\square$

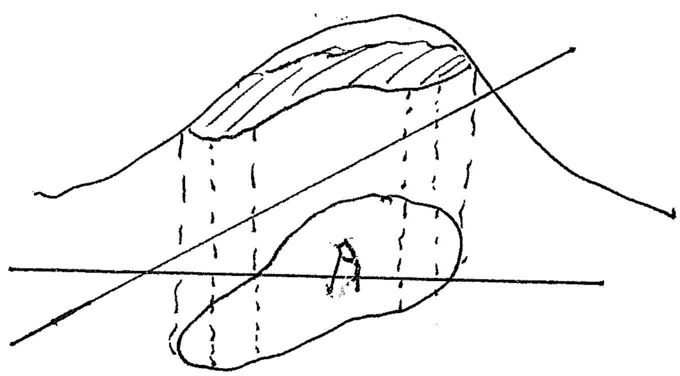
In the **continuous case** where $\Omega = \mathbf{R}^n$, let $f(\cdot)$ be a function on $\mathbf{R}^n$ that satisfies

(a) $f(x_1, \ldots, x_n) \geq 0 \quad \forall (x_1, \ldots, x_n) \in \mathbf{R}^n$,

(b) $\int \cdots \int_{\mathbf{R}^n} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n = 1$.

Then $f(\cdot)$ *defines* a probability measure P on $\mathbf{R}^n$ by

$$(1.5) \qquad P(A) = \int \cdots \int_A f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n \quad \forall A \subseteq \mathbf{R}^n.$$



The function $f(\cdot)$ is called a *probability density function (pdf)* on $\mathbf{R}^n$. Note that in the continuous case, unlike the discrete case, it follows from (1.5) that singleton events $\{x\}$ have probability 0.

Example 1.8. The following $f(x)$ are pdfs on $\mathbf{R}^1$ or $\mathbf{R}^n$ [verify all]:

(1.6) $\lambda e^{-\lambda x} I_{(0,\infty)}(x) \quad (\lambda > 0)$ [Exponential(λ)];

(1.7) $\dfrac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2} \quad (\sigma > 0)$ [Normal(μ, σ^2); see Example 6.6];

(1.8) $\dfrac{1}{\pi\sigma} \dfrac{1}{1 + (x-\mu)^2/\sigma^2} \quad (\sigma > 0)$ [Cauchy(μ, σ^2)];

(1.9) $\dfrac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x) \quad (\alpha, \lambda > 0)$ [Gamma(α, λ)];

$$(1.10)\quad \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}I_{(0,1)}(x) \quad (\alpha,\beta>0) \qquad [\text{Beta}(\alpha,\beta)];$$

$$(1.11)\quad \frac{e^x}{(1+e^x)^2} = \frac{e^{-x}}{(1+e^{-x})^2} \qquad [\text{standard Logistic}];$$

$$(1.12)\quad \frac{1}{b-a}I_{(a,b)}(x) \quad ((a,b)\subset \mathbf{R}^1) \qquad [\text{Uniform}(a,b)].$$

$$(1.13)\quad \frac{1}{\text{volume}(C)}I_C(x) \ (x=(x_1,\ldots,x_n), C\subset \mathbf{R}^n) \qquad [\text{Uniform}(C)].$$

Here, I_A is the indicator function of the set A: $I_A(x) = 1(0)$ if $x \in (\notin)A$. For the Uniform(C) pdf in (1.13), it follows from (1.5) that for any $A \subseteq C$,

$$(1.14)\qquad P(A) = \frac{\text{volume}(A)}{\text{volume}(C)}.$$

The exponential distribution appears as the distribution of waiting times between events in a *Poisson process* – cf. §3.6. According to the *Central Limit Theorem* (cf. §3.5), the normal ≡ Gaussian distribution occurs as the limiting distribution of sample averages (suitably standarized).

1.2. Random variables, pmfs, cdfs, and pdfs.

Often it is convenient to represent a feature of the outcome of a random experiment by a *random variable (rv)*, usually denoted by a capital letter X, Y, Z, etc. Thus in Example 1.1, $X \equiv$ the total number of Heads in the n trials and $Y \equiv$ the length of the longest run of Tails in the same n trials are both random variables. This shows already that two or more random variables may arise from the same random experiment. Additional random variables may be constructed by arithmetic operations, e.g., $Z \equiv X + Y$ and $W \equiv XY^3$ are also random variables arising in Example 1.1.

Formally, a random variable $X \equiv X(\omega)$ arising from a probability model (Ω, P) is simply a function defined on Ω. Each random variable X determines its own *induced* probability model (Ω_X, P_X), where Ω_X is the *range* of possible values of X and P_X is the probability distribution induced on Ω_X from P by X: for any $B \subseteq \Omega$,

$$(1.15)\qquad P_X(B) \equiv P[X \in B] := P[X^{-1}(B)] \equiv P[\{\omega \in \Omega \mid X(\omega) \in B\}].$$

If in Example 1.2 we define $X :=$ the number of trials needed to obtain the first Head, then the induced probability model for X is the geometric distribution in (1.3).

A multivariate rv $(X_1, \ldots, X_n)$ is called a *random vector (rvtr)*. It is important to realize that the individual rvs $X_1, \ldots, X_n$ are related in that they must arise from the *same* random experiment. Thus they may (or may not) be correlated. [Example: $(X, Y) =$ (height, weight); other examples?] One goal of statistical analysis is to study the relationship among correlated rvs for purposes of prediction.

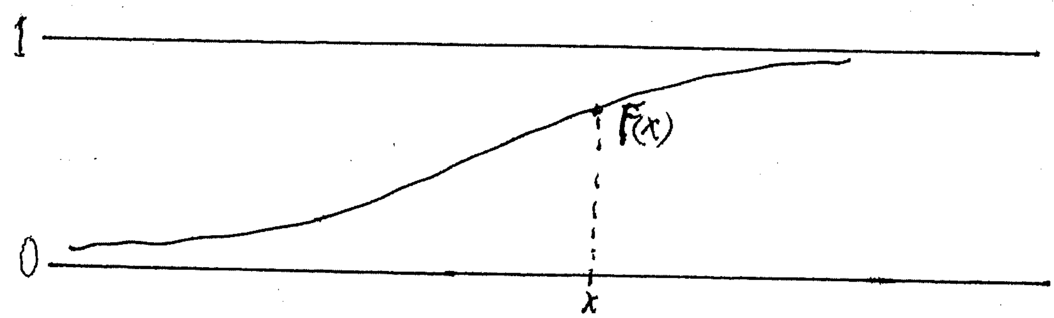
The random variable (or random vector) X is called *discrete* if its range Ω_X is discrete, and *continuous* if Ω_X is continuous. As in (1.1), the probability distribution P_X of a discrete random variable is completely determined by its pmf

(1.16) $$f_X(x) := P[X = x], \qquad x \in \Omega_X.$$

The probability distribution P_X of a univariate continuous random variable is determined by a pdf f_X on $\mathbf{R}^1$. It is useful to define the *cumulative distribution function (cdf)* F_X as follows:

(1.17) $$F_X(x) := P[X \le x] = \int_{-\infty}^{x} f_X(t)dt, \qquad x \in \mathbf{R}^1,$$

$F:$



The pdf f_X can be recovered from the cdf F_X as follows:

(1.18) $$f_X(x) = \frac{d}{dx} F_X(x), \qquad x \in \mathbf{R}^1.$$

Clearly F_X directly determines the probabilities of all intervals in $\mathbf{R}^1$:

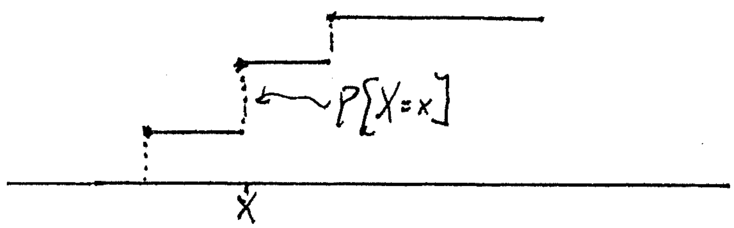
$$(1.19) \qquad P_X[\,(a,b]\,] \equiv \Pr[X \in (a,b]\,] = F_X(b) - F_X(a).$$

In fact, F_X completely determines[1] the probability distribution P_X on $\mathbf{R}^1$.

Note: The cdf F_X is also defined for *univariate discrete* random variables by (1.17). Now F_X determines the pmf f_X not by (1.18) but by

$$(1.20) \qquad f_X(x) \equiv P[X = x] = F_X(x) - F_X(x-), \quad x \in \mathbf{R}^1 \quad \text{[verify]}.$$

F :



Basic properties of a cdf F on $\mathbf{R}^1$:

(i) $F(-\infty) = 0 \le F(x) \le 1 = F(+\infty)$.

(ii) $F(\cdot)$ is non-decreasing and right-continuous: $F(x) = F(x+)$.

For a continuous *multivariate* rvtr $(X_1, \ldots, X_n)$ the *joint cdf* is

$$(1.21) \qquad F_{X_1,\ldots,X_n}(x_1, \ldots, x_n) := P[X_1 \le x_1, \ldots, X_n \le x_n],$$

from which the *joint pdf* f is recovered as follows:

$$(1.22) \qquad f_{X_1,\ldots,X_n}(x_1, \ldots, x_n) : \frac{\partial^n}{\partial x_1 \cdots \partial x_n} F_{X_1,\ldots,X_n}(x_1, \ldots, x_n).$$

Exercise 1.1. Extend (1.19) to show that for $n = 2$, the cdf F directly determines the probabilities of all rectangles in $\mathbf{R}^2$. □

[1] Since any Borel set $B \subset R^1$ can be approximated by finite disjoint unions of intervals.

For a discrete *multivariate* rvtr $(X_1, \dots, X_n)$, the *joint cdf* $F_{X_1,\dots,X_n}$ is again defined by (1.21). The *joint pmf* is given by

$$(1.23) \qquad f_{X_1,\dots,X_n}(x_1, \dots, x_n) := P[X_1 = x_1, \dots, X_n = x_n],$$

from which all joint probabilities can be determined as in (1.1).

The *marginal* pmf or pdf of any X_i can be recovered from the joint pmf or pdf by summing or integrating over the other variables. The marginal cdf can also be recovered from the joint cdf. In the bivariate case ($n = 2$), for example, if the rvtr (X, Y) has joint pmf $f_{X,Y}$ or joint pdf $f_{X,Y}$, and joint cdf $F_{X,Y}$, then, respectively,

(1.24) $f_X(x) = \sum_y f_{X,Y}(x, y);$ [verify via countable additivity]

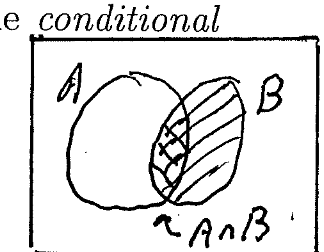
(1.25) $f_X(x) = \int f_{X,Y}(x, y)\, dy.$ [verify via (1.18) and (1.17)]

(1.26) $F_X(x) = F_{X,Y}(x, \infty).$ [verify via (1.21)]

The joint distribution contains information about X and Y beyond their marginal distributions, i.e., information about the nature of any dependence between them. Thus, the joint distribution determines all marginal distributions but not conversely (except under independence – cf. (1.32), (1.33).)

1.3. Conditional probability.

Consider a probability model (Ω, P). Let $B \subseteq \Omega$ be an event such that $P(B) > 0$. If we are told that B has occurred but given no other information, then the original probability model is reduced to the *conditional probability model* $(\Omega, P[\cdot \mid B])$, where for any event $A \subseteq \Omega$,



$$(1.27) \qquad P[A \mid B] = \frac{P(A \cap B)}{P(B)}.$$

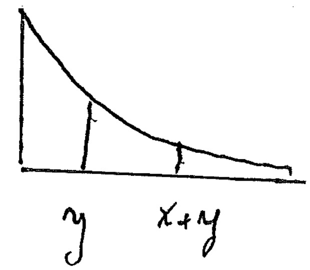
Then $P[\cdot \mid B]$ is also a probability measure [verify] and $P[B \mid B] = 1$, i.e., $P[\cdot \mid B]$ assigns probability 1 to B. Thus Ω *is reduced to* B and, by (1.27), events within B retain the same *relative* probabilities.

Example 1.9. Consider the Uniform(C) probability model $(\mathbf{R}^n, P_C)$ determined by (1.13). If $B \subset C$ and volume$(B) > 0$, then the conditional distribution $P_C[\cdot|B] = P_B$, the Uniform(B) distribution [verify via (1.14)].

Example 1.10. Let X be a random variable whose distribution on $[0, \infty)$ is determined by the exponential pdf $\lambda e^{-\lambda x}$ in (1.6). Then for $x, y > 0$,

$$P[X > x + y \mid X > y] = \frac{P[X > x + y]}{P[X > y]}$$

$$= \frac{\int_{x+y}^{\infty} \lambda e^{-\lambda t}\, dt}{\int_{y}^{\infty} \lambda e^{-\lambda t}\, dt} = \frac{e^{-\lambda(x+y)}}{e^{-\lambda y}} = e^{-\lambda x}.$$



Because $e^{-\lambda x} = P[X > x]$, this can be interpreted as follows: the exponential distribution is *memory-free*; i.e., given that we have waited at least y time units, the probability of having to wait an additional x time units is the same as the unconditional probability of waiting at least x units from the start. □

Exercise 1.2. Show that the exponential distribution is the *only* continuous distribution on $(0, \infty)$ with this memory-free property. That is, show that if X is a continuous rv on $(0, \infty)$ such that $P[X > x + y \mid X > y] = P[X > x]$ for every $x, y > 0$, then $f_X(x) = \lambda e^{-\lambda x}$ for some $\lambda > 0$.

1.4. Conditional pmfs and pdfs.

Let (X, Y) be a *discrete* bivariate rvtr with joint pmf $f_{X,Y}$. For any $x \in \Omega_X$ such that $P[X = x] > 0$, the *conditional pmf* of Y given $X = x$ is defined by

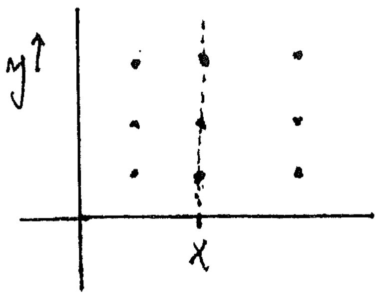
$$f_{Y|X}(y|x) \equiv P[Y = y \mid X = x] = \frac{f_{X,Y}(x, y)}{f_X(x)}, \tag{1.28}$$

where the second equality follows from (1.27). As in (1.1), the conditional pmf completely determines the conditional distribution of Y given $X = x$:

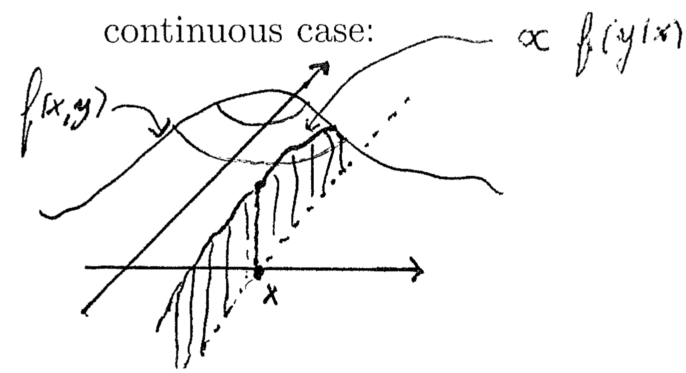
$$P[Y \in B \mid X = x] = \sum_{y \in B} f_{Y|X}(y|x) \quad \forall B. \qquad \text{[verify]} \tag{1.29}$$

"Slicing":

discrete case:



continuous case:



Next let (X, Y) be a *continuous* bivariate rvtr with joint pdf $f_{X,Y}$. By analogy with (1.28), for any $x \in \Omega_X$ such that the marginal pdf $f_X(x) > 0$, the *conditional pdf* of Y given $X = x$ might be defined by

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}. \tag{1.30}$$

As in (1.29), the conditional pdf (1.30) completely determines the conditional distribution of Y given $X = x$:

$$P[Y \in B \mid X = x] = \int_B f_{Y|X}(y|x)\, dy \qquad \forall B. \tag{1.31}$$

Note that $P[Y \in B \mid X = x]$ *cannot* be interpreted as a conditional probability for events via (1.27), since $P[X = x] = 0$ for every x in the continuous case. Instead, (1.31) will be given a more accurate definition in §4.

1.5. Independence.

Two *events* $A, B \subseteq \Omega$ are *independent* under the probability model (Ω, P), denoted as $A \perp\!\!\!\perp B\ [P]$ or simply $A \perp\!\!\!\perp B$, if any of the following five equivalent [verify!] conditions hold:

$$P[A \cap B] = P[A]P[B]; \tag{1.32}$$

$$P[A \mid B] = P[A]; \qquad P[A \mid B] = P[A \mid B^c], \tag{1.33}$$

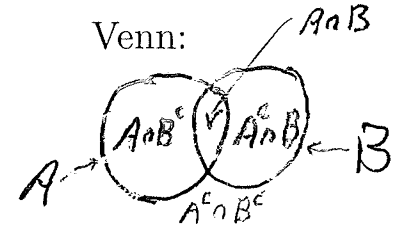
$$P[B \mid A] = P[B]; \qquad P[B \mid A] = P[B \mid A^c]. \tag{1.34}$$

Intuitively, $A \perp\!\!\!\perp B$ means that information about the occurrence (or non-occurrence!) of either event does not change the probability of occurrence or non-occurrence for the other.

Exercise 1.3. Show that $A \perp\!\!\!\perp B \Leftrightarrow A \perp\!\!\!\perp B^c \Leftrightarrow A^c \perp\!\!\!\perp B \Leftrightarrow A^c \perp\!\!\!\perp B^c$.

	B	B^c
A	$A \cap B$	$A \cap B^c$
A^c	$A^c \cap B$	$A^c \cap B^c$

Venn:



□

Two *rvs* X and Y are *independent* under the model (Ω, P), denoted as $X \perp\!\!\!\perp Y\ [P]$ or simply $X \perp\!\!\!\perp Y$, if $\{X \in A\}$ and $\{Y \in B\}$ are independent for each pair of events $A \subseteq \Omega_X$ and $B \subseteq \Omega_Y$. It is straightforward to show that for a *jointly discrete or jointly continuous bivariate* rvtr (X, Y), $X \perp\!\!\!\perp Y$ iff any of the following four equivalent conditions hold [verify]:

(1.35) $$f_{X,Y}(x, y) = f_X(x) f_Y(y) \quad \forall (x, y) \in \Omega_{X,Y};$$

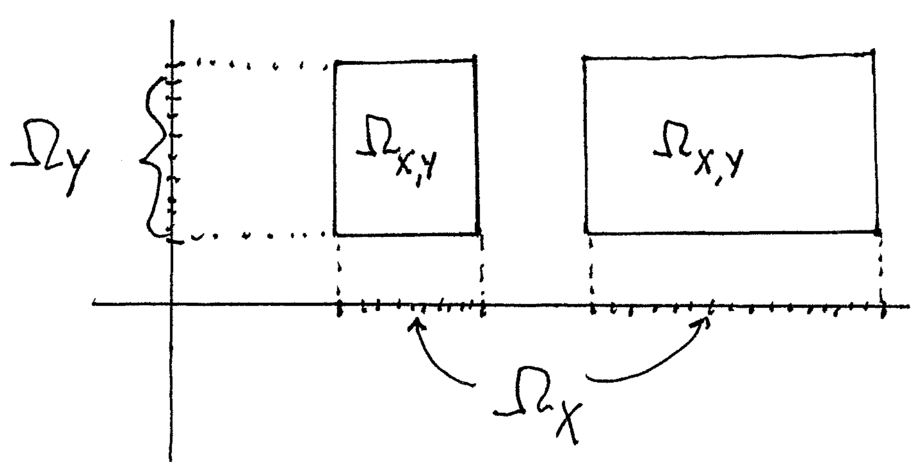
(1.36) $$f_{Y|X}(y|x) = f_Y(y) \quad \forall (x, y) \in \Omega_{X,Y};$$

(1.37) $$f_{X|Y}(x|y) = f_X(x) \quad \forall (x, y) \in \Omega_{X,Y};$$

(1.38) $$F_{X,Y}(x, y) = F_X(x) F_Y(y) \quad \forall (x, y) \in \Omega_{X,Y}.$$

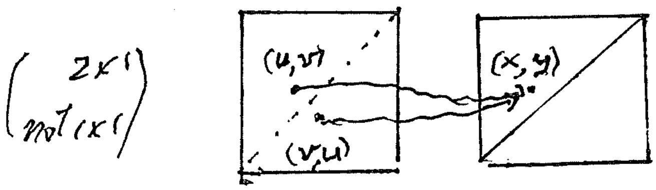
Intuitively, it follows from (1.36) and (1.37) that independence of rvs means that information about the values of one of the rvs does not change the probability distribution of the other rv. It is important to note that this requires that the *joint range of* (X, Y) *is the Cartesian product of the marginal ranges:*

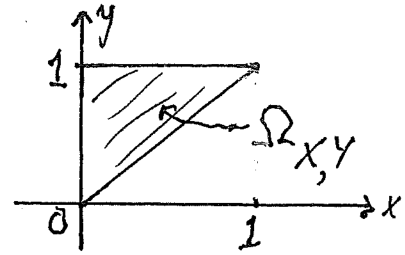
(1.39) $$\Omega_{X,Y} = \Omega_X \times \Omega_Y.$$



Example 1.11. Let U, V be independent Uniform(0,1) rvs and set $X = \min(U, V)$, $Y = \max(U, V)$. Then the range of (X, Y) is given by

$$\Omega_{X,Y} = \{(x, y) \mid 0 \le x \le y \le 1\}; \tag{1.40}$$



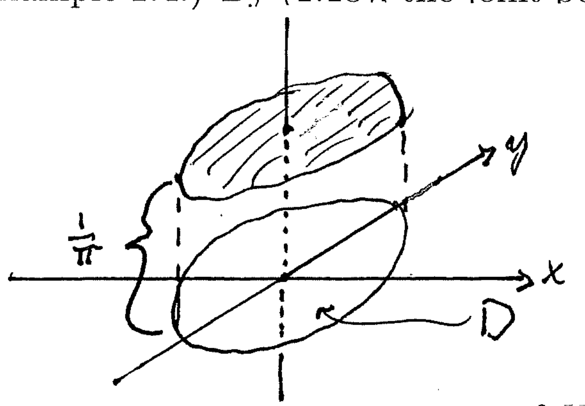


Because $\Omega_{X,Y}$ is not a Cartesian product set, X and Y cannot be independent. [In fact, they are *positively correlated* – why?] □

Exercise 1.4. The condition $\Omega_{X,Y} = \Omega_X \times \Omega_Y$ is necessary for mutual independence. Show by counterexample that it is not sufficient.

Example 1.12. Let $(X, Y) \sim$ Uniform(D), where $D \equiv \{x^2 + y^2 \le 1\}$ denotes the unit disk in $\mathbf{R}^2$. (Recall Example 1.4.) By (1.13), the joint pdf of X, Y is

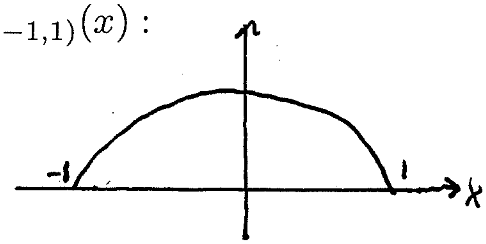
$$f_{X,Y}(x, y) = \frac{1}{\pi} I_D(x, y) : \tag{1.41}$$



In particular, the range of (X, Y) is D. However, the marginal ranges of X and Y are both $[-1, 1]$, so $\Omega_{X,Y} \ne \Omega_X \times \Omega_Y$, hence $X \not\perp\!\!\!\perp Y$.

More precisely, it follows from (1.41) that the marginal pdf of X is

$$f_X(x) = \frac{1}{\pi} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy = \frac{2}{\pi} \sqrt{1 - x^2} I_{(-1,1)}(x) : \tag{1.42}$$



and similarly $f_Y(y) = \frac{2}{\pi}\sqrt{1 - y^2} I_{(-1,1)}(y)$. Thus $f(x, y) \ne f(x)f(y)$ by (1.41) and (1.41), hence X and Y are not independent. [But they are uncorrelated: no *linear* trend – verify.]

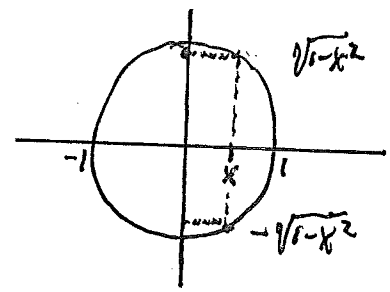
The dependence of X and Y can also be seen from the conditional pdf $f_{Y|X}$ (recall (1.36)). From (1.30), (1.41), and (1.42),

$$f_{Y|X}(y|x) = \frac{1}{2\sqrt{1-x^2}} I_{(-\sqrt{1-x^2},\sqrt{1-x^2})}(y) \neq f_Y(y), \tag{1.43}$$

so (1.36) fails and X and Y are not independent. Note that (1.43) is equivalent to the statement that the conditional distribution of $Y|X$ is uniform on the interval $\big(-\sqrt{1-x^2}, \sqrt{1-x^2}\big)$, i.e.,

$$Y|X{=}x \sim \text{Uniform}\big(-\sqrt{1-x^2}, \sqrt{1-x^2}\big), \tag{1.44}$$

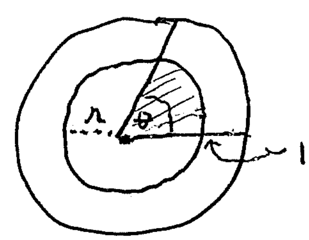
which is already obvious from this figure:



From (1.44), $\frac{Y}{\sqrt{1-X^2}} \,\Big|\, X \sim \text{Unif}(-1,1)$, so $\frac{Y}{\sqrt{1-X^2}} \perp\!\!\!\perp X$.
Similarly, $\frac{X}{\sqrt{1-Y^2}} \,\Big|\, Y \sim \text{Unif}(-1,1)$, so $\frac{X}{\sqrt{1-Y^2}} \perp\!\!\!\perp Y$.

Exercise 1.5*. Let $S = \frac{Y}{\sqrt{1-X^2}}$, $T = \frac{X}{\sqrt{1-Y^2}}$. Is $S \perp\!\!\!\perp T$?
[Half credit: find $f(s,t)$; full credit: find $F(s,t)$.] □

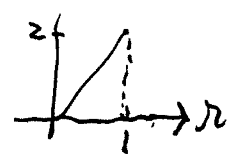
If we represent the rvtr (X, Y) in polar coordinates as (R, Θ), then $R \perp\!\!\!\perp \Theta$. This is readily verified: clearly $\Omega_{R,\Theta} = \Omega_R \times \Omega_\Theta$ [verify], while by (1.41) (uniformity),



$$\begin{aligned} F_{R,\Theta}(r,\theta) &\equiv P[0 \le R \le r,\ 0 \le \Theta \le \theta] \\ &= \frac{\pi r^2 \cdot [\theta/(2\pi)]}{\pi} \\ &= r^2 \cdot [\theta/(2\pi)] \\ &= P[0 \le R \le r] P[0 \le \Theta \le \theta] \\ &\equiv F_R(r) \cdot F_\Theta(\theta) \end{aligned} \tag{1.45}$$

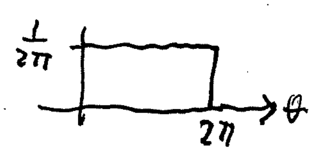
so $R \perp\!\!\!\perp \Theta$. It follows too that

$$f_R(r) = 2r I_{(0,1)}(r); \tag{1.46a}$$



$$f_\Theta(\theta) = \frac{1}{2\pi} I_{[0,2\pi)}(\theta); \tag{1.46b}$$

the latter states that $\Theta \sim \text{Uniform}[0, 2\pi)$.



□

Mutual independence. Events $A_1, \ldots, A_n$ ($n \geq 3$) are *mutually independent* iff the following 2^n conditions hold:

$$(1.47) \qquad P(A_1^{\epsilon_1} \cap \cdots \cap A_n^{\epsilon_n}) = P(A_1^{\epsilon_1}) \cdots P(A_n^{\epsilon_n}) \ \forall (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n,$$

where $A^1 := A$ and $A^0 := A^c$. A finite family[2] of rvs $X_1, \ldots, X_n$ are *mutually independent* iff $\{X_1 \in B_1\}, \ldots, \{X_n \in B_n\}$ are mutually independent events in Ω for every choice of events $B_1, \ldots, B_n$ in $\mathbf{R}^1$. An infinite family $X_1, X_2, \ldots$ of rvs are *mutually independent* iff every finite subfamily is mutually independent. Intuitively, mutual independence of rvs means that information about the values of some of the rvs does not change the (joint) probability distribution of the other rvs. (This extends directly to rvtrs.)

Exercise 1.6. (i) For $n \geq 3$ events $A_1, \ldots, A_n$, show that mutual independence implies pairwise independence. (ii) Show by counterexample that pairwise independence does not imply mutual independence. (iii) Show that

$$(1.48) \qquad P(A \cap B \cap C) = P(A)P(B)P(C)$$

is not by itself sufficient for mutual independence of A, B, C.

Example 1.13. In Example 1.1, suppose that the n trials are mutually independent and that $p : P(H)$ and $q \equiv (1-p) \equiv P(T)$ do not vary from trial to trial. Let X denote the total number of Heads in the n trials. Then by independence, $X \sim$ Binomial(n, p), i.e., the pmf p_X of X is given by (1.2). [Verify!].

Example 1.14. In Example 1.2, suppose that the entire infinite sequence of trials are mutually independent and that $p := P(H)$ does not vary from trial to trial. Let X denote the number of trials needed to obtain the first Head. Then by independence, $X \sim$ Geometric(p), i.e., the pmf p_X is given by (1.3). [Verify!] □

Mutual independence of rvs $X_1, \ldots, X_n$ can be expressed in terms of their joint pmf (discrete case), joint pdf (continuous case), or joint cdf (both

---

[2] Here $X_1, \ldots, X_n$ must have a joint distribution, i.e., must arise from the same random experiment (Ω, P).

cases): $X_1, \ldots, X_n$ are mutually independent iff either

$$f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = f_{X_1}(x_1)\cdots f_{X_n}(x_n); \tag{1.49}$$

$$F_{X_1,\ldots,X_n}(x_1,\ldots,x_n) = F_{X_1}(x_1)\cdots F_{X_n}(x_n). \tag{1.50}$$

Again, these conditions implicity require that the *joint range of* $X_1, \ldots, X_n$ *is the Cartesian product of the marginal ranges:*

$$\Omega_{X_1,\ldots,X_n} = \Omega_{X_1} \times \cdots \times \Omega_{X_n}. \tag{1.51}$$

Example 1.15. Continuing Example 1.13, let $X_1, \ldots, X_n$ be indicator variables ($\equiv$ *Bernoulli variables*) that denote the outcomes of trials $1, \ldots, n$: $X_i = 1$ or 0 according to whether Heads or Tails occurs on the ith trial. Here $X_1, \ldots, X_n$ are *mutually independent and identically distributed (i.i.d.)* rvs, and X can be represented as their sum: $X = X_1 + \cdot + X_n$. Therefore we expect that X and X_1 are *not* independent. In fact, the joint range

$$\Omega_{X,X_1} = \{(x, x_1) \mid x = 0, 1, \ldots, n,\ x_1 = 0, 1,\ x \geq x_1\}.$$

The final inequality implies that $\Omega_{X,X_1} \neq \Omega_X \times \Omega_{X_1}$, so X and X_1 cannot be independent.

1.6. Composite events and total probability.

Equation (1.27) can be rewritten in the following useful form(s):

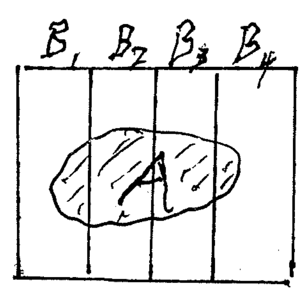
$$P(A \cap B) = P[A \mid B]\, P(B) \ \big(= P[B \mid A]\, P(A) \big). \tag{1.52}$$

By (1.28) and (1.30), similar formulas hold for joint pmfs and pdfs:

$$f(x, y) = f(x|y) f(y) \ \big(= f(y|x) f(x) \big). \tag{1.53}$$

Now suppose that the sample space Ω is partitioned into a finite or countable set of disjoint events: $\Omega = \cup_{i=1}^{\infty} B_i$. Then by the countable additivity of P and (1.52), we have the *law of total probability*:

$$\begin{aligned} P(A) &= P\big[A \cap \big(\cup_{i=1}^{\infty} B_i\big)\big] = P\big[\cup_{i=1}^{\infty} (A \cap B_i)\big] \\ &= \sum_{i=1}^{\infty} P[A \cap B_i] = \sum_{i=1}^{\infty} P[A \mid B_i]\, P(B_i). \end{aligned} \tag{1.54}$$



Example 1.16. Let $X \sim \text{Poisson}(\lambda)$, where $\lambda > 0$ is unknown and is to be estimated. (For example, λ might be the decay rate of a radioactive process and X the number of emitted particles recorded during a unit time interval.) We shall later see that the *expected value* of X is given by $\mathrm{E}(X) = \lambda$, so if X were observed then we would estimate λ by $\hat{\lambda} \equiv X$.

Suppose, however, that we do not observe X but instead only observe the value of Y, where

$$Y|X\!=\!x \;\sim \text{Binomial}(n = x,\, p). \tag{1.55}$$

(This would occur if each particle emitted has probability p of being observed, independently of the other emitted particles.) If p is known then we may still obtain a reasonable estimate of λ based on Y, namely $\tilde{\lambda} = \frac{1}{p}Y$.

To obtain the distribution of Y, apply (1.54) as follows (set $q = (1-p)$):

$$\begin{aligned}
P[Y = y] &= P[Y = y,\ X = y, y+1, \ldots] && \text{[since } X \geq Y\text{]}\\
&= \sum_{x=y}^{\infty} P[Y = y \mid X = x] \cdot P[X = x] && \text{[by (1.54)]}\\
&= \sum_{x=y}^{\infty} \binom{x}{y} p^y q^{x-y} \cdot e^{-\lambda}\frac{\lambda^x}{x!} && \text{[by (1.2), (1.4)]}\\
&= \frac{e^{-\lambda}(p\lambda)^y}{y!} \sum_{k=0}^{\infty} \frac{(q\lambda)^k}{k!} && \text{[let } k = x - y\text{]}\\
&= \frac{e^{-p\lambda}(p\lambda)^y}{y!}. && (1.56)
\end{aligned}$$

This implies that $Y \sim \text{Poisson}(p\lambda)$, so

$$\mathrm{E}(\tilde{\lambda}) \equiv \mathrm{E}(\frac{1}{p}Y) = \frac{1}{p}\mathrm{E}(Y) = \frac{1}{p}(p\lambda) = \lambda,$$

which shows that $\tilde{\lambda}$ is an *unbiased* estimate of λ based on Y. □

[What if p is unknown? Knee-jerk Bayesian: "assume $p \sim \text{Uniform}(0,1)$." Then $\mathrm{E}(Y) = \mathrm{E}[\mathrm{E}(Y \mid p)] = \mathrm{E}(p\lambda) = \frac{1}{2}\lambda$, so $\mathrm{E}(2Y) = \lambda$. Use $\tilde{\lambda} = 2Y$??]

1.7. Bayes formula.

If $P(A) > 0$ and $P(B) > 0$, then (1.27) yields *Bayes formula for events:*

$$P[A \mid B] = \frac{P[A \cap B]}{P(B)} = \frac{P[B \mid A]P(A)}{P(B)}. \tag{1.57}$$

Similarly, (1.28) and (1.30) yield *Bayes formula for joint pmfs and pdfs:*

$$f(x|y) = \frac{f(y|x)f(x)}{f(y)} \qquad \text{if } f(x), f(y) > 0. \tag{1.58}$$

[See §4 for extensions to the *mixed* cases where X is discrete and Y is continuous, or vice versa.]

Example 1.17. In Example 1.16, what is the conditional distribution of X given that $Y = y$? By (1.58), the conditional pmf of $X \mid Y = y$ is

$$\begin{aligned} f(x|y) &= \frac{\binom{x}{y} p^y q^{x-y} \cdot e^{-\lambda}\lambda^x/x!}{e^{-p\lambda}(p\lambda)^y/y!} \\ &= \frac{e^{-q\lambda}(q\lambda)^{x-y}}{(x-y)!}, \qquad x = y,\ y+1, \ldots. \end{aligned}$$

Thus, if we set $Z = X - Y$, then

$$P[Z = z \mid Y = y] = \frac{e^{-q\lambda}(q\lambda)^z}{z!}, \quad z = 0, 1, \ldots, \tag{1.59}$$

so $Z \mid Y{=}y \sim \text{Poisson}(q\lambda)$. Because this conditional distribution does not depend on y, it follows from (1.36) that $X - Y \perp\!\!\!\perp Y$. (In the radioactivity scenario, this states that the number of uncounted particles is independent of the number of counted particles.)

Note: this also shows that if $U \sim \text{Poisson}(\mu)$ and $V \sim \text{Poisson}(\nu)$ with U and V independent, then $U + V \sim \text{Poisson}(\mu + \nu)$. [Why?] □

Exercise 1.7. (i) Let X and Y be independent Bernoulli rvs with

$$P[X = 1] = p, \qquad P[X = 0] = 1 - p;$$
$$P[Y = 1] = r, \qquad P[Y = 0] = 1 - r.$$

Let $Z = X + Y$, a discrete rv with range $\{0, 1, 2\}$. Do there exist p, r such that Z is uniformly distributed on its range, i.e., such that $P[Z = k] = \frac{1}{3}$ for $k = 0, 1, 2$? (Prove or disprove.)

(ii)* (unfair dice.) Let X and Y be independent discrete rvs, each having range $\{1, 2, 3, 4, 5, 6\}$, with pmfs

$$p_X(k) = p_k, \quad p_Y(k) = r_k, \quad k = 1, \dots, 6.$$

Let $Z = X + Y$, a discrete rv with range $\{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$. First note that if X and Y are the outcomes of tossing two fair dice, i.e. $p_X(k) = p_Y(k) = \frac{1}{6}$ for $k = 1, \dots, 6$, then the pmf of Z is given by

$$\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}, \frac{5}{36}, \frac{6}{36}, \frac{5}{36}, \frac{4}{36}, \frac{3}{36}, \frac{2}{36}, \frac{1}{36},$$

which is *not* uniform over its range. Do there exist *unfair* dice such that Z *is* uniformly distributed over its range, i.e., such that $P[Z = k] = \frac{1}{11}$ for $k = 2, 3, \dots, 12$? (Prove or disprove.) $\square$

1.8. Conditional independence.

Consider three events A, B, C with $P(C) > 0$. We say that A and B are *conditionally independent given* C, written $A \perp\!\!\!\perp B \mid C$, if any of the following three equivalent conditions hold (recall (1.32) - (1.34)):

$$P[A \cap B \mid C] = P[A \mid C] P[B \mid C]; \tag{1.60}$$
$$P[A \mid B,\ C] = P[A \mid C]; \tag{1.61}$$
$$P[B \mid A,\ C] = P[B \mid C]. \tag{1.62}$$

As with ordinary independence, $A \perp\!\!\!\perp B \mid C \Leftrightarrow A \perp\!\!\!\perp B^c \mid C \Leftrightarrow A^c \perp\!\!\!\perp B \mid C \Leftrightarrow A^c \perp\!\!\!\perp B^c \mid C$ (see Exercise 1.3). However [construct counterexamples]

$$A \perp\!\!\!\perp B \mid C \not\Leftrightarrow A \perp\!\!\!\perp B \mid C^c,$$
$$A \perp\!\!\!\perp B \mid C,\ A \perp\!\!\!\perp B \mid C^c \not\Leftrightarrow A \perp\!\!\!\perp B.$$

Let A and B be events and Z a random variable. Then A and B are *conditionally independent given* Z, denoted as $A \perp\!\!\!\perp B \mid Z$, if

$$(1.63) \qquad P[A \cap B \mid Z = z] = P[A \mid Z = z]\, P[B \mid Z = z] \quad \forall z.$$

(If Z is continuous, $P[\cdot \mid Z = z]$ must be defined as in (4.12)-(4.13).) The rvs X and Y are *conditionally independent given* Z, written $X \perp\!\!\!\perp Y \mid Z$, if

$$(1.64) \qquad \{X \in A\} \perp\!\!\!\perp \{Y \in B\} \mid Z$$

for each pair of events A, B. It is straightforward to show that for a *jointly discrete or jointly continuous trivariate* rvtr (X, Y, Z), $X \perp\!\!\!\perp Y \mid Z$ iff any of the following four equivalent conditions hold [verify]:

$$(1.65) \qquad f(x, y \mid z) = f(x \mid z) f(y \mid z);$$
$$(1.66) \qquad f(y \mid x, z) = f(y \mid z);$$
$$(1.67) \qquad f(x \mid y, z) = f(x \mid z);$$
$$(1.68) \qquad f(x, y, z) f(z) = f(x, z) f(y, z).$$

Exercise 1.8. Conditional independence $\not\Leftrightarrow$ independence.

(i) Construct (X, Y, Z) such that $X \perp\!\!\!\perp Y \mid Z$ but $X \not\perp\!\!\!\perp Y$.

(ii) Construct (X, Y, Z) such that $X \perp\!\!\!\perp Y$ but $X \not\perp\!\!\!\perp Y \mid Z$.

(This is related to Simpson's Paradox, see Example 3.3.) □

The graphical Markov model representation of $X \perp\!\!\!\perp Y \mid Z$:

$$(1.69) \qquad X \longleftarrow Z \longrightarrow Y.$$

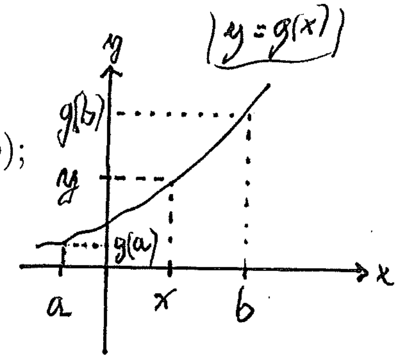
This can be realized by the linear model $X = Z + \epsilon_X$, $Y = Z + \epsilon_Y$, where $Z, \epsilon_X, \epsilon_Y$ are mutually independent rvs.

2. Transforming Continuous Distributions.

2.1. One function of one random variable.

Let X be a *continuous* rv with pdf f_X on the range $\Omega_X = (a,b)$ $(-\infty \leq a < b \leq \infty)$. Define the new rv $Y = g(X)$, where g is a *strictly increasing and smooth* function on (a,b). Then the pdf f_Y is determined as follows:

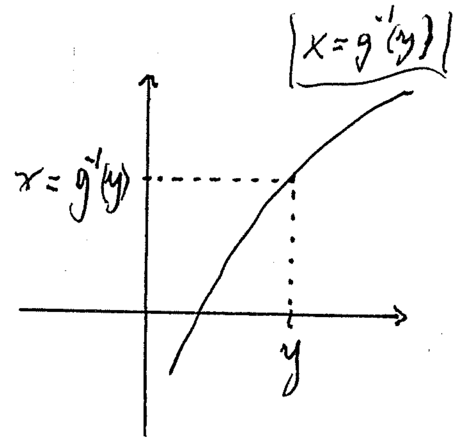
Theorem 2.1.



(2.1)
$$f_Y(y) = \begin{cases} \frac{f_X(g^{-1}(y))}{g'(g^{-1}(y))}, & g(a) < y < g(b); \\ 0, & \textit{otherwise}. \end{cases}$$

Proof.

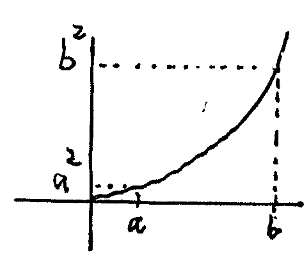
(2.2)
$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) \\ &= \frac{d}{dy} P[Y \leq y] \\ &= \frac{d}{dy} P[g(X) \leq y] \\ &= \frac{d}{dy} P[X \leq g^{-1}(y)] \\ &= \frac{d}{dy} F_X(g^{-1}(y)) \\ &= f_X(g^{-1}(y)) \, \frac{d}{dy} g^{-1}(y) \\ &= f_X(g^{-1}(y)) \, \frac{1}{g'(g^{-1}(y))}. \end{aligned}$$



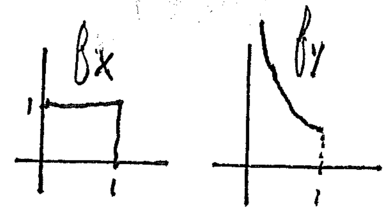
[why?]

Example 2.1. Consider $g(x) = x^2$. In order that this g be strictly increasing we must have $0 \leq a$. Then $g'(x) = 2x$ and $g^{-1}(y) = \sqrt{y}$, so from (2.1) with $Y = X^2$,

(2.3)
$$f_Y(y) = \frac{1}{2\sqrt{y}} f_X(\sqrt{y}), \qquad a^2 < y < b^2.$$



In particular, if $X \sim$ Uniform$(0,1)$ then $Y \equiv X^2$ has pdf



(2.4) $$f_Y(y) = \frac{1}{2\sqrt{y}}, \quad 0 < y < 1. \qquad \text{[decreasing]}$$

Example 2.2a. If X has cdf F then $Y \equiv F(X) \sim$ Uniform$(0,1)$. [cf.§18.11]

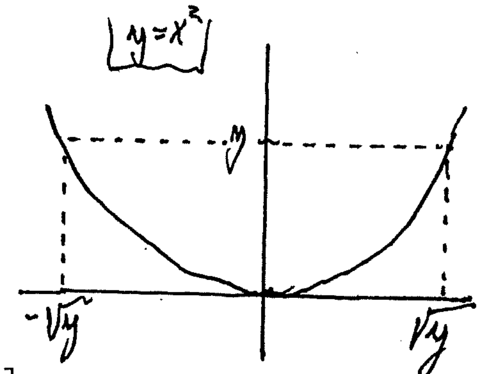
Example 2.2b. *How to generate a rv Y with a pre-specified pdf f:*

Solution: Let F be the cdf corresponding to f. Use a computer to generate $X \sim$ Uniform$(0,1)$ and set $Y = F^{-1}(X)$. Then Y has cdf F. [Verify] $\square$

Note: If g is strictly decreasing then (2.1) remains true with $g'(g^{-1}(y))$ replaced by $|g'(g^{-1}(y))|$ [Verify].

Now suppose that g is *not monotone.* Still $f_Y(y) = \frac{d}{dy}P[g(X) \le y]$, but the region $\{x \mid g(x) \le y\}$ must be determined before proceeding.

Example 2.3. Again let $Y = X^2$, but now suppose that the range of X is $(-\infty, \infty)$. Then for $y > 0$,

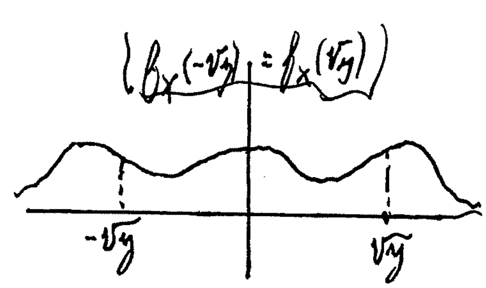


$$\begin{aligned} f_Y(y) &= \frac{d}{dy}P[Y \le y] \\ &= \frac{d}{dy}P[X^2 \le y] \\ &= \frac{d}{dy}P[-\sqrt{y} \le X \le \sqrt{y}] \\ &= \frac{d}{dy}\Big[F_X(\sqrt{y}) - F_X(-\sqrt{y})\Big] \end{aligned}$$

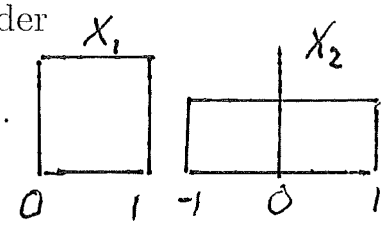
(2.5) $$= \frac{1}{2\sqrt{y}}\Big[f_X(\sqrt{y}) + f_X(-\sqrt{y})\Big].$$

If in addition the distribution of X is symmetric about 0, i.e., $f_X(x) = f_X(-x)$, then (2.5) reduces to

(2.6) $$f_Y(y) = \frac{1}{\sqrt{y}} f_X(\sqrt{y}).$$



Note that this is similar to (2.3) (where the range of X was restricted to $(0, \infty)$) but without the factor $\frac{1}{2}$. To understand this, consider



$$X_1 \sim \text{Uniform}(0,1), \qquad X_2 \sim \text{Uniform}(-1,1).$$

Then

$$f_{X_1}(x) = I_{(0,1)}(x), \qquad f_{X_2}(x) = \tfrac{1}{2} I_{(-1,1)}(x),$$

but $Y_i = X_i^2$ has pdf $\frac{1}{2\sqrt{y}} I_{(0,1)}$ for $i = 1, 2$. [Verify – recall (2.4)]. □

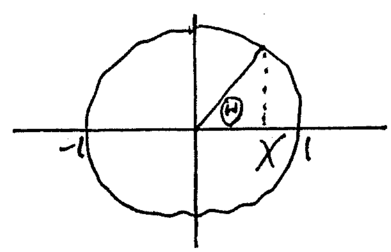
Example 2.4. Let $X \sim N(0,1)$, i.e., $f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ and $Y = X^2$. Then by (2.6),

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2} I_{(0,\infty)}(y), \tag{2.7}$$

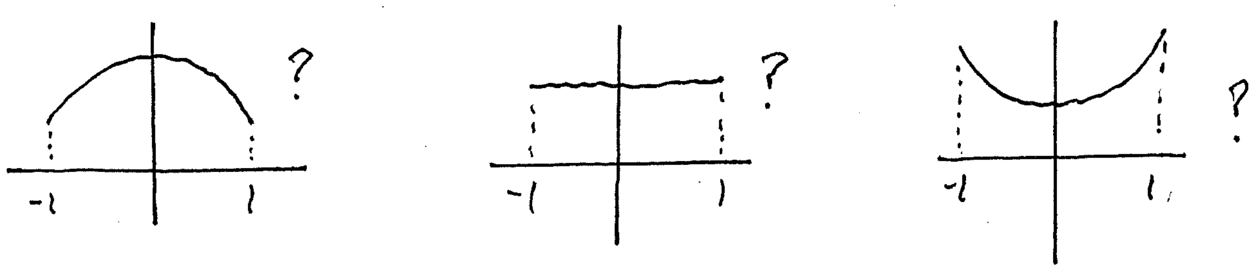
the Gamma$(\frac{1}{2}, \frac{1}{2})$ pdf, which is also called the *chi-square pdf with one degree of freedom,* denoted as χ_1^2 (see Remark 6.3). Note that (2.7) shows that

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}. \tag{2.8}$$

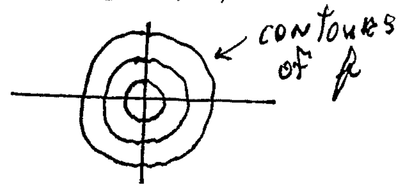
Exercise 2.1. Let $\Theta \sim \text{Uniform}(0, 2\pi)$, so we may think of Θ as a *random angle.* Define $X = \cos\Theta$. Find the pdf f_X.



Hint: Always begin by specifying the range of X, which is $[-1, 1]$ here. On this range, what shape do you expect f_X to have, among the following three possibilities? (Compare this f_X to that in Example 1.12, p. 11.) □



Exercise 2.1 suggests the following problem. A bivariate pdf $f_{X,Y}$ on $\mathbf{R}^2$ is called *radial* if it has the form



$$f_{X,Y}(x,y) = g(x^2 + y^2) \tag{2.9}$$

for some (non-negative) function g on $(0, \infty)$. Note that the condition

$$\iint_{\mathbf{R}^2} f(x,y)\,dxdy = 1$$

requires that

$$\int_0^\infty r g(r^2)\,dr = \frac{1}{2\pi} \qquad \text{[why?]} \tag{2.10}$$

Exercise 2.2*. (a) Does there exist a radial pdf $f_{X,Y}$ on the unit disk in $\mathbf{R}^2$ such that the marginal distribution of X is Uniform$(-1,1)$? More precisely, does there exist g on $(0,1)$ that satisfies (2.10) and

$$f_X(x) \equiv \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} g(x^2+y^2)\,dy = \tfrac{1}{2} I_{(-1,1)}(x)? \tag{2.11}$$

(b) What about $\mathbf{R}^3$? $\mathbf{R}^4$?

Note: if such a radial pdf $f_{X,Y}$ on the unit disk exists, it could be called a *bivariate uniform distribution*, since both X and Y (by symmetry) have the Uniform$(-1,1)$ distribution. Of course, there are simpler bivariate distributions with these uniform marginal distributions but which are not radial on the unit disk. (Can you think of two?) [Also see §26.] □

2.2. One function of two or more random variables.

Let (X, Y) be a *continuous* bivariate rvtr with pdf $f_{X,Y}$ on a subset of $\mathbf{R}^2$. Define a new rv

$$U = g(X, Y), \text{ e.g., } U = X + Y, \;\; X - Y, \;\; \frac{X}{Y}, \;\; \frac{1 + \exp(X+Y)}{1 + \exp(X-Y)}.$$

Then the pdf f_U can be determined via two methods:

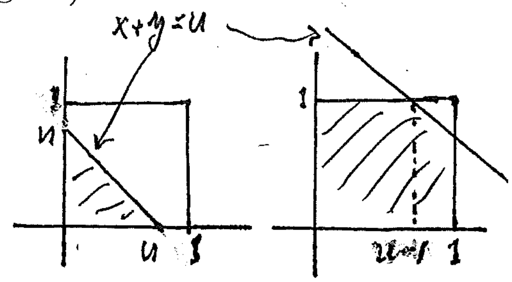
Method One: Apply

$$f_U(u) = \frac{d}{du} F_U(u) = \frac{d}{du} P[U \le u] = \frac{d}{du} P[g(X,Y) \le u], \tag{2.12}$$

then determine the region $\{(X,Y) \mid g(X,Y) \le u\}$.

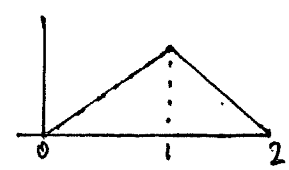
Example 2.5. Let (X, Y) be uniformly distributed on the unit square. [Note that this is equivalent to assuming that X and Y are independent Uniform$(0,1)$ rvs – why?] To find the pdf f_U of $U = X + Y$, begin by noting that the range of U is the interval $[0, 2]$. Then (see Figure)

$$P[X + Y \le u] = \begin{cases} \frac{1}{2}u^2, & 0 < u < 1; \\ 1 - \frac{1}{2}(2-u)^2, & 1 < u < 2; \end{cases}$$



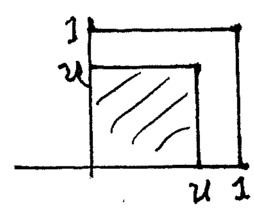
so

$$f_U(u) = \begin{cases} u, & 0 < u < 1; \\ 2 - u, & 1 < u < 2. \end{cases} \tag{2.13}$$



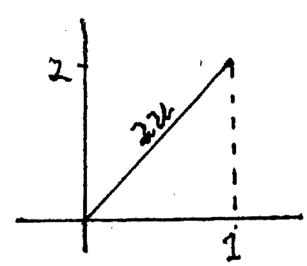
Next let $U = \max(X, Y)$. The range of U is $(0, 1)$. For $0 < u < 1$,

$$\begin{aligned} P[\max(X,Y) \le u] &= P[X \le u, \; Y \le u] \\ &= P[X \le u] P[Y \le u] \\ &= u^2, \end{aligned}$$

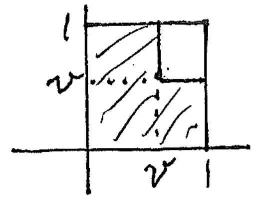


so

$$f_U(u) = 2u I_{(0,1)}(u). \tag{2.14}$$

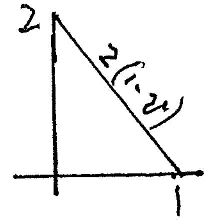


Finally, let $V = \min(X, Y)$. Again the range of V is $[0, 1]$, and for $0 < v < 1$,



$$
\begin{aligned}
P[\min(X,Y) \le v] &= 1 - P[\min(X,Y) > v] \\
&= 1 - P[X > v,\ Y > v] \\
&= 1 - P[X > v]P[Y > v] \\
&= 1 - (1-v)^2,
\end{aligned}
$$

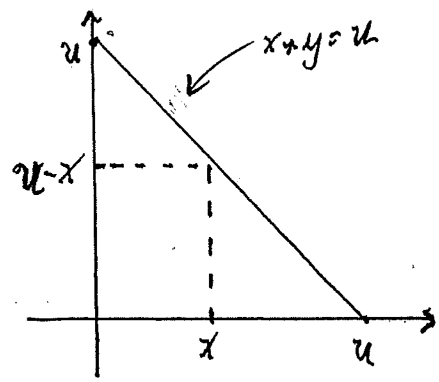
so



$$
f_V(v) = 2(1-v)I_{(0,1)}(v). \tag{2.15}
$$

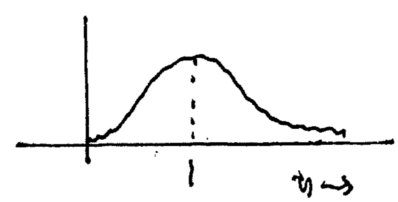
Exercise 2.3*. Let X, Y, Z be independent, identically distributed (i.i.d.) Uniform(0,1) rvs. Find the pdf of $U \equiv X + Y + Z$. [What is range(U)?]

Example 2.6. Let X, Y be i.i.d. Exponential(1) rvs and set $U = X + Y$. Then for $0 < u < \infty$,



$$
\begin{aligned}
P[X + Y \le u] &= \iint_{x+y\le u} e^{-x-y}\,dx\,dy \\
&= \int_0^u e^{-x}\Big[\int_0^{u-x} e^{-y}dy\Big]dx \\
&= \int_0^u e^{-x}\Big[1 - e^{-(u-x)}\Big]dx \\
&= \int_0^u e^{-x}dx - e^{-u}\int_0^u dx \\
&= 1 - e^{-u} - ue^{-u},
\end{aligned}
$$

so $U \sim$ Gamma(2, 1), since



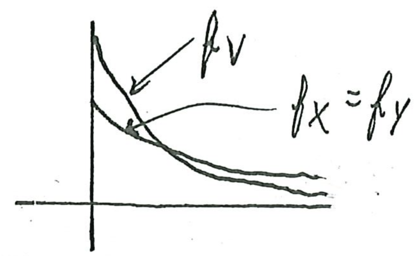
$$
f_U(u) = \frac{d}{du}[1 - e^{-u} - ue^{-u}] = ue^{-u}. \tag{2.16}
$$

Next let $V = \min(X, Y)$. Then for $0 < v < \infty$,

$$
\begin{aligned}
P[\min(X,Y) \le v] &= 1 - P[\min(X,Y) > v] \\
&= 1 - P[X > v,\ Y > v] \\
&= 1 - \Big[\int_v^\infty e^{-x}dx\Big]\Big[\int_v^\infty e^{-y}dy\Big] \\
&= 1 - e^{-2v},
\end{aligned}
$$

so

(2.17) $$f_V(v) = 2e^{-2v},$$



that is, $V \equiv \min(X, Y) \sim \text{Exponential}(2)$.

More generally: If $X_1, \ldots, X_n$ are i.i.d. Exponential(λ) rvs, then [verify!]

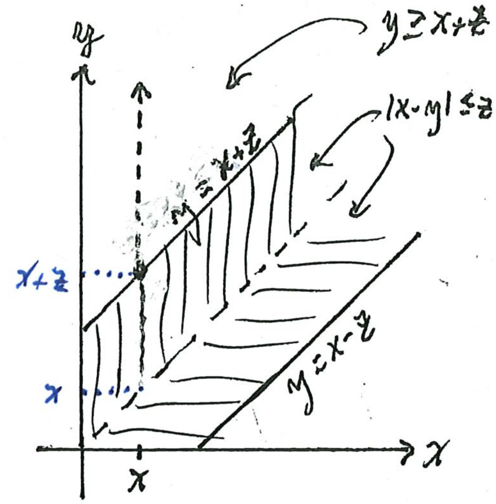
(2.18) $$\min(X_1, \ldots, X_n) \sim \text{Exponential}(n\lambda).$$

However: if $T = \max(X, Y)$, then T is *not* an exponential rv [verify!]:

(2.19) $$f_T(t) = 2\big(e^{-t} - e^{-2t}\big).$$

Now let $Z = |X - Y| \equiv \max(X, Y) - \min(X, Y)$. The range of Z is $(0, \infty)$. For $0 < z < \infty$,

$$\begin{aligned}
&P[|X - Y| \le z] \\
=&1 - P[Y \ge X + z] - P[Y \le X - z] \\
=&1 - 2P[Y \ge X + z] \text{ [by symmetry]} \\
=&1 - 2\int_0^\infty e^{-x}\Big[\int_{x+z}^\infty e^{-y}dy\Big]dx \\
=&1 - 2\int_0^\infty e^{-x}e^{-(x+z)}dx \\
=&1 - 2e^{-z}\int_0^\infty e^{-2x}dx \\
=&1 - e^{-z},
\end{aligned}$$



so

(2.20) $$f_Z(z) = e^{-z}.$$

That is, $Z \equiv \max(X, Y) - \min(X, Y) \sim \text{Exponential}(1)$, the same as X and Y themselves.

Note: It will be shown in Example 6.2 that $V \perp\!\!\!\perp Z$, so we have another "memory-free" property of the exponential distribution. It is stronger in that it involves a *random* starting time, namely $\min(X, Y)$.

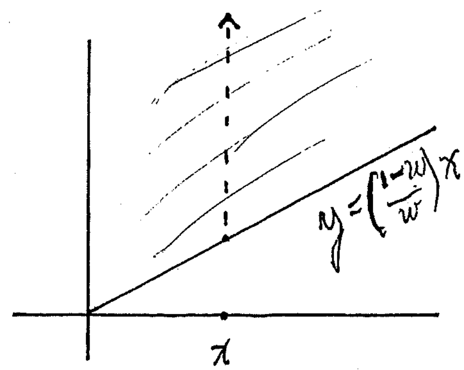
Exercise 2.4. From the above,

$$\mathrm{E}[\max(X,Y)] = \mathrm{E}[\min(X,Y)] + \mathrm{E}|X-Y| = \tfrac{1}{2} + 1.$$

Find $\mathrm{E}[\max(X_1, \ldots, X_n)]$, where the X_i are i.i.d. Exponential(1) rvs. □

Finally, let $W = \frac{X}{X+Y}$. The range of W is $(0,1)$. For $0 < w < 1$,

$$\begin{aligned}
&P\Big[\frac{X}{X+Y} \le w\Big] \\
=&P[X \le w(X+Y)] \\
=&P\Big[Y \ge \Big(\frac{1-w}{w}\Big)X\Big] \\
=&\int_0^\infty \Big[\int_{\left(\frac{1-w}{w}\right)x}^\infty e^{-y}dy\Big] e^{-x}dx \\
=&\int_0^\infty \Big[e^{-\left(\frac{1-w}{w}\right)x} e^{-x}dx\Big] \\
=&\int_0^\infty e^{-\frac{x}{w}}dx \\
=&\, w,
\end{aligned}$$



so

(2.21) $\quad f_W(w) = I_{(0,1)}(w),$

that is, $W \equiv \frac{X}{X+Y} \sim \text{Uniform}(0,1)$. □

Note: In Example 6.3 we shall show that $\frac{X}{X+Y} \perp\!\!\!\perp (X+Y)$. Then (2.21) can be viewed as a "backward" memory-free property of the exponential distribution: given $X+Y$, the location of X is uniformly distributed over the interval $(0, X+Y)$.

Method Two: Introduce a second rv $V = h(X,Y)$, where h is chosen cleverly so that it is relatively easy to find the joint pdf $f_{U,V}$ via the "Jacobian method", then marginalize to find f_U. (This method appears in §6.2.)

3. Expected Value of a RV: Mean, Variance, Covariance; Moment Generating Function; Normal & Poisson Approximations.

3.1. The **expected value** (expectation, mean) of a rv X is defined by

$$\text{(3.1)} \qquad \mathrm{E}X = \sum_x x f_X(x), \qquad \text{[discrete case]}$$

$$\text{(3.2)} \qquad \mathrm{E}X = \int x f_X(x)\,dx, \qquad \text{[continuous case]}$$

provided that the sum or integral exists, that is, is not $= \infty - \infty$.

The *Law of Large Numbers* states that if EX exists (possibly infinite), then for i.i.d. copies $X_1, X_2, \ldots,$ of X the sample averages $\bar{X}_n \equiv \frac{1}{n}\sum_{i=1}^n X_i$ converge to EX as $n \to \infty$. If the sum or integral does not exist then $\bar{X}_n$ will oscillate indefinitely [verify? what if asymmetric?]

If the probability distribution of X is thought of as a (discrete or continuous) mass distribution on $\mathbf{R}^1$, then EX is just the *center of gravity* of the mass. With this interpretation, we can often use *symmetry* to find the expected value without actually calculating the sum or integral; however, absolute convergence still must be verified! [Eg. Cauchy distribution]

Example 3.1: [verify, including convergence; for Var see (3.9) and (3.10)]

$$X \sim \text{Binomial}(n,p) \Rightarrow \mathrm{E}X = np,\ \ \mathrm{Var}X = np(1-p); \qquad \text{[sum]}$$
$$X \sim \text{Geometric}(p) \Rightarrow \mathrm{E}X = 1/p,\ \ \mathrm{Var}X = (1-p)/p^2; \qquad \text{[sum]}$$
$$X \sim \text{Poisson}(\lambda) \Rightarrow \mathrm{E}X = \lambda,\ \ \mathrm{Var}X = \lambda; \qquad \text{[sum]}$$
$$X \sim \text{Exponential}(\lambda) \Rightarrow \mathrm{E}X = 1/\lambda,\ \ \mathrm{Var}X = 1/\lambda^2; \qquad \text{[integrate]}$$
$$X \sim \text{Normal } N(0,1) \Rightarrow \mathrm{E}X = 0,\ \ \mathrm{Var}X = 1; \qquad \text{[symmetry, integrate]}$$
$$X \sim \text{Cauchy } C(0,1) \Rightarrow \mathrm{E}X \text{ and } \mathrm{Var}X \text{ do not exist;}$$
$$X \sim \text{Gamma}(\alpha,\lambda) \Rightarrow \mathrm{E}X = \alpha/\lambda, \mathrm{Var}X = \alpha/\lambda^2; \qquad \text{[integrate]}$$
$$X \sim \text{Beta}(\alpha,\beta) \Rightarrow \mathrm{E}X = \tfrac{\alpha}{\alpha+\beta},\ \mathrm{Var}X = \tfrac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \qquad \text{[integrate]}$$
$$X \sim \text{std. Logistic} \Rightarrow \mathrm{E}X = 0 \qquad \text{[symmetry]}$$
$$X \sim \text{Uniform}(a,b) \Rightarrow \mathrm{E}X = \frac{a+b}{2}, \mathrm{Var}X = \frac{(b-a)^2}{12}; \qquad \text{[symm., integ.]}$$

The expected value $E[g(X)]$ of a function of a rv X is defined similarly:

$$(3.3) \qquad E[g(X)] = \sum_x g(x) f_X(x), \qquad \text{[discrete case]}$$

$$(3.4) \qquad E[g(X)] = \int g(x) f_X(x)\, dx, \qquad \text{[continuous case]}$$

In particular, the *rth moment* of X (if it exists) is defined as $E(X^r)$, $r \in \mathbf{R}^1$.

Expectations of functions of random vectors are defined similarly. For example in the bivariate case,

$$(3.5) \quad E[g(X,Y)] = \sum_x \sum_y g(x,y) f_{X,Y}(x,y), \qquad \text{[discrete case]}$$

$$(3.6) \quad E[g(X,Y)] = \int\int g(x,y) f_{X,Y}(x,y)\, dxdy, \quad \text{[continuous case]}$$

Linearity: It follows from (3.5) and (3.6) that expectation is *linear*:

$$(3.7) \; E[ag(X,Y) + bh(X,Y)] = aE[g(X,Y)] + bE[h(X,Y)]. \qquad \text{[verify]}$$

Order-preserving: $X \geq 0 \Rightarrow EX \geq 0$ (and $EX = 0$ iff $X \equiv 0$).
$X \geq Y \Rightarrow EX \geq EY$ (and $EX = EY$ iff $X \equiv Y$). [Pf: $EX - EY = E(X - Y)$]

Linearity ($\equiv$ *additivity*) simplifies many calculations:

Binomial mean: We can find the expected value of $X \sim \text{Binomial}(n,p)$ easily as follows: Because X is the total number of successes in n independent *Bernoulli trials*, i.e., trials with exactly two outcomes (H,T, or S,F, etc.), we can represent X as

$$(3.8) \qquad X = X_1 + \cdots + X_n,$$

where $X_i = 1$ (or 0) if S (or F) occurs on the ith trial. (Recall Example 1.15.) Thus by linearity,

$$EX = E(X_1 + \cdots + X_n) = EX_1 + \cdots + EX_n = p + \cdots + p = np.$$

Variance. The *variance* of X is defined to be

$$\text{Var}X = \text{E}[(X - \text{E}X)^2], \tag{3.9}$$

the average of the square of the deviation of X about its mean. The *standard deviation* of X is

$$\text{sd}\,(X) = \sqrt{\text{Var}X}.$$

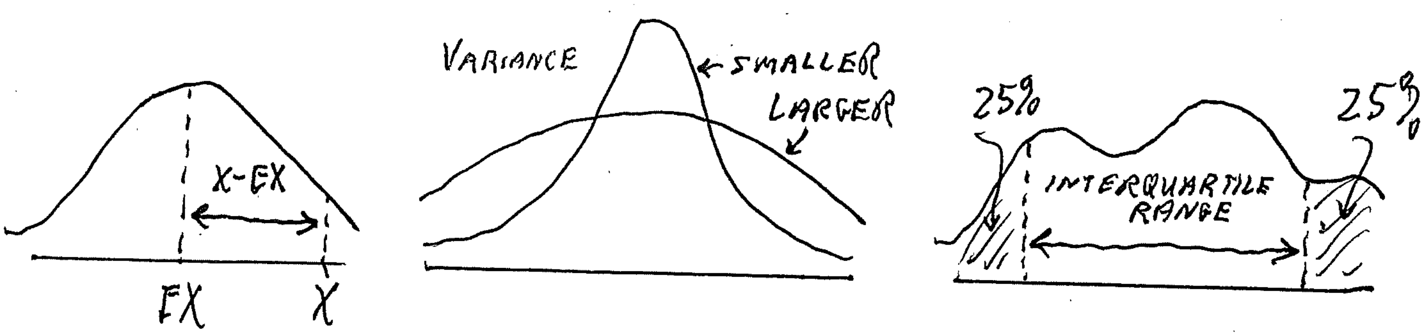
Properties of Variance.

(a) Var $X \geq 0$; equality holds iff X is degenerate (constant).

(b) *location – scale :*
$$\text{Var}(aX + b) = a^2\,\text{Var}X;$$
$$\text{sd}\,(aX + b) = |a| \cdot \text{sd}\,(X).$$

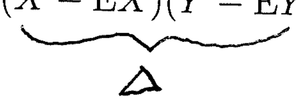
(c)
$$\begin{aligned} \text{Var}X \equiv \text{E}[(X - \text{E}X)^2] &= \text{E}[X^2 - 2X\text{E}X + (\text{E}X)^2] \\ &= \text{E}(X^2) - 2(\text{E}X)(\text{E}X) + (\text{E}X)^2 \\ &= \text{E}(X^2) - (\text{E}X)^2. \qquad (3.10) \end{aligned}$$

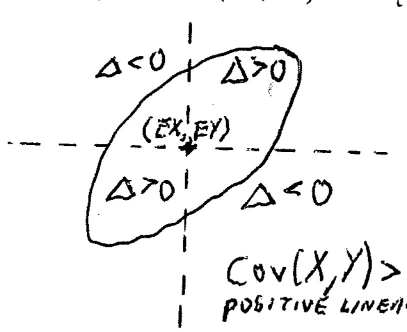
The standard deviation is a measure of the *spread* $\equiv$ *dispersion* of the distribution of X about its mean value. An alternative measure of spread is $\text{E}[|X - \text{E}X|]$. Another measure of spread is the difference between the 75th and 25th *percentiles* of the distribution of X.

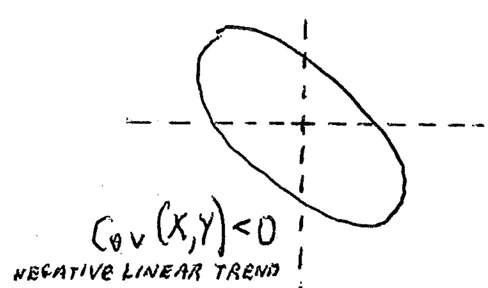


Covariance: The *covariance* between X and Y indicates the nature of the *linear* dependence (if any) between X and Y:

(3.11) $\text{Cov}(X, Y) = \text{E}[(X - \text{E}X)(Y - \text{E}Y)]$. [interpret; also see §4]







Properties of covariance:

(a) $\text{Cov}(X,\ Y) = \text{Cov}(Y,\ X)$.

(b)
$$\begin{aligned}\text{Cov}(X,Y) &= \text{E}[XY - X\text{E}Y - Y\text{E}X + (\text{E}X)(\text{E}Y)] \\ &= \text{E}(XY) - 2(\text{E}X)(\text{E}Y) + (\text{E}X)(\text{E}Y) \\ &= \text{E}(XY) - (\text{E}X)(\text{E}Y). \qquad (3.12)\end{aligned}$$

(c) $\text{Cov}(X,\ X) = \text{Var}X$.

(d) If X or Y is a degenerate rv (a constant), then $\text{Cov}(X,\ Y) = 0$.

(e) *Bilinearity:* $\text{Cov}(aX,\ bY + cZ) = ab\,\text{Cov}(X,\ Y) + ac\,\text{Cov}(X,\ Z)$.

$$\text{Cov}(aX + bY,\ cZ) = ac\,\text{Cov}(X,\ Z) + bc\,\text{Cov}(Y,\ Z).$$

(f) *Variance of a sum or difference:*

(3.13) $\text{Var}(X \pm Y) = \text{Var}X + \text{Var}Y \pm 2\,\text{Cov}(X,Y)$. [verify]

(g) *Product rule.* If X and Y are *independent* it follows from (1.35), (3.5) and (3.6) that

(3.14) $\text{E}[g(X)h(Y)] = \text{E}[g(X)] \cdot \text{E}[h(Y)]$. [verify]

Thus, by (3.12) and (3.13),

(3.15) $X \perp\!\!\!\perp Y \Rightarrow \text{Cov}(X,Y) = 0$,

(3.16) $X \perp\!\!\!\perp Y \Rightarrow \text{Var}(X \pm Y) = \text{Var}X + \text{Var}Y$.

Exercise 3.1. Show by counterexample that the converse of (3.15) is not true. [Example 1.12 provides one counterexample: Suppose that (X,Y) is uniformly distributed over the unit disk D. Then by the symmetry of D, $(X,Y) \sim (X,-Y)$. Thus $\text{Cov}(X,Y) = \text{Cov}(X,-Y) = -\text{Cov}(X,Y)$, so $\text{Cov}(X,Y) = 0$. But we have already seen that $X \not\perp\!\!\!\perp Y$.] □

Binomial variance: We can find the variance of $X \sim \text{Binomial}(n,p)$ easily as follows (recall Example 1.12):

$$\begin{aligned}\text{Var}X &= \text{Var}(X_1 + \cdots + X_n) && \text{[by (3.8)]} \\ &= \text{Var}X_1 + \cdots + \text{Var}X_n && \text{[by (3.16)]} \\ &= p(1-p) + \cdots + p(1-p) && \text{[by (3.10)]} \\ &= np(1-p). && (3.17)\end{aligned}$$

Variance of a sample average $\equiv$ *sample mean:* Let $X_1, \ldots, X_n$ be i.i.d. rvs, each with mean μ and variance $\sigma^2 < \infty$ and set $\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n)$. Then by linearity and independence,

$$(3.18) \qquad \mathrm{E}(\bar{X}_n) = \mu, \qquad \mathrm{Var}(\bar{X}_n) = \frac{\mathrm{Var}(X_1 + \cdots + X_n)}{n^2} = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}.$$

Theorem 3.1. The Weak Law of Large Numbers (WLLN).

Let $X_1, \ldots, X_n$ be i.i.d. rvs, each with mean μ and variance $\sigma^2 < \infty$. Then $\bar{X}_n$ converges to μ in probability ($\bar{X}_n \xrightarrow{p} \mu$), that is, for each $\epsilon > 0$,

$$P[|\bar{X}_n - \mu| \leq \epsilon] \to 1 \quad \text{as } n \to \infty.$$

Proof. By Chebyshev's Inequality (below) and (3.18),

$$P[|\bar{X}_n - \mu| \geq \epsilon] \leq \frac{\mathrm{Var}(\bar{X}_n)}{\epsilon^2} = \frac{\sigma^2}{n\epsilon^2} \to 0 \quad \text{as } n \to \infty.$$

Lemma 3.1. Chebyshev's Inequality. *Let* $\mathrm{E}Y = \nu$, $\mathrm{Var}Y = \tau^2$. Then

$$(3.19) \qquad P[|Y - \nu| \geq \epsilon] \leq \frac{\tau^2}{\epsilon^2}.$$

Proof. Let $X = Y - \nu$, so $\mathrm{E}(X) = 0$. Assume that X is continuous with pdf f. (The discrete case is similar, with sums replacing integrals.) Then

$$\begin{aligned}\tau^2 \equiv \mathrm{E}(X^2) &= \int_{|x| \geq \epsilon} x^2 f(x)dx + \int_{|x| < \epsilon} x^2 f(x)dx \\ &\geq \int_{|x| \geq \epsilon} \epsilon^2 f(x)dx = \epsilon^2 P[|X| \geq \epsilon].\end{aligned}$$

Example 3.2. Sampling without replacement - the hypergeometric distribution.

Suppose an urn contains r red balls and w white balls. Draw n balls at random from the urn and let X denote the number of red balls obtained. If the balls are sampled *with* replacement, then clearly $X \sim \mathrm{Binomial}(n, p)$, where $p = r/(r + w)$, so $\mathrm{E}X = np$, $\mathrm{Var}X = np(1 - p)$.

Suppose, however, that the balls are sampled *without* replacement. Note that we now require that $n \leq r + w$. The probability distribution of X is described as follows: its range is $\max(0, n - w) \leq x \leq \min(r, n)$ [why?], and its pmf is given by

$$(3.20) \qquad P[X = x] = \frac{\binom{r}{x}\binom{w}{n-x}}{\binom{r+w}{n}}, \qquad \max(0, n - w) \leq x \leq \min(r, n).$$

[Verify the range and verify the pmf. This probability distribution is called *hypergeometric* because these ratios of binomial coefficients occurs as the coefficients in the expansion of hypergeometric functions.]

To determine EX and Var$X$, rather than using (3.20) it is easier again to use the representation

$$X = X_1 + \cdots + X_n,$$

where, as in the binominal case, $X_i = 1$ (or 0) if a red (or white) ball is obtained on the ith trial. Here, however, clearly $X_1, \ldots, X_n$ are *not mutually independent.* [Why?] Nevertheless, the joint distribution of $X \equiv (X_1, \ldots, X_n)$ is *exchangeable* $\equiv$ *symmetric* $\equiv$ *permutation-invariant*:

Definition 3.1. A random vector $Y \equiv (Y_1, \ldots, Y_n)$ is *exchangeable* if

$$(Y_1, \ldots, Y_n) \sim (Y_{i_1}, \ldots, Y_{i_n})$$

for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$. Equivalently,

$$f(y_1, \ldots, y_n) = f(y_{i_1}, \ldots, y_{i_n}) \quad \forall\ (i_1, \ldots, i_n). \qquad \square$$

It suffices to show that $(X_1, \ldots, X_n)$ is exchangeable when $n = r + w$ [why?]. Here, if $x_1 + \cdots + x_n = r$ with each $x_i = 0$ or 1 then

$$P[(X_1, \ldots, X_n) = (x_1, \ldots, x_n)] = \frac{1}{\binom{n}{r}} = \frac{1}{\binom{n}{x_1 + \cdots + x_n}},$$

which is obviously invariant under all permutations of $x_1, \ldots, x_n$.

It follows from exchangeability that $X_1 \sim \cdots \sim X_n$. In particular,

$$\begin{aligned} P[X_2 = 1] &= P[X_2 = 1 | X_1 = 1] P[X_1 = 1] + P[X_2 = 1 | X_1 = 0] P[X_1 = 0] \\ &= \frac{r-1}{r+w-1} \cdot \frac{r}{r+w} + \frac{r}{r+w-1} \cdot \frac{w}{r+w} \\ (3.21) \qquad &= \frac{r}{r+w} \equiv P[X_1 = 1], \end{aligned}$$

so $X_1 \sim X_2$. Note too that since X_1X_2 has range $\{0,1\}$,

$$\begin{aligned}
\mathrm{E}(X_1X_2) &= P[X_1X_2 = 1] \\
&= P[X_1 = 1, X_2 = 1] \\
&= P[X_2 = 1 | X_1 = 1] P[X_1 = 1] \\
&= \frac{r-1}{r+w-1} \frac{r}{r+w},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Cov}(X_1, X_2) &= \mathrm{E}(X_1X_2) - (\mathrm{E}X_1)(\mathrm{E}X_2) \\
&= \frac{r-1}{r+w-1} \frac{r}{r+w} - \left[\frac{r}{r+w}\right]^2 \\
&= \frac{r}{r+w} \left[\frac{r-1}{r+w-1} - \frac{r}{r+w}\right] \\
&= \frac{-rw}{(r+w)^2(r+w-1)}. \qquad (3.22)
\end{aligned}$$

[Thus X_1 and X_2 are *negatively correlated*, which is intuitively clear. [Why?]

By (3.21) and exchangeability, $\mathrm{E}X_i = \frac{r}{r+w} \equiv p$ for $i = 1, \ldots, n$, so

$$\mathrm{E}X = \mathrm{E}X_1 + \cdots \mathrm{E}X_n = n\left(\frac{r}{r+w}\right) \equiv np, \qquad (3.23)$$

the same as for sampling with replacement.

Key question: do we expect $\mathrm{Var}X$ also to be the same as for sampling with replacement, namely, $np(1-p)$? Larger? Smaller?

Answer: By (3.22) and exchangeability,

$$\begin{aligned}
\mathrm{Var}X &= \sum_{i=1}^{n} \mathrm{Var}X_i + 2 \sum_{1 \le i < j \le n} \mathrm{Cov}(X_i, X_j) \\
&= np(1-p) + n(n-1)\mathrm{Cov}(X_1, X_2) \\
&= n\frac{r}{r+w}\frac{w}{r+w} + n(n-1)\left[\frac{-rw}{(r+w)^2(r+w-1)}\right] \\
&= \frac{nrw}{(r+w)^2}\left[1 - \frac{n-1}{r+w-1}\right] \\
&= np(1-p)\left[1 - \frac{n-1}{N-1}\right], \qquad (3.24)
\end{aligned}$$

where $N \equiv r + w$ is the total number of balls in the urn. [Discuss $\frac{n-1}{N-1}$.]

By comparing (3.24) to $np(1-p)$, we see that *sampling without replacement from a finite population reduces the variability of the outcome.* This is to be expected from the representation $X = X_1 + \cdots + X_n$ and the fact that each pair (X_i, X_j) is *negatively correlated* (by (3.22) and exchangeability).

3.2. Correlated and conditionally correlated events.

The events A and B are *positively correlated* if any of the following five equivalent [verify!] conditions hold:

$$P[A \cap B] > P[A]P[B]; \tag{3.25}$$
$$P[A \mid B] > P[A]; \tag{3.26}$$
$$P[B \mid A] > P[B]; \tag{3.27}$$
$$P[A \mid B] > P[A \mid B^c]; \tag{3.28}$$
$$P[B \mid A] > P[B \mid A^c]. \tag{3.29}$$

Because $P(C) = E(I_C)$, (3.25) is equivalent to $\text{Cov}(I_A, I_B) > 0$. Negative correlation is defined similarly with $>$ replaced by $<$.

The events A and B are *(conditionally) positively correlated given* C if any of the following five equivalent [verify!] conditions hold:

$$P[A \cap B \mid C] > P[A \mid C]P[B \mid C]; \tag{3.30}$$
$$P[A \mid B,\, C] > P[A \mid C]; \tag{3.31}$$
$$P[B \mid A,\, C] > P[B \mid C]; \tag{3.32}$$
$$P[A \mid B,\, C] > P[A \mid B^c,\, C]; \tag{3.33}$$
$$P[B \mid A,\, C] > P[B \mid A^c,\, C]. \tag{3.34}$$

Example 3.3: Simpson's paradox.

$$\left\{ \begin{array}{c} P[A \mid B,\, C] > P[A \mid B^c,\, C] \\ P[A \mid B,\, C^c] > P[A \mid B^c,\, C^c] \end{array} \right\} \not\Rightarrow P[A \mid B] > P[A \mid B^c]\ ! \tag{3.35}$$

To see this, consider the famous Berkeley Graduate School Admissions data. To simplify, assume there are only two graduate depts, Physics and English.

Physics	Accept	Reject		
Female	60	40	$P[A \mid F,\ Ph] = 0.6$	(1)
Male	50	50	$P[A \mid M,\ Ph] = 0.5$	(3)

English	Accept	Reject		
Female	250	750	$P[A \mid F,\ En] = 0.25$	(2)
Male	20	80	$P[A \mid M,\ En] = 0.2$	(4)

Total	Accept	Reject	
Female	310	790	$P[A \mid F] = 0.28$
Male	70	130	$P[A \mid M] = 0.35$

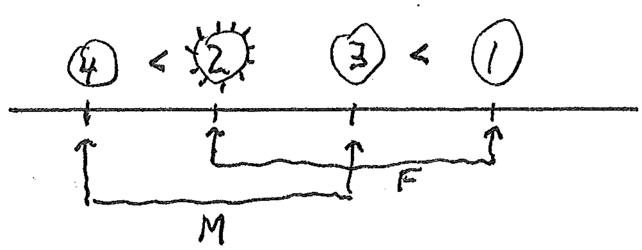
$$\text{Note :} \qquad P[A \mid F] < P[A \mid M].$$

Is this evidence of gender discrimination?

$$\text{No :} \quad \begin{array}{l} P[A \mid F,\ Ph] > P[A \mid M,\ Ph], \\ P[A \mid F,\ En] > PA \mid M,\ En], \end{array}$$

so F's are more likely to be accepted into each dept Ph and En separately!

Explanation: Most F's applied to English, where the acceptance rate is low:



$$P[A \mid F] = P[A \mid F,\ Ph]P[Ph \mid F] + P[A \mid F,\ En]P[En \mid F],$$

$$P[A \mid M] = P[A \mid M,\ Ph]P[Ph \mid M] + P[A \mid M,\ En]P[En \mid M]. \qquad \square$$

Exercise 3.2. Show that the implication (3.35) does hold if $B \perp\!\!\!\perp C$. $\square$

3.3. Moment generating functions: they uniquely determine moments, distributions, and convergence of distributions.

The *moment generating function (mgf) of (the distribution of)* the rv X is:

(3.36) $$m_X(t) = \mathrm{E}\big(e^{tX}\big), \qquad -\infty < t < \infty.$$

Clearly $m_X(0) = 1$ and $0 < m_X(t) \le \infty$, with ∞ possible. If $m_X(t) < \infty$ for $|t| < \delta$ then the Taylor series expansion of e^{tX} yields

(3.37) $$m_X(t) = \mathrm{E}\Big[\sum_{k=0}^{\infty} \frac{(tX)^k}{k!}\Big] = \sum_{k=0}^{\infty} \frac{t^k}{k!}\mathrm{E}(X^k), \quad |t| < \delta,$$

a power series whose coefficients are the kth moments of X. In this case the moments of X are recovered from the mgf m_X by differentiation at $t = 0$:

(3.38) $$\mathrm{E}(X^k) = m_X^{(k)}(0), \quad k = 1, 2, \ldots. \qquad \text{[verify]}$$

Location-scale:

(3.39) $$m_{aX+b}(t) = \mathrm{E}\big(e^{t(aX+b)}\big) = e^{bt}\mathrm{E}\big(e^{atX}\big) = e^{bt}m_X(at).$$

Multiplicativity: X, Y *independent* $\Rightarrow$

(3.40) $$m_{X+Y}(t) = \mathrm{E}\big(e^{t(X+Y)}\big) = \mathrm{E}\big(e^{tX}\big)\mathrm{E}\big(e^{tY}\big) = m_X(t)m_Y(t).$$

In particular, if $X_1, \ldots, X_n$ are i.i.d. then

(3.41) $$m_{X_1+\cdots+X_n}(t) = [m_{X_1}(t)]^n.$$

Example 3.4.

Bernoulli (p): Let $X = \begin{cases} 1, & \text{with probability } p; \\ 0, & \text{with probability 1-p.} \end{cases}$ Then

(3.42) $$m_X(t) = pe^t + (1-p).$$

Binomial (n, p): We can represent $X = X_1 + \cdots + X_n$, where $X_1, \ldots, X_n$ are i.i.d. Bernoulli(p) 0-1 rvs. Then by (3.41) and (3.42),

(3.43) $$m_X(t) = \big[pe^t + (1-p)\big]^n.$$

Poisson (λ):

$$m_X(t) = \sum_{k=0}^{\infty} e^{tk}\, e^{-\lambda} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}. \tag{3.44}$$

Standard univariate normal $N(0,1)$:

$$\begin{aligned} m_X(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{x^2}{2}}\, dx \\ &= e^{t^2/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\, dx \\ &= e^{\frac{t^2}{2}}. \end{aligned} \tag{3.45}$$

General univariate normal $N(\mu, \sigma^2)$: We can represent $X = \sigma Z + \mu$ where $Z \sim N(0,1)$. Then by (3.39) (location-scale) and (3.45),

$$m_X(t) = e^{\mu t} m_Z(\sigma t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}. \tag{3.46}$$

Gamma(α, λ) *(includes Exponential*(λ) *when* $\alpha = 1$*)*:

$$\begin{aligned} m_X(t) &= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^{\infty} e^{tx} \cdot x^{\alpha-1} e^{-\lambda x}\, dx \\ &= \frac{\lambda^\alpha}{(\lambda - t)^\alpha} \cdot \frac{(\lambda - t)^\alpha}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} e^{-(\lambda - t)x}\, dx \\ &= \frac{\lambda^\alpha}{(\lambda - t)^\alpha}, \qquad -\infty < t < \lambda. \end{aligned} \tag{3.47}$$

□

Uniqueness: *Suppose that* $X \geq 0$ (≤ 0)*. If* $m_X(t) < \infty$ *for some interval* $0 < t < \delta$ $(-\delta < t < 0)$*, then* m_X *uniquely determines the distribution of* X*. More generally, if* $m_X(t) = m_Y(t) < \infty$ *for* $|t| < \delta$*, then* $X \stackrel{\text{distn}}{=} Y$*, i.e.,* $P[X \in A] = P[Y \in A]$ *for all events* A*. [See §3.3.1]*

Application 3.3.1: The sum of independent Poisson rvs is Poisson. Let $X_1, \ldots, X_n$ be independent, $X_i \sim \text{Poisson}(\lambda_i)$. Then by independence,

$$m_{X_1 + \cdot + X_n}(t) = e^{\lambda_1(e^t - 1)} \cdots e^{\lambda_n(e^t - 1)} = e^{(\lambda_1 + \cdots + \lambda_n)(e^t - 1)},$$

so $X_1 + \cdot + X_n \sim \text{Poisson}(\lambda_1 + \cdots + \lambda_n)$. □

Application 3.3.2: The sum of independent normal rvs is normal. Suppose $X_1, \ldots, X_n$ are independent, $X_i \sim N(\mu_i, \sigma_i^2)$. Then by (3.40) and (3.46),

$$m_{X_1+\cdot+X_n}(t) = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} \cdots e^{\mu_n t + \frac{\sigma_n^2 t^2}{2}} = e^{(\mu_1+\cdots+\mu_n)t + \frac{(\sigma_1^2+\cdots+\sigma_n^2)t^2}{2}},$$

so $X_1 + \cdot + X_n \sim N(\mu_1 + \cdots + \mu_n,\ \sigma_1^2 + \cdots + \sigma_n^2)$. □

Application 3.3.3: The sum of independent Gamma rvs with the same scale parameter is Gamma. Suppose that $X_1, \ldots, X_n$ are independent rvs with $X_i \sim G(\alpha_i, \lambda)$. Then by (3.40) and (3.47),

$$m_{X_1+\cdot+X_n}(t) = \frac{\lambda^{\alpha_1}}{(\lambda - t)^{\alpha_1}} \cdots \frac{\lambda^{\alpha_n}}{(\lambda - t)^{\alpha_n}} = \frac{\lambda^{\alpha_1+\cdots+\alpha_n}}{(\lambda - t)^{\alpha_1+\cdots+\alpha_n}}, \quad -\infty < t < \lambda,$$

so $X_1 + \cdot + X_n \sim \text{Gamma}(\alpha_1 + \cdots + \alpha_n,\ \lambda)$. □

Convergence in distribution: A sequence of rvs $\{X_n\}$ *converges in distribution to* X, denoted as $X_n \xrightarrow{d} X$, if $P[X_n \in A] \to P[X \in A]$ for every[3] event A. Then if $m_X(t) < \infty$ for $|t| < \delta$, we have

$$(3.48) \qquad X_n \xrightarrow{d} X \iff m_{X_n}(t) \to m_X(t) \quad \forall |t| < \delta. \qquad \text{[See §3.3.1]}$$

Application 3.3.4: The normal approximation to the binomial distribution (≡ the Central Limit Theorem for Bernoulli rvs).

Let $S_n \sim \text{Binomial}(n, p)$, that is, S_n represents the total number of successes in n independent trials with $P[\text{Success}] = p$, $0 < p < 1$. (This is called a sequence of *Bernoulli trials.*) Since $\text{E}(S_n) = np$ and $\text{Var}(S_n) = np(1-p)$,the *standardized* version of S_n is

$$Y_n \equiv \frac{S_n - np}{\sqrt{np(1-p)}},$$

so $\text{E}(Y_n) = 0$, $\text{Var}(Y_n) = 1$. We apply (3.48) to show that $Y_n \xrightarrow{d} N(0, 1)$, or equivalently, that

$$(3.49) \qquad X_n \equiv \sqrt{p(1-p)}\, Y_n \xrightarrow{d} N(0, p(1-p)):$$

[3] Actually A must be restricted to be such that $P[X \in \partial A] = 0$; see §10.2.

$$
\begin{aligned}
m_{X_n}(t) &= \mathrm{E}\Big[e^{\frac{t(S_n - np)}{\sqrt{n}}}\Big] \\
&= e^{-tp\sqrt{n}}\mathrm{E}\Big[e^{\frac{tS_n}{\sqrt{n}}}\Big] \\
&= e^{-tp\sqrt{n}}\Big[pe^{\frac{t}{\sqrt{n}}} + (1-p)\Big]^n && \text{[by (3.43)]} \\
&= \Big[pe^{\frac{t(1-p)}{\sqrt{n}}} + (1-p)e^{-\frac{tp}{\sqrt{n}}}\Big]^n \\
&= \Big[p\Big(1 + \frac{t(1-p)}{\sqrt{n}} + \frac{t^2(1-p)^2}{2n} + O(\frac{1}{n^{3/2}})\Big) \\
&\quad + (1-p)\Big(1 - \frac{tp}{\sqrt{n}} + \frac{t^2p^2}{2n} + O(\frac{1}{n^{3/2}})\Big)\Big]^n \\
&= \Big[1 + \frac{t^2p(1-p)}{2n} + O(\frac{1}{n^{3/2}})\Big]^n \\
&\to e^{\frac{t^2p(1-p)}{2}} && \text{[by CB Lemma 2.3.14.]}
\end{aligned}
$$

Since $e^{\frac{t^2p(1-p)}{2}}$ is the mgf of $N(0, p(1-p))$, $X_n \xrightarrow{d} N(0, p(1-p))$. □

Application 3.3.5: The Poisson approximation to the binomial distribution for "rare" events.

Let $X_n \sim \text{Binomial}(n, p_n)$, where $p_n = \frac{\lambda}{n}$ for some $\lambda \in (0, \infty)$. Thus $\mathrm{E}(X_n) \equiv np_n = \lambda$ remains fixed while $P[\text{Success}] \equiv p_n \to 0$, so "Success" becomes a rare event as $n \to \infty$. From (3.43),

$$
\begin{aligned}
m_{X_n}(t) &= \Big[\Big(\frac{\lambda}{n}\Big)e^t + \Big(1 - \frac{\lambda}{n}\Big)\Big]^n \\
&= \Big[1 + \Big(\frac{\lambda}{n}\Big)(e^t - 1)\Big]^n \\
&\to e^{\lambda(e^t - 1)},
\end{aligned}
$$

so $X_n \xrightarrow{d} \text{Poisson}(\lambda)$ as $n \to \infty$.

Note: This also can be proved directly: for $k = 0, 1, \ldots,$

$$
\begin{aligned}
P[X_n = k] &= \tbinom{n}{k}\big(\tfrac{\lambda}{n}\big)^k\big(1 - \tfrac{\lambda}{n}\big)^{n-k} \\
&= \tfrac{n(n-1)\cdots(n-k+1)}{n^k}\tfrac{\lambda^k}{k!}\big(1 - \tfrac{\lambda}{n}\big)^{n-k} \to \tfrac{\lambda^k}{k!}e^{-\lambda} \text{ as } n \to \infty.
\end{aligned}
$$
□

3.3.1. Proofs of the uniqueness and convergence properties of mgfs for discrete distributions with finite support.

Consider rvs X, Y, and $\{X_n\}$ with a *common, finite support* $\{1, 2 \ldots, s\}$. The probability distributions of X, Y, and X_n are given by the vectors

$$\mathbf{p}_X \equiv \begin{pmatrix} p_1 \\ \vdots \\ p_s \end{pmatrix}, \quad \mathbf{p}_Y \equiv \begin{pmatrix} q_1 \\ \vdots \\ q_s \end{pmatrix} \quad \mathbf{p}_{X_n} \equiv \begin{pmatrix} p_{n,1} \\ \vdots \\ p_{n,s} \end{pmatrix},$$

respectively, where $p_j = P[X = j]$, $q_j = P[Y = j]$, $p_{n,j} = P[X_n = j]$, $j = 1, \ldots, s$. Choose any s distinct points $t_1 < \cdots < t_s$ and let

$$\mathbf{m}_X = \begin{pmatrix} m_X(t_1) \\ \vdots \\ m_X(t_s) \end{pmatrix}, \quad \mathbf{m}_Y = \begin{pmatrix} m_Y(t_1) \\ \vdots \\ m_Y(t_s) \end{pmatrix}, \quad \mathbf{m}_{X_n} = \begin{pmatrix} m_{X_n}(t_1) \\ \vdots \\ m_{X_n}(t_s) \end{pmatrix}.$$

Since

$$m_X(t_i) \equiv \mathrm{E}e^{t_i X} = \sum_{j=1}^{s} e^{t_i j} p_j \equiv (e^{t_i}, e^{2t_i} \ldots, e^{st_i})\mathbf{p}_X,$$

etc., we can write [verify!]

$$\mathbf{m}_X = A\,\mathbf{p}_X, \quad \mathbf{m}_Y = A\,\mathbf{p}_Y, \quad \mathbf{m}_{X_n} = A\,\mathbf{p}_{X_n}, \tag{3.50}$$

where A is the $s \times s$ matrix given by

$$A = \begin{pmatrix} e^{t_1} & e^{2t_1} & \ldots & e^{st_1} \\ \vdots & \vdots & & \vdots \\ e^{t_s} & e^{2t_s} & \ldots & e^{st_s} \end{pmatrix}.$$

Exercise 3.3*. Show that A is a nonsingular matrix, so A^{-1} exists. □

Thus

$$\mathbf{p}_X = A^{-1}\mathbf{m}_X \quad \text{and } \mathbf{p}_Y = A^{-1}\mathbf{m}_Y, \tag{3.51}$$

so if $m_X(t_i) = m_Y(t_i)\ \forall i = 1, \ldots, s$ then $\mathbf{m}_X = \mathbf{m}_Y$, hence $\mathbf{p}_X = \mathbf{p}_Y$ by (3.51), which establishes the uniqueness property of mgfs in this special case. Also, if $m_{X_n}(t_i) \to m_X(t_i)\ \forall$ then $\mathbf{m}_{X_n} \to \mathbf{m}_X$, hence $\mathbf{p}_{X_n} \to \mathbf{p}_X$ by (3.51), which established the convergence property of mgfs in this case.

Remark 3.1. (3.50) simply shows that the mgf is a *nonsingular linear transform* of the probability distribution. In engineering, the mgf would be called the *Laplace transform* of the probability distribution. An alternative transformation is the *Fourier transform* defined by $\phi_X(t) = \mathrm{E}(e^{itX})$, where $i = \sqrt{-1}$, which we call the *characteristic function* of (the distribution of) X. ϕ_X is complex-valued, but has the advantage that it is always finite. In fact, since $|e^{iu}| = 1$ for all real u, $|\phi_X(t)| \leq 1$ for all real t. □

3.4. Multivariate moment generating functions.

Let $X \equiv \begin{pmatrix} X_1 \\ \vdots \\ X_p \end{pmatrix}$ and $t \equiv \begin{pmatrix} t_1 \\ \vdots \\ t_p \end{pmatrix}$. The *moment generating function (mgf) of (the distribution of)* the rvtr X is

$$m_X(t) = \mathrm{E}\big(e^{t'X}\big) \equiv \mathrm{E}\big(e^{t_1X_1+\cdots+t_pX_p}\big). \tag{3.52}$$

Again, $m_X(0) = 1$ and $0 < m_X(t) \leq \infty$, with ∞ possible. Note that if $X_1, \ldots, X_p$ are independent, then

$$\begin{aligned} m_X(t) &\equiv \mathrm{E}\big(e^{t_1X_1+\cdots+t_pX_p}\big) \\ &= \mathrm{E}\big(e^{t_1X_1}\big)\cdots\mathrm{E}\big(e^{t_pX_p}\big) \\ &\equiv m_{X_1}(t_1)\cdots m_{X_p}(t_p). \end{aligned} \tag{3.53}$$

All properties of the mgf, including uniqueness, convergence in distribution, the location-scale formula, and multiplicativity, extend to the multivariate case. For example:

If $m_X(t) < \infty$ for $\|t\| < \delta$ then the multiple Taylor series expansion of $e^{t'X}$ yields

$$m_X(t) = \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_p=k} \frac{t_1^{k_1}\cdots t_p^{k_p}\mathrm{E}(X_1^{k_1}\cdots X_p^{k_p})}{k_1!\cdots k_p!}, \quad \|t\| < \delta, \tag{3.54}$$

so

$$(3.55) \qquad \mathrm{E}(X_1^{k_1} \cdots X_p^{k_p}) = m_X^{(k_1,\ldots,k_p)}(0), \quad k_1,\ldots,k_p \geq 0. \qquad \text{[verify]}$$

Multivariate location-scale: For any fixed $q \times p$ matrix A and $q \times 1$ vector b,

$$(3.56) \qquad m_{AX+b}(t) = \mathrm{E}\big(e^{t'(AX+b)}\big) = e^{t'b}\mathrm{E}\big(e^{t'AX}\big) = e^{t'b} m_X(A't).$$

Example 3.5. The multivariate normal distribution $N_p(\mu, \Sigma)$.

First suppose that $Z_1, \ldots, Z_q$ are i.i.d. standard normal $N(0,1)$ rvs. Then by independence the rvtr $Z \equiv (Z_1, \ldots, Z_q)'$ has mgf

$$(3.57) \qquad m_Z(t) = e^{t_1^2/2} \cdots e^{t_q^2/2} = e^{t't/2}.$$

Now let $X = AZ + \mu$ with $A : p \times q$, and $\mu : p \times 1$. Then by (3.56), (3.57),

$$\begin{aligned} m_X(t) &= e^{t'\mu} m_Z(A't) \\ &= e^{t'\mu} e^{(A't)'(A't)/2} \\ &= e^{t'\mu} e^{t'(AA')t/2} \\ (3.58) \qquad &\equiv e^{t'\mu + t'\Sigma t/2}, \end{aligned}$$

where $\Sigma = AA'$. We shall see in §8.3 that $\Sigma = \mathrm{Cov}(X)$, the *covariance matrix* of X. Thus the distribution of $X \equiv AZ + \mu$ depends on (μ, A) only through (μ, Σ), so we denote this distribution by $N_p(\mu, \Sigma)$, *the p-dimensional multivariate normal distribution (MVND) with mean vector μ and covariance matrix Σ.* We shall derive its pdf in §8.3. However, we can use the representation $X = AZ + \mu$ to derive its basic linearity property:

Linearity of $N_p(\mu, \Sigma)$: If $X \sim N_p(\mu, \Sigma)$ then for $C : r \times p$ and $d : r \times 1$,

$$\begin{aligned} Y \equiv CX + d &= (CA)Z + (C\mu + d) \\ &\sim N_r\big(C\mu + d, (CA)(CA)'\big) \\ (3.59) \qquad &= N_r(C\mu + d,\ C\Sigma C'). \end{aligned}$$

In particular, if $r = 1$ then for $c : p \times 1$ and $d : 1 \times 1$,

$$(3.60) \qquad c'X + d \sim N_1(c'\mu + d,\ c'\Sigma c). \qquad \square$$

3.5. The Central Limit Theorem (CLT) ≡ normal approximation.

The normal approximation to the binomial distribution (see Application 3.3.4) can be viewed as an approximation to the distribution of the sum of i.i.d. Bernoulli (0-1) rvs. This extends to any sum of i.i.d. rvs with finite second moments.

Theorem 3.2. *Let $\{Y_n\}$ be a sequence of i.i.d. rvs with finite mean μ and variance σ^2. Set $S_n = Y_1 + \cdots + Y_n$ and $\bar{Y}_n = \frac{S_n}{n}$. Their standardized distributions converge to the standard normal $N(0,1)$ distribution: for any $a < b$,*

$$(3.61)\qquad P\Big[a \le \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \le b\Big] \to P[a \le N(0,1) \le b] \equiv \Phi(b) - \Phi(a),$$

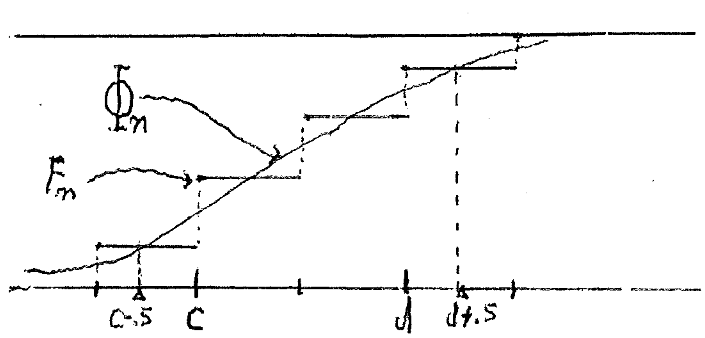
$$(3.62)\qquad P\Big[a \le \frac{\sqrt{n}\,(\bar{Y}_n - \mu)}{\sigma} \le b\Big] \to P[a \le N(0,1) \le b] \equiv \Phi(b) - \Phi(a),$$

where Φ is the cdf of $N(0,1)$. Thus, if n is "large", for any $c < d$ we have

$$\begin{aligned} P[\,c \le S_n \le d\,] &= P\Big[\frac{c - n\mu}{\sqrt{n}\,\sigma} \le \frac{S_n - n\mu}{\sqrt{n}\,\sigma} \le \frac{d - n\mu}{\sqrt{n}\,\sigma}\Big] \\ &\approx \Phi\Big[\frac{d - n\mu}{\sqrt{n}\,\sigma}\Big] - \Phi\Big[\frac{c - n\mu}{\sqrt{n}\,\sigma}\Big]. \end{aligned} \qquad (3.63)$$

Continuity correction: Suppose that $\{Y_n\}$ are integer-valued, hence so is S_n. Then if c, d are integers, the accuracy of (3.63) can be improved significantly as follows:

$$\begin{aligned} P[\,c \le S_n \le d\,] &= P[\,c - 0.5 \le S_n \le d + 0.5\,] \\ &\approx \Phi\Big[\frac{d + 0.5 - n\mu}{\sqrt{n}\,\sigma}\Big] - \Phi\Big[\frac{c - 0.5 - n\mu}{\sqrt{n}\,\sigma}\Big]. \end{aligned} \qquad (3.64)$$



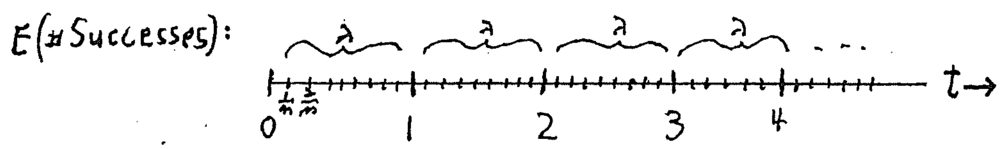
3.6. The Poisson process.

We shall first construct the *Poisson process* (PP) in one dimension. Recall the Poisson approximation to the binomial distribution (Application 3.3.5):

Lemma 3.2. *Let* $N^{(n)} \sim \text{Binomial}(n, p_n)$, *where* $n \to \infty$ *and* $p_n \to 0$ *s.t.* $\mathrm{E}(N^{(n)}) \equiv np_n = \lambda > 0$. *Then* $N^{(n)} \xrightarrow{d} \text{Poisson}\,(\lambda)$ *as* $n \to \infty$.

[Note that the range of $N^{(n)}$ is $\{0, 1, \ldots, n\}$, which converges to the Poisson range $\{0, 1, \ldots\}$ as $n \to \infty$.] □

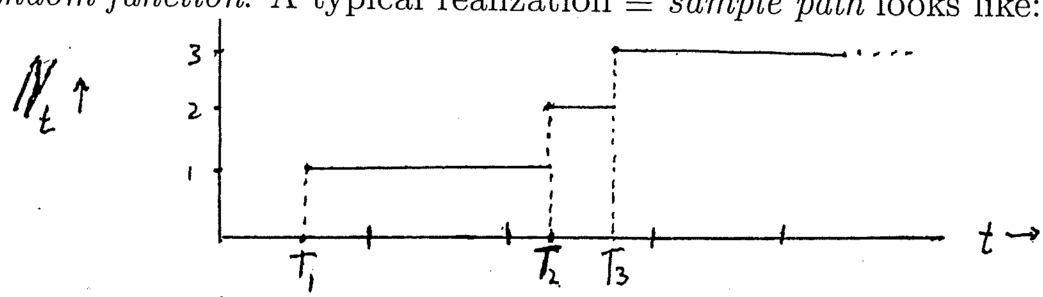
This result says that if a very large number n of elves toss identical coins independently, each with a very small success probability p_n, so that the expected number of successes $np_n = \lambda$, then the total number of successes approximately follows a Poisson distribution. Suppose now that these n elves are spread uniformly over the unit interval $(0, 1]$, and that n more elves with identical coins are spread uniformly over the interval $(1, 2]$, and n more spread over $(2, 3]$, and so on:



For $0 < t < \infty$, let $N_t^{(n)}$ denote the total number of successes occurring in the interval $(0, t]$ (set $N_0^{(n)} = 0$.) Then $\mathrm{E}(N_t^{(n)}) = \lambda t$, and as $n \to \infty$,

$$N_t^{(n)} \xrightarrow{d} N_t \sim \text{Poisson}(\lambda t), \tag{3.65}$$
$$\mathrm{E}(N_t) = \lambda t. \tag{3.66}$$

Considered as a function of t, $\{N_t \mid 0 \leq t < \infty\}$ is a *stochastic process* $\equiv$ *random function*. A typical realization $\equiv$ *sample path* looks like:



Here the jump points $0 < T_1 < T_2 < \cdots$ are *random variables*.

Because of (3.65)-(3.66), the process $\{N_t \mid 0 \le t < \infty\}$ is called a *homogeneous Poisson process* (PP) with *intensity* λ. Its sample paths are nondecreasing step functions with jump size 1. Such a process is called a *point process* because it is completely determined by the locations of the jump points $T_1, T_2, \ldots$.

Because all the elves are probabilistically independent and independence is preserved under limits in distribution [cf. §10], for any fixed points $0 \equiv t_0 < t_1 < t_2 \cdots$, the increments $N_{t_1} - N_{t_0}$, $N_{t_2} - N_{t_1}$, ..., are mutually independent. Thus the PP has *independent increments* with

$$N_{t_i} - N_{t_{i-1}} \sim \text{Poisson}(\lambda(t_i - t_{i-1})), \tag{3.67}$$

$$\text{E}[N_{t_i} - N_{t_{i-1}}] = \lambda(t_i - t_{i-1}). \tag{3.68}$$

A *non-homogeneous* PP also can be defined, with λ replaced by an *intensity function* $\lambda(t) \ge 0$. A non-homogeneous PP retains all the above properties of a homogeneous PP except that in (3.67) and (3.68), $(t_i - t_{i-1})\lambda$ is replaced by $\int_{t_{i-1}}^{t_i} \lambda(t)dt$. A non-homogeneous PP can be thought of as the limit of non-homogeneous elf-coin-tossing processes, where the elves are distributed non-uniformly along the line.

There is a duality between a homogeneous PP $\{N_t \mid 0 \le t < \infty\}$ and sums of i.i.d. exponential random variables. This is seen as follows. Let $T_1, T_2, \ldots$ be the times of the jumps of the PP.[4]

Proposition 3.1. *T_1, $T_2 - T_1$, $T_3 - T_2, \ldots$ are i.i.d. Exponential (λ) rvs. In particular, $\text{E}(T_i - T_{i-1}) = \frac{1}{\lambda}$, which reflects the intuitive fact that the expected waiting time to the next success is inversely proportional to the intensity rate λ.*

Partial Proof. $P[T_1 > t] = P[\text{no successes occur in } (0, t]] = P[N_t = 0] = e^{-\lambda t}$, since $N_t \sim \text{Poisson}(\lambda t)$. (The proof continues with Exercise 6.5.) □

Note: $T_k \sim \text{Gamma}(k, \lambda)$, because the sum of i.i.d. exponential rvs has a Gamma distribution. Since $\{T_k > t\} = \{N_t \le k-1\}$, this implies a relation

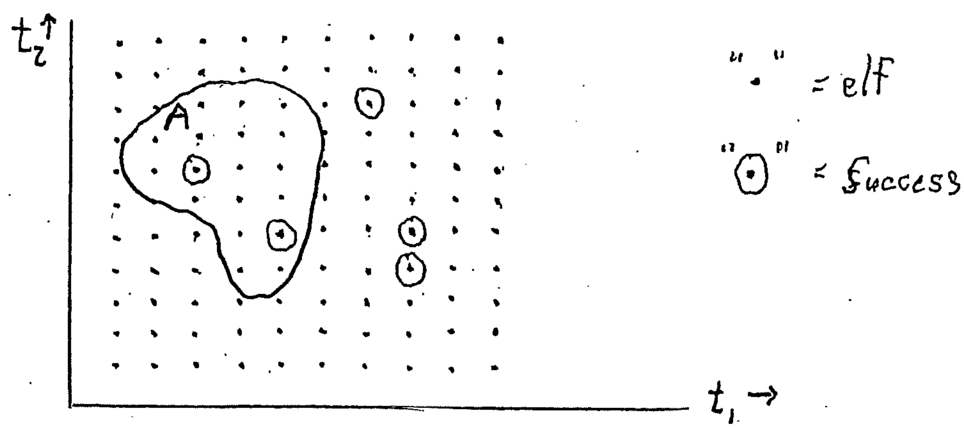
[4] There must be infinitely many jumps in $(0, \infty)$. This follows from the Borel-Cantelli Theorem, which says that if $\{A_n\}$ is a sequence of *independent* events, then $P[\text{infinitely many } A_n \text{ occur}]$ is 0 or 1 according to whether $\sum P(A_n)$ is $< \infty$ or $= \infty$. Now let A_n be the event that at least one success occurs in the interval $(n-1, n]$.

between the cdfs of the Gamma and Poisson distributions: see CB Example 3.3.1 and CB Exercise 3.19; also Ross Exercise 26, Ch. 4.]

In view of Proposition 3.1 and the fact that a PP is completely determined by the location of the jumps, a PP can be constructed from a sequence of i.i.d. Exponential(λ) rvs $V_1, V_2, \ldots$: just define $T_1 = V_1$, $T_2 = V_1 + V_2$, $T_3 = V_1 + V_2 + V_3$, etc. Then $T_1, T_2, T_3, \ldots$ determine the jump points of the PP, from which the entire sample path can be constructed.

PPs arise in many applications, for example as a model for radioactive decay over time. Here, an "elf" is an individual atom – each atom has a tiny probability p of decaying (a "success") in unit time, but there are a large number n of atoms. PPs also serve as models for the number of traffic accidents over time (or location) on a busy freeway.

PPs can be extended in several ways: from homogeneous to non-homogeneous as mentioned above, and/or to point processes in more than one dimension. In general, a point process on an open region $R \subseteq \mathbf{R}^n$ is a random set function $\{N(A) \mid A \subseteq R\}$, where $N(A)$ is the (random) number of points that occur in the subset A.



This constitutes a *Poisson process with intensity function* $\lambda(t) \geq 0$ if

$$(3.69) \qquad N(A) \sim \text{Poisson}\Big(\int_A \lambda(t) dt \Big),$$

$$(3.70) \qquad N(A_1) \perp\!\!\!\perp \cdots \perp\!\!\!\perp N(A_k) \quad \text{if } A_1, \ldots, A_k \text{ are disjoint.}$$

This can be thought of as a limit of elf-coin-tossing processes where many elves are distributed in R according to density function $\lambda(t)$. The PP is homogeneous if $\lambda(t) \equiv \lambda > 0$ (a constant), in which case $\int_A \lambda(t)dt$ reduces to $\lambda \cdot \text{Volume}(A)$; otherwise it is non-homogeneous.

Examples of random processes that may be PPs include the spatial distribution of weeds in a field, of ore deposits in a region, of erroneous pixels in a picture transmitted from a Mars orbiter, or of galaxies in the cosmos. These are called "spatial" processes because the random points occur in at random locations in a region. These "may be" PPs because the independence property may not hold if spatial correlation is present.

The waiting-time paradox for a homogeneous Poisson Process.

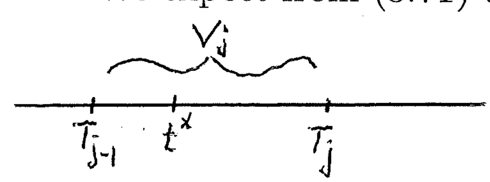
Does it seem that your waiting time for a bus is usually longer than you had expected? This can be explained by the memory-free property of the exponential distribution of the waiting times.

We will model the bus arrival times as the jump times $T_1 < T_2 < \cdots$ of a homogeneous PP $\{N_t\}$ on $[0, \infty)$ with intensity λ. Thus the interarrival times $V_i \equiv T_i - T_{i-1}$ $(i \geq 1,\ T_0 \equiv 0)$ are i.i.d Exponential (λ) rvs and

$$\text{(3.71)} \qquad \mathrm{E}(V_i) = \frac{1}{\lambda}, \qquad i \geq 1.$$

Now suppose that you arrive at the bus stop at a fixed time $t^* > 0$. Let the index $j \geq 1$ be such that $T_{j-1} < t^* < T_j$ $(j \geq 1)$, so V_j is the length of the interval that contains your arrival time. We expect from (3.71) that

$$\text{(3.72)} \qquad \mathrm{E}(V_j) = \frac{1}{\lambda}.$$



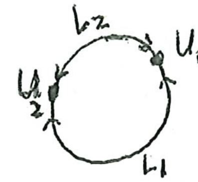
Paradoxically, however,

$$\text{(3.73)} \qquad \mathrm{E}(V_j) = \mathrm{E}(T_j - t^*) + \mathrm{E}(t^* - T_{j-1}) \ > \ \mathrm{E}(T_j - t^*) \ = \ \frac{1}{\lambda},$$

since $T_j - t^* \sim$ Expo (λ) by the memory-free property of the exponential distribution: if the next bus has not arrived by time t^* then the additional waiting time to the next bus still has the Expo (λ) distribution. Thus you appear always to be unlucky to arrive at the bus stop during a longer-than-average interarrival time!

This paradox is resolved by noting that *the index j is random, not fixed: it is the random index such that V_j includes t^*. The fact that this interval includes a prespecified point t^* tends to make V_j larger than average: a larger interval is more likely to include t^* than a shorter one!* Thus it is not so suprising that $\mathrm{E}(V_j) > \frac{1}{\lambda}$.

Exercise 3.4. A simpler example of this phenomenon can be seen as follows. Let U_1 and U_2 be two random points chosen independently and uniformly on the (circumference of the) unit circle and let L_1 and L_2 be the lengths of the two arcs thus obtained:

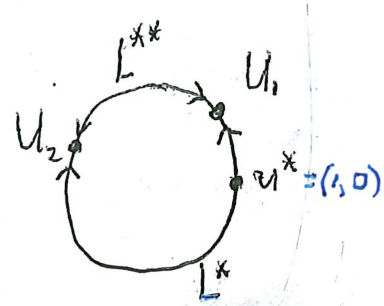


Thus, $L_1 + L_2 = 2\pi$ and $L_1 \stackrel{d}{=} L_2$ by symmetry, so

$$\text{(3.74)} \qquad \mathrm{E}(L_1) = \mathrm{E}(L_2) = \pi.$$

(i) Find the distributions of L_1 and of L_2.

(ii) Let L^* denote the length of the arc that contains the point $u^* \equiv (1, 0)$ and let L^{**} be the length of the other arc.



Find the distributions of L^* and L^{**}. Find $\mathrm{E}(L^*)$ and $\mathrm{E}(L^{**})$ and show that $\mathrm{E}(L^*) > \mathrm{E}(L^{**})$.

Hint: There is a simplifying geometric trick. □

Remark 3.2. In (3.73), it is tempting to apply the memory-free property in reverse to assert that also $t^* - T_{j-1} \sim \text{Expo}(\lambda)$. This is actually true whenever $j \geq 2$, but not when $j = 1$: $t^* - T_0 \equiv t^* \not\sim \text{Expo}(\lambda)$. However this may be achieved by assuming that the bus arrival times $\ldots, T_{-2}, T_{-1}, T_0, T_1, T_2, \ldots$ follow a "doubly-infinite" homogeneous PP on the entire real line $(-\infty, \infty)$. Just as the PP on $(0, \infty)$ can be thought of in terms of many coin-tossing elves spread homogeneously over $(0, \infty)$, this PP can be thought of in terms of many coin-tossing elves spread homogeneously over $(-\infty, \infty)$. The PP properties remain the same, in particular, the interarrival times $T_i - T_{i-1}$ are i.i.d. Exponential (λ) rvs. In this case it *is* true that $t^* - T_{j-1} \sim \text{Expo}\,(\lambda)$, hence we have the *exact* result that

$$\text{E}(V_j) = \frac{2}{\lambda}. \tag{3.75}$$

(In fact, $V_j \sim \text{Expo}\,(\lambda) + \text{Expo}\,(\lambda) \stackrel{\text{d}}{=} \text{Gamma}(2,\, \lambda)$.) □

4. Conditional Expectation and Conditional Distribution.

Let (X, Y) be a bivariate random vector (rvtr) with joint pmf (discrete case) or joint pdf (continuous case) $f(x, y)$. The *conditional expectation of* Y given X is defined by

$$(4.1) \qquad \mathrm{E}[Y \mid X = x] = \sum_y y f(y|x), \qquad \text{[jointly discrete]}$$

$$(4.2) \qquad \mathrm{E}[Y \mid X](x) = \int y f(y|x)\, dy, \qquad \text{[jointly continuous]}$$

provided that the sum or integral exists, where $f(y|x)$ is given by (1.28) or (1.30). More generally, for any (measurable) function $g(y)$,

$$(4.3) \qquad \mathrm{E}[g(Y) \mid X = x] = \sum_y g(y) f(y|x), \qquad \text{[jointly discrete]}$$

$$(4.4) \qquad \mathrm{E}[g(Y) \mid X](x) = \int g(y) f(y|x)\, dy, \qquad \text{[jointly continuous]}$$

Note that (1.29) and (1.31) are special cases of (4.3) and (4.4), respectively, with $g(y) = I_B(y)$. For simplicity, we often shorten the notation to $E[\cdot \mid X]$ in both cases.

Because $f(\cdot|x)$ is a bona fide pmf or pdf, conditional expectation enjoys all the properties of ordinary expectation, in particular, *linearity:*

$$(4.5) \qquad \mathrm{E}[ag(Y) + bh(Y) \mid X] = a\mathrm{E}[g(Y) \mid X] + b\mathrm{E}[h(Y) \mid X].$$

The key *Iteration Formula*, which extends the Law of Total Probability, follows from (4.3), (1.28), (3.5) (discrete) or (4.4), (1.30), (3.6) (continuous):

$$(4.6) \qquad \mathrm{E}[g(Y)] = \mathrm{E}\big(\mathrm{E}[g(Y) \mid X]\big). \qquad \text{[verify]}$$

As a special case (set $g(y) = I_B(y)$), for any (measurable) event B,

$$(4.7) \qquad P[Y \in B] = \mathrm{E}\big(P[Y \in B \mid X]\big).$$

We now discuss the extension of these results to the two "mixed" cases.

(i) *X is discrete and Y is continuous.*

First, the Iteration Formulas continue to hold: (4.7) follows immediately from the law of total probability (1.54) [verify], then (4.6) follows since any (measurable) g can be approximated as $g(y) \approx \sum b_i I_{B_i}(y)$. Thus, although we cannot calculate $P[Y \in B]$ or $\mathrm{E}[g(Y)]$ directly since we do not have a joint pmf or joint pdf, we can obtain them by the iteration formulas in (4.6) and (4.7). For this we need to determine $f(y|x)$ as follows:

For any event B and any x s.t. $P[X = x] > 0$,

$$P[Y \in B \mid X = x] \equiv \frac{P[Y \in B,\ X = x]}{P[X = x]} \tag{4.8}$$

is well defined. Thus we can define

$$f(y|x) = \frac{d}{dy} F(y|x) \equiv \frac{d}{dy} P[Y \leq y \mid X = x]. \tag{4.9}$$

Clearly $f(y|x) \geq 0$ and $\int f(y|x)dy = 1$ for each x s.t. $P[X = x] > 0$. Thus for each such x, $f(\cdot|x)$ determines a bona fide pdf. This $f(\cdot|x)$ does in fact determine the conditional distribution of Y given X because (4.9) extends to all (measurable) sets B:

$$P[Y \in B \mid X = x] = \int_B f(y|x)\, dy = \int I_B(y) f(y|x) dy. \tag{4.10}$$

Now use the approximation $g(y) \approx \sum b_i I_{B_i}(y)$ to extend (4.10) to obtain

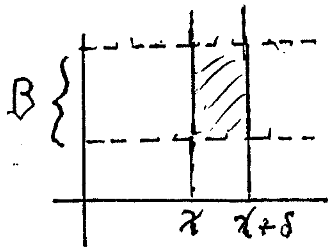
$$\mathrm{E}[g(Y) \mid X = x] = \int g(y) f(y|x) dy. \tag{4.11}$$

Note: In most applications, $f(y|x)$ is not found via (4.9) but instead is either specified directly or else is found using $f(x|y)$ and Bayes formula (4.14) – see Example 4.3.

(ii) *X is continuous and Y is either discrete or continuous.*

For any (measurable) event B and any x such that $f(x) > 0$, we define the conditional probability

$$(4.12) \qquad P[Y \in B \mid X](x) = \lim_{\delta \downarrow 0} P[Y \in B \mid x \le X \le x + \delta]$$



$$= \lim_{\delta \downarrow 0} \frac{P[Y \in B,\ x \le X \le x + \delta]}{P[x \le X \le x + \delta]}$$

$$= \frac{1}{f(x)} \lim_{\delta \downarrow 0} \frac{P[Y \in B,\ x \le X \le x + \delta]}{\delta}$$

$$(4.13) \qquad \equiv \frac{1}{f(x)} \frac{d}{dx} P[Y \in B,\ X \le x].$$

Then the iteration formulas (4.6) and (4.7) continue to hold. For (4.7),

$$\begin{aligned} \mathrm{E}\big(P[Y \in B \mid X]\big) &= \int_{-\infty}^{\infty} \Big(\frac{1}{f(x)} \frac{d}{dx} P[Y \in B,\ X \le x]\Big) f(x) dx \\ &= P[Y \in B]. \end{aligned}$$

Again (4.6) follows by the approximation $g(y) \approx \sum b_i I_{B_i}(x, y)$.

In particular, *if X is continuous and Y is discrete,* then by (4.13), $f(y|x)$ is given by

$$\begin{aligned} f(y|x) &\equiv P[Y = y \mid X](x) \\ &= \frac{1}{f(x)} \frac{d}{dx} P[Y = y,\ X \le x] \\ &= \frac{1}{f(x)} \frac{d}{dx} P[X \le x \mid Y = y] \cdot P[Y = y] \end{aligned}$$

$$(4.14) \qquad \equiv \frac{f(x|y) f(y)}{f(x)} \qquad \text{[by (4.9)].}$$

This is *Bayes formula for pmfs/pdfs in the mixed case*, and extends (1.58).

Remark 4.1. By (4.14),

$$(4.15) \qquad f(y|x) f(x) = f(x|y) f(y),$$

even in the mixed cases where a joint pmf or pdf $f(x, y)$ does not exist. In such cases, the joint distribution is specified either by specifying $f(y|x)$ and $f(x)$, or $f(x|y)$ and $f(y)$ – see Example 4.3. $\square$

Remark 4.2. If (X, Y) is jointly continuous then we now have two definitions of $f(y|x)$: the "slicing" definition (1.30): $f(y|x) = \frac{f(x,y)}{f(x)}$, and the following definition (4.16) obtained from (4.12)-(4.13):

$$\begin{aligned}
(4.16) \qquad f(y|x) &\equiv \frac{d}{dy} P[Y \leq y \mid X](x) \\
&= \frac{d}{dy}\Big[\frac{1}{f(x)}\frac{d}{dx} P[Y \leq y,\ X \leq x]\Big] \qquad \text{[by (4.13)]} \\
&= \frac{1}{f(x)}\frac{\partial^2}{\partial x \partial y} F(x, y) \\
&\equiv \frac{f(x, y)}{f(x)}.
\end{aligned}$$

Thus the two definitions coincide in this case. $\square$

Exercise 4.1. If X is continuous and Y discrete, show that $X \perp\!\!\!\perp Y \iff f(y|x) = f(y)$.

Remark 4.3. This useful result illustrates the Iteration Formula (4.6):

$$\begin{aligned}
\mathrm{Cov}(g(X),\ h(Y)) &= \mathrm{E}(g(X)h(Y)) - (\mathrm{E}g(X))(\mathrm{E}h(Y)) \\
&= \mathrm{E}\big(\mathrm{E}[\,g(X)h(Y) \mid X]\big) - (\mathrm{E}g(X))\big[\mathrm{E}(\mathrm{E}[\,h(Y) \mid X])\big] \\
&= \mathrm{E}\big(g(X)\mathrm{E}[h(Y) \mid X]\big) - (\mathrm{E}g(X))\big[\mathrm{E}(\mathrm{E}[\,h(Y) \mid X])\big] \\
(4.17) \qquad &= \mathrm{Cov}(g(X),\ \mathrm{E}[\,h(Y)|X]).
\end{aligned}$$

Here we have used the *Product Formula* [verify]:

$$(4.18) \qquad \mathrm{E}[g(X)\, h(Y) \mid X] = g(X)\mathrm{E}[h(Y) \mid X]. \qquad \square$$

Example 4.1. (Example 1.12 revisited.) Let $(X, Y) \sim \mathrm{Uniform}(D)$, where D is the unit disk in $\mathbf{R}^2$. In (1.44) we saw that

$$Y|X \sim \ \mathrm{Uniform}\big(-\sqrt{1-X^2}, \sqrt{1-X^2}\big),$$

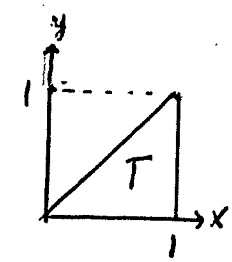
which immediately implies $\mathrm{E}[Y|X] \equiv 0$. Thus the iteration formula (4.6) and the covariance formula (4.17) yield, respectively,

$$\mathrm{E}(Y) = \mathrm{E}\big(\mathrm{E}[Y|X]\big) = \mathrm{E}(0) = 0,$$
$$\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, \mathrm{E}[Y|X]) = \mathrm{Cov}(X,0) = 0,$$

which are also clear from considering the joint distribution of (X,Y). □

Example 4.2. Let $(X,Y) \sim \mathrm{Uniform}(T)$, where T is the triangle below, so

$$f(x,y) = \begin{cases} 2, & 0 < x < 1,\ 0 < y < x; \\ 0, & \text{otherwise.} \end{cases} \tag{4.19}$$



Thus

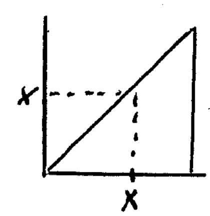
$$f(x) = 2xI_{(0,1)}(x), \tag{4.20}$$

hence

$$f(y|x) = \begin{cases} \frac{1}{x}, & 0 < y < x; \\ 0, & \text{otherwise.} \end{cases}$$

That is,

$$Y|X \sim \ \mathrm{Uniform}(0,\ X) \tag{4.21}$$



[verify from the figure by "slicing"], so

$$\mathrm{E}[Y \mid X] = \frac{X}{2}. \tag{4.22}$$

From (4.20) we have $\mathrm{E}(X) = \frac{2}{3}$ [verify], so the iteration formula gives

$$\mathrm{E}(Y) = \mathrm{E}\big(\mathrm{E}[Y|X]\big) = \mathrm{E}\Big(\frac{X}{2}\Big) = \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}. \tag{4.23}$$

Also from (4.17), (4.22), and the bilinearity of covariance,

$$\mathrm{Cov}(X,\ Y) = \frac{1}{2}\,\mathrm{Cov}(X,\ X) \equiv \frac{1}{2}\,\mathrm{Var}(X).$$

But [verify from (4.20)]

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2 = \frac{1}{2} - \Big(\frac{2}{3}\Big)^2 = \frac{1}{18}, \tag{4.24}$$

so $\mathrm{Cov}(X,\ Y) = \frac{1}{36} > 0$. Thus X and Y are *positively correlated.* □

Example 4.3. (A "mixed" joint distribution: Binomial-Uniform). Suppose that the joint distribution of (X, Y) is specified by the conditional distribution of $X|Y$ and the marginal distribution of $Y (\equiv \text{"}p\text{"})$ as follows:

$$\begin{aligned} X|Y &\sim \text{Binomial}(n, Y), && \text{(discrete)} \\ Y &\sim \text{Uniform}\,(0,1), && \text{(continuous)} \end{aligned} \tag{4.25}$$

so

$$\begin{aligned} f(x|y) &= \binom{n}{x} y^x (1-y)^{n-x}, && x = 0, \ldots, n, \\ f(y) &= 1, && 0 < y < 1. \end{aligned} \tag{4.26}$$

Here, X is discrete and Y is continuous, and their joint range is

$$\Omega_{X,Y} = \Omega_X \times \Omega_Y = \{0, 1, \ldots, n\} \times (0,\, 1). \tag{4.27}$$

However, (4.25) shows that $X \not\perp\!\!\!\perp Y$, since the conditional distribution of $X|Y$ varies with Y. In particular,

$$\mathrm{E}[X \mid Y] = nY. \tag{4.28}$$

Suppose that only X is observed and we wish to estimate Y by $\mathrm{E}[Y|X]$. For this we first need to find $f(y|x)$ via Bayes formula (4.14). First,

$$\begin{aligned} f(x) \equiv P[X = x] &= \mathrm{E}\big(P[X = x \mid Y]\big) \\ &= \mathrm{E}\Big[\binom{n}{x} Y^x (1-Y)^{n-x}\Big] \\ &= \binom{n}{x} \int_0^1 y^x (1-y)^{n-x} dy \\ &= \binom{n}{x} \int_0^1 y^{(x+1)-1} (1-y)^{(n-x+1)-1} dy \\ &= \binom{n}{x} \frac{\Gamma(x+1)\Gamma(n-x+1)}{\Gamma(n+2)} \qquad \text{[see (1.10)]} \\ &= \frac{n!}{x!(n-x)!} \cdot \frac{x!(n-x)!}{(n+1)!} \\ &= \frac{1}{n+1}, \qquad x = 0, 1, \ldots, n. \end{aligned} \tag{4.29}$$

This shows that, marginally, X has the *discrete uniform distribution* over the integers $0, \ldots, n$. Then from (4.14),

$$f(y|x) = \frac{\binom{n}{x} y^x (1-y)^{n-x} \cdot 1}{\frac{1}{n+1}}$$

$$= \frac{(n+1)!}{x!(n-x)!}\, y^x (1-y)^{n-x}$$

$$\equiv \frac{\Gamma(n+2)}{\Gamma(x+1)\Gamma(n-x+1)}\, y^{(x+1)-1}(1-y)^{(n-x+1)-1}, \quad 0 < y < 1. \tag{4.30}$$

Thus, the conditional ($\equiv$ *posterior*) distribution of Y given X is

$$Y|X \sim \text{Beta}\,(X+1,\; n-X+1), \tag{4.31}$$

so the *posterior* $\equiv$ *Bayes estimator* of $Y|X$ is given by

$$\mathrm{E}[Y \mid X] = \int_0^1 y \cdot \left[\frac{\Gamma(n+2)}{\Gamma(X+1)\Gamma(n-X+1)}\, y^{(X+1)-1}(1-y)^{(n-X+1)-1}\right]$$

$$= \frac{\Gamma(n+2)}{\Gamma(X+1)\Gamma(n-X+1)} \int_0^1 y^{(X+2)-1}(1-y)^{(n-X+1)-1}$$

$$= \frac{\Gamma(n+2)}{\Gamma(X+1)\Gamma(n-X+1)} \cdot \frac{\Gamma(X+2)\Gamma(n-X+1)}{\Gamma(n+3)}$$

$$= \frac{(n+1)!(X+1)!}{X!(n+2)!}$$

$$= \frac{X+1}{n+2}. \tag{4.32}$$

□

Remark 4.4. If we observe $X = n$ successes (so no failures), then the Bayes estimator is $\frac{n+1}{n+2}$, not 1. In general, the Bayes estimator can be written as

$$\frac{X+1}{n+2} = \frac{n}{n+2}\left(\frac{X}{n}\right) + \frac{2}{n+2}\left(\frac{1}{2}\right), \tag{4.33}$$

which is a *convex combination* of the usual estimate $\frac{X}{n}$ and the *a priori* estimate $\frac{1}{2} \equiv \mathrm{E}(Y)$. Thus the Bayes estimator adjusts the usual estimate to reflect the *a priori* assumption that $Y \sim$ Uniform(0, 1). Note, however, that the weight $\frac{n}{n+2}$ assigned to $\frac{X}{n}$ increases to 1 as the sample size $n \to \infty$, i.e., the prior information becomes less influential as $n \to \infty$. (See §16.) □

Example 4.4: Borel's Paradox. (This example shows the need for the limit operation (4.12) in the definition of the conditional probability $P[(X, Y) \in C | X = x]$ when X is continuous:)

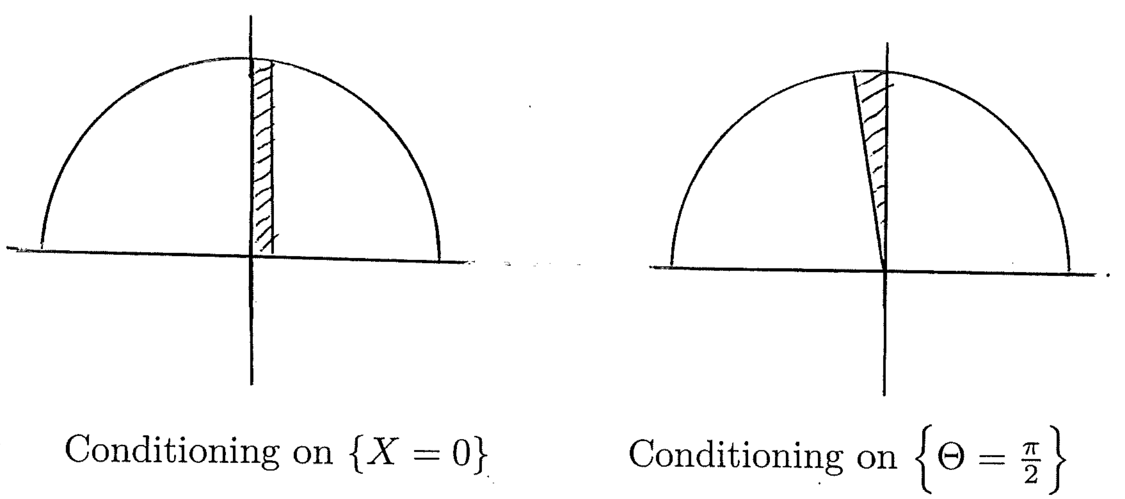
Similar to Examples 1.12 and 4.1, let (X, Y) be uniformly distributed over the upper half H of the unit disk $\mathbf{R}^2$ and consider the conditional distribution of Y given $X = 0$. The "slicing" formula (1.30) applied to $f(x, y) \equiv 2\pi^{-1} I_H(x, y)$ gives

$$Y \mid \{X = 0\} \sim \text{Uniform}(0, 1). \tag{4.34}$$

However, if we represent (X, Y) in terms of polar coordinates (R, Θ) as in Example 1.12, then the event $\{X = 0\}$ is equivalent to $\{\Theta = \frac{\pi}{2}\}$, while under this event, $Y = R$. However, $R \perp\!\!\!\perp \Theta$ and $f(r) = 2r I_{(0,1)}(r)$ (use the same argument as in (1.45) and (1.46a)), hence the "slicing" formula (1.30) applied to $f(r, \theta) \equiv f(r) f(\theta)$ shows that

$$R \,\Big|\, \left\{\Theta = \frac{\pi}{2}\right\} \sim f(r) \neq \text{Uniform}(0, 1). \tag{4.35}$$

Because the left sides of (4.34) and (4.35) appear identical, this yields *Borel's Paradox.* The paradox is resolved by noting that, according to (4.12), *conditioning on X is not equivalent to conditioning on Θ*:



Conditioning on $\{X = 0\}$ Conditioning on $\left\{\Theta = \frac{\pi}{2}\right\}$ □

5. Correlation, Prediction, and Regression.

5.1. Correlation. The covariance $\mathrm{Cov}(X,\ Y)$ indicates the nature (positive or negative) of the linear relationship (if any) between X and Y, but does not indicate the strength, or exactness, of this relationship. The *Pearson correlation coefficient*

$$(5.1) \qquad \mathrm{Cor}(X,\ Y) = \frac{\mathrm{Cov}(X,\ Y)}{\sqrt{\mathrm{Var}X}\sqrt{\mathrm{Var}Y}} \equiv \rho_{X,Y},$$

the standardized version of $\mathrm{Cov}(X,\ Y)$, does serve this purpose.

Properties:

(a) $\mathrm{Cor}(X,\ Y) = \mathrm{Cor}(Y,\ X)$.

(b) *location-scale:* $\mathrm{Cor}(aX + b,\ cY + d) = \mathrm{sgn}(ac) \cdot \mathrm{Cor}(X,\ Y)$.

(c) $-1 \le \rho_{X,Y} \le 1$. Equality holds ($\rho_{X,Y} = \pm 1$) iff $Y = aX + b$ for some a, b, i.e., iff X and Y are perfectly linearly related.

Proof. Let $U = X - \mathrm{E}X$, $V = Y - \mathrm{E}Y$, and

$$g(t) = \mathrm{E}\big[(tU + V)^2\big] = t^2\mathrm{E}(U^2) + 2t\mathrm{E}(UV) + \mathrm{E}(V^2).$$

Since this quadratic function is always ≥ 0, its discriminant is ≤ 0, i.e.

$$(5.2) \qquad \big[\mathrm{E}(UV)\big]^2 \le \mathrm{E}(U^2) \cdot \mathrm{E}(V^2),$$

so

$$(5.3) \qquad \big[\mathrm{Cov}(X,\ Y)\big]^2 \le \mathrm{Var}X \cdot \mathrm{Var}Y,$$

which is equivalent to $\rho^2_{X,Y} \le 1$. [(5.2) is the *Cauchy-Schwartz Inequality.*]

Next, equality holds in (5.2) iff the discriminant of g is 0, so $g(t_0) = 0$ for some t_0. But $g(t_0) = \mathrm{E}\big[(t_0U + V)^2\big]$, hence $t_0U + V \equiv 0$, so V must be exactly a linear function of U, i.e., Y is exactly a linear function of X.

Property (c) suggests that the closer $\rho^2_{X,Y}$ is to 1, the closer the relationship between X and Y is to exact linearity. (Also see (5.22).)

5.2. Mean-square error prediction (general regression).

For any rv Y s.t. $\mathrm{E}(Y^2) < \infty$ and any $-\infty < c < \infty$,

$$\begin{aligned}\mathrm{E}\big[(Y-c)^2\big] &= \mathrm{E}\Big(\big[(Y-\mathrm{E}Y)+(\mathrm{E}Y-c)\big]^2\Big)\\ &= \mathrm{E}\big[(Y-\mathrm{E}Y)^2\big]+(\mathrm{E}Y-c)^2+2(\mathrm{E}Y-c)\mathrm{E}(Y-\mathrm{E}Y)\\ &= \mathrm{Var}\,Y+(\mathrm{E}Y-c)^2. \end{aligned}\tag{5.4}$$

Thus $c = \mathrm{E}Y$ is the best predictor of Y w.r.to *mean-square error (MSE)* in the absence of any other information, and the minimum MSE is

$$\min_{-\infty<c<\infty} \mathrm{E}\big[(Y-c)^2\big] = \mathrm{Var}\,Y. \tag{5.5}$$

Now consider a bivariate rvtr (X, Y) with $\mathrm{E}(Y^2) < \infty$. How can we best use the information in X to obtain a better predictor $g(X)$ of Y than $\mathrm{E}Y$? That is, what function $g(X)$ minimizes the MSE

$$\mathrm{E}[(Y-g(X))^2] = \mathrm{E}\big(\mathrm{E}\big[(Y-g(X))^2 \mid X\big]\big)? \tag{5.6}$$

But this follows immediately if we hold X fixed and in (5.4) and (5.5), replace the marginal distribution of Y by the conditional distribution of Y given X and replace c by $g(X)$:

$$\mathrm{E}\big[(Y-g(X))^2 \mid X\big] = \mathrm{Var}[Y|X] + \big(\mathrm{E}[Y|X]-g(X)\big)^2; \tag{5.7}$$

$$\min_{-\infty<g(X)<\infty} \mathrm{E}\big[(Y-g(X))^2 \mid X\big] = \mathrm{Var}[Y|X]. \tag{5.8}$$

Thus $g(X) = \mathrm{E}[Y|X]$ is the best predictor of Y based on X, and from (5.6) and (5.8) the minimum (unconditional) MSE is

$$\min_{g(X)} \mathrm{E}\big[(Y-g(X))^2\big] = \mathrm{E}\big(\mathrm{Var}[Y|X]\big). \tag{5.9}$$

The *best predictor* $\mathrm{E}[Y|X]$ is often called the *regression function* [explain] of Y on X. The prediction error $Y - \mathrm{E}[Y|X]$ is called the *residual.* The *basic decomposition formula* for the prediction of Y by X is:

$$Y = \mathrm{E}[Y|X] + \big(Y-\mathrm{E}[Y|X]\big) \equiv \text{best predictor} + \text{residual}. \tag{5.10}$$

Note that: E(best predictor)=EY and E(residual)=0.

Proposition 5.1. *(Variance decomposition). The best predictor and the residual are uncorrelated:*

$$\text{Cov}\big(\text{E}[Y|X],\ Y - \text{E}[Y|X]\big) = 0. \tag{5.11}$$

Therefore the variance of Y *can be decomposed as follows:*

$$\begin{aligned} \text{Var}\,Y &= \text{Var}\big(\text{E}[Y|X]\big) + \text{Var}\big(Y - \text{E}\,[Y|X]\big) \qquad (5.12)\\ &\equiv \text{Var(best predictor)} + \text{Var(residual)}\\ &= \text{Var}\big(\text{E}[Y|X]\big) + \text{E}\big(\text{Var}\,[Y|X]\big) \qquad (5.13)\end{aligned}$$

Proof. From (4.17) and bilinearity,

$$\text{Cov}\big(\text{E}[Y|X],\ Y - \text{E}[Y|X]\big) = \text{Cov}\big(\text{E}[Y|X],\ \underbrace{\text{E}[Y|X] - \text{E}[Y|X]}_{=0}\big) = 0,$$

which establishes (5.11) and therefore (5.12). Finally,

$$\begin{aligned} \text{Var}\big(Y - \text{E}[Y|X]\big) &= \text{E}\big[\big(Y - \text{E}[Y|X]\big)^2\big] \qquad (5.14)\\ &= \text{E}\Big(\text{E}\big[\big(Y - \text{E}[Y|X]\big)^2 \mid X\big]\Big)\\ &= \text{E}\big(\text{Var}\,[Y|X]\big), \end{aligned}$$

which gives (5.13). □

Exercise 5.1. Prove (5.11) directly from (3.11) or (3.12).

Remark 5.1. Consider a rvtr $(X_1, \ldots, X_k, Y)$ with $\text{E}(Y^2) < \infty$. Then the best predictor $g(X_1, \ldots, X_k)$ of $Y \mid (X_1, \ldots, X_k)$ is $\text{E}[Y \mid X_1, \ldots, X_k]$. All of the results above remain valid with X replaced by $X_1, \ldots, X_k$. For example, (5.9) becomes

$$\min_{g(X_1,\ldots,X_k)} \text{E}\big[(Y - g(X_1, \ldots, X_k))^2\big] = \text{E}\big(\text{Var}[Y|X_1, \ldots, X_k]\big),$$

which implies that

$$\text{E}\big(\text{Var}[Y|X_1, \ldots, X_k]\big) \le \cdots \le \text{E}\big(\text{Var}[Y|X_1]\big) \le \text{Var}\,Y. \tag{5.15}$$

5.3. Linear prediction ($\equiv$ linear regression).

In practice, the best predictor $\equiv$ regression function $\mathrm{E}[Y|X]$ is unavailable, since to find it would require knowing the entire joint distribution of (X, Y). As a first step, we might ask to find the *linear* prediction function $g(X) = a + bX$ that minimizes the MSE

$$\mathrm{E}\big([Y - (a + bX)]^2\big). \tag{5.16}$$

First hold b fixed. From (5.4) with Y replaced by $Y - bX$, the MSE is minimized when

$$a = \hat{a}(b) = \mathrm{E}(Y - bX) = \mathrm{E}(Y) - b\mathrm{E}(X), \tag{5.17}$$

so
$$\begin{aligned}\min_a \mathrm{E}\big[(Y - (a + bX)]^2\big) &= \mathrm{E}\big([Y - \mathrm{E}Y] - b[X - \mathrm{E}X]\big)^2 \\ &= \mathrm{Var}\,Y - 2b\,\mathrm{Cov}(X,\,Y) + b^2\mathrm{Var}\,X.\end{aligned}$$

This is a quadratic in b and is minimized when

$$\hat{b} = \frac{\mathrm{Cov}(X,\,Y)}{\mathrm{Var}\,X}, \tag{5.18}$$

so
$$\min_{a,b} \mathrm{E}\big([Y - (a + bX)]^2\big) = \mathrm{Var}\,Y - 2\frac{[\mathrm{Cov}(X,\,Y)]^2}{\mathrm{Var}\,X} + \frac{[\mathrm{Cov}(X,\,Y)]^2}{\mathrm{Var}\,X}$$

$$= \mathrm{Var}\,Y - \frac{[\mathrm{Cov}(X,\,Y)]^2}{\mathrm{Var}\,X}. \tag{5.19}$$

Thus from (5.17) and (5.18), the *best linear predictor (BLP)* of $Y|X$ is

$$\hat{a}(\hat{b}) + \hat{b}X = \Big[\mathrm{E}\,Y - \Big(\frac{\mathrm{Cov}(X,\,Y)}{\mathrm{Var}\,X}\Big)\mathrm{E}\,X\Big] + \Big[\frac{\mathrm{Cov}(X,\,Y)}{\mathrm{Var}\,X}\Big]X$$

$$= \mathrm{E}\,Y + \Big(\frac{\mathrm{Cov}(X,\,Y)}{\mathrm{Var}\,X}\Big)[X - \mathrm{E}\,X]. \tag{5.20}$$

$$\equiv \mu_Y + \rho_{X,Y}\Big(\frac{\sigma_Y}{\sigma_X}\Big)(X - \mu_X), \tag{5.21}$$

where $\mu_X = \mathrm{E}\,X$, $\mu_Y = \mathrm{E}\,Y$, $\sigma_X = \mathrm{sd}(X)$, $\sigma_Y = \mathrm{sd}(Y)$. Then $\mathrm{E}(\hat{a} + \hat{b}X) = \mu_Y$ and, from (5.19), the MSE of the BLP $\hat{a} + \hat{b}X$ is

$$\mathrm{E}\big([Y - (\hat{a} + \hat{b}X)]^2\big) = \sigma_Y^2 - \frac{\rho_{X,Y}^2\sigma_X^2\sigma_Y^2}{\sigma_X^2}$$

$$= (1 - \rho_{X,Y}^2)\sigma_Y^2. \tag{5.22}$$

The error of linear prediction $Y - (\hat{a} + \hat{b}X)$ is again called the *residual.* [See scatter plots.] The *basic decomposition formula* for linear prediction is:

$$Y = (\hat{a} + \hat{b}X) + [Y - (\hat{a} + \hat{b}X)] \tag{5.23}$$
$$\equiv \text{best linear predictor} + \text{residual.}$$

Again: E(best linear predictor)=EY and E(residual)=0. [verify]

Proposition 5.2. *(Linear variance decomposition). The best linear predictor and residual are uncorrelated:*

$$\mathrm{Cov}\big[\hat{a} + \hat{b}X,\ Y - (\hat{a} + \hat{b}X)\big] = 0. \tag{5.24}$$

Therefore the variance of Y can be decomposed as follows:

$$\mathrm{Var}\,Y = \mathrm{Var}(\hat{a} + \hat{b}X) + \mathrm{Var}[Y - (\hat{a} + \hat{b}X)] \tag{5.25}$$
$$\equiv \mathrm{Var}(\text{best linear predictor}) + \mathrm{Var}(\text{residual})$$
$$= \rho_{X,Y}^2\sigma_Y^2 + (1 - \rho_{X,Y}^2)\sigma_Y^2. \tag{5.26}$$

Proof. From bilinearity and (5.18),

$$\mathrm{Cov}\big[\hat{a} + \hat{b}X,\ Y - (\hat{a} + \hat{b}X)\big] = \hat{b}\big[\mathrm{Cov}(X, Y) - \hat{b}\mathrm{Var}X\big] = 0,$$

which establishes (5.24) and therefore (5.25). Finally, (5.26) follows from (5.21) and (5.22). □

Exercise 5.2. Prove (5.25) directly, then use it to deduce (5.24). □

Remark 5.2. Note that (5.22) gives another proof that $-1 \leq \rho_{X,Y} \leq 1$, or equivalently, $\rho_{X,Y}^2 \leq 1$. Also it follows from (5.26) that $\rho_{X,Y}^2$, not $|\rho_{X,Y}|$, expresses the strength of the linear relationship between X and Y: if $\rho_{X,Y}^2 = 1$ then there is an exact linear relationship, while if $\rho_{X,Y}^2 = 0$ then $\hat{b} = 0$ and there is no overall linear relationship, the BLP reduces to the constant EY. Note that it is possible that $\rho_{X,Y}^2 < 1$ even if there is an exact non-linear relationship between X and Y; for example if $Y = e^X$. (However, there is an exact linear relationship between $\log Y$ and X.) □

Figure 1. Scatter diagram for the heights of 1,078 fathers and sons, showing the positive association between son's height and father's height. Families where the height of the son equals the height of the father are plotted along the 45-degree line $y = x$. Families where the father is 72 inches tall (to the nearest inch) are plotted in the vertical strip.

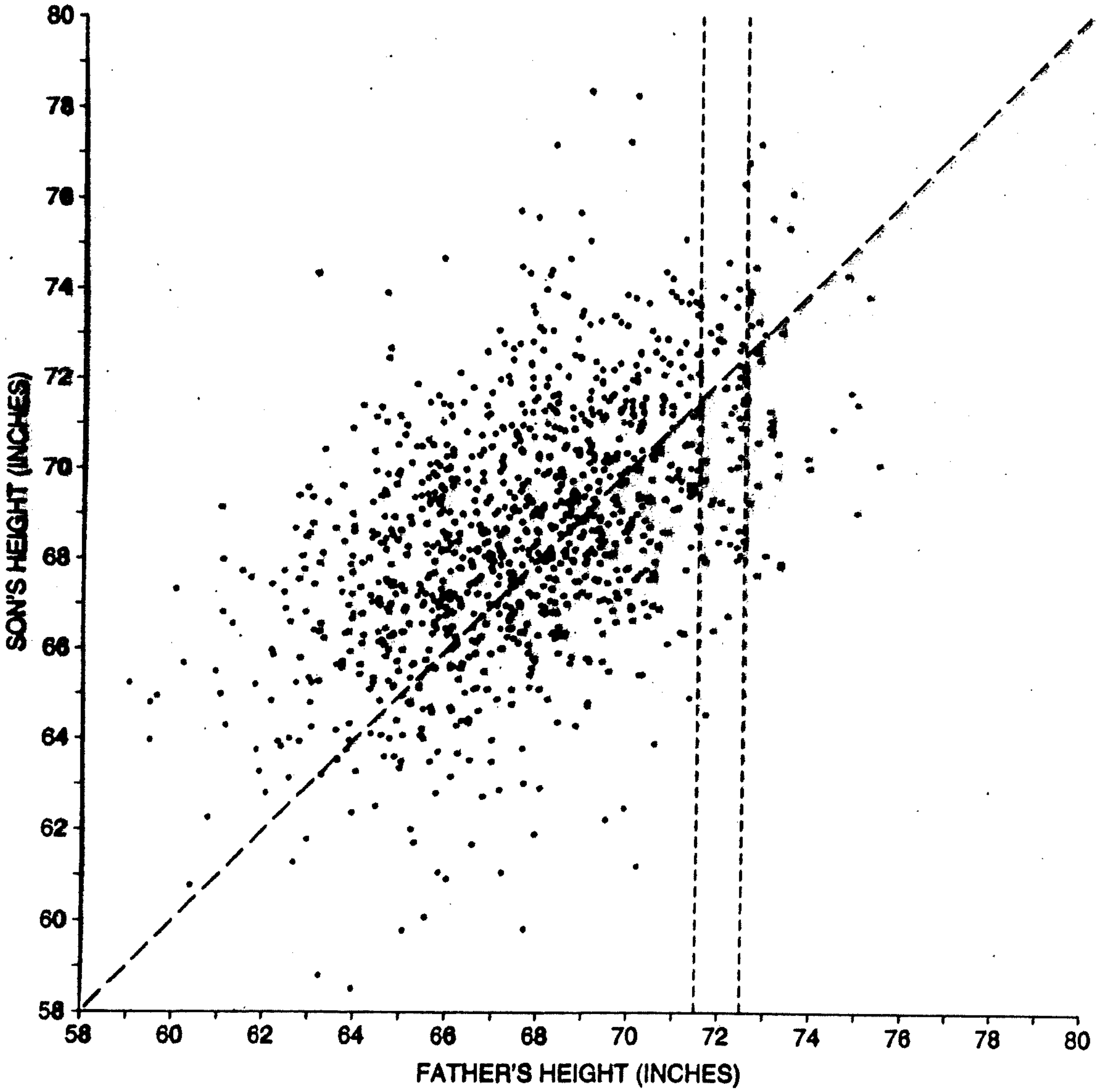
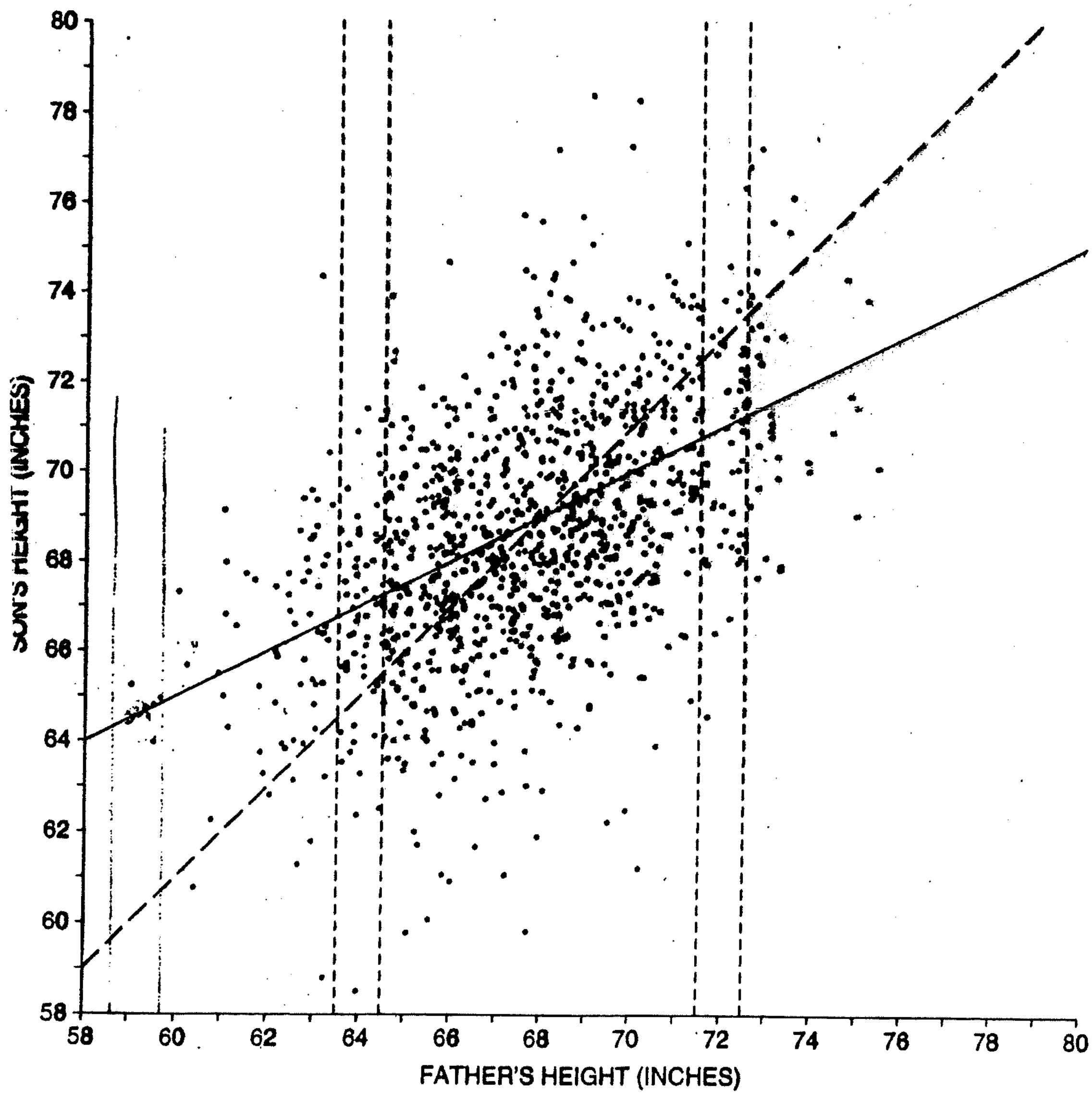


Figure 5. The regression effect. If a son is 1 inch taller than his father, the family is plotted along the dashed line. The points in the strip over 72 inches correspond to the families where the father is 72 inches tall, to the nearest inch; most of these points are below the dashed line. The points in the strip over 64 inches correspond to families where the father is 64 inches tall, to the nearest inch; most of these points are above the dashed line. The solid regression line picks off the centers of all the vertical strips, and is flatter than the dashed line.



Remark 5.3. Suppose we *know* that E$[Y|X]$ is a linear function, i.e., E$[Y|X] = c + dX$. (This holds in multinomial and multivariate normal distributions – see §7.5 and §8.3). Then necessarily E$[Y|X] = \hat{a} + \hat{b}X$, i.e., the best general predictor must coincide with the best linear predictor [why?]. In this case, all the results in §5.2 reduce to those in §5.3.

Exercise 5.3. Prove or disprove: E$[Y|X]$ is linear $\Rightarrow$ E$[X|Y]$ is linear.

Example 5.1. (Another Bayesian example.) Let X = height of father, Y = height of son. Suppose that the joint distribution of (X, Y) is specified by the conditional distribution of $Y|X$ and the marginal distribution of X as follows:

$$Y|X \sim \text{Normal}(a + bX, \tau^2), \tag{5.27}$$

$$X \sim \text{Normal}(\mu, \sigma^2), \tag{5.28}$$

so

$$\begin{aligned} f(y|x) &= \frac{1}{\sqrt{2\pi}\tau} e^{-\frac{(y-a-bx)^2}{2\tau^2}}, \\ f(x) &= \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}. \end{aligned} \tag{5.29}$$

Suppose that we observe a son's height Y and want to estimate ($\equiv$ predict) his father's height X using $E[X|Y]$. For this we find $f(x|y)$ via Bayes' formula (4.14):

$$\begin{aligned} f(x|y) &= \frac{f(y|x)f(x)}{f(y)} \\ &= \frac{\text{const}}{f(y)} \cdot e^{-h(x)}, \end{aligned}$$

where

$$h(x) \equiv \frac{(y-a-bx)^2}{2\tau^2} + \frac{(x-\mu)^2}{2\sigma^2} \tag{5.30}$$

is a quadratic in x with leading term

$$\frac{x^2}{2}\left(\frac{b^2}{\tau^2} + \frac{1}{\sigma^2}\right) \equiv \frac{x^2}{2}\left(\frac{1}{\gamma^2}\right). \tag{5.31}$$

Since $h(x)$ is minimized when

$$h'(x) \equiv \frac{b(a+bx-y)}{\tau^2} + \frac{x-\mu}{\sigma^2} = 0,$$

i.e., when

$$x = \frac{\frac{b(y-a)}{\tau^2} + \frac{\mu}{\sigma^2}}{\frac{b^2}{\tau^2} + \frac{1}{\sigma^2}} \equiv c(y), \tag{5.32}$$

it follows that

$$h(x) = \frac{(x-c(y))^2}{2\gamma^2} + d(y),$$

where $d(y)$ does not involve x. Thus $f(x|y)$ must have the following form:

$$f(x|y) = \frac{\text{const} \cdot e^{-d(y)}}{f(y)} \cdot e^{-\frac{(x-c(y))^2}{2\gamma^2}}.$$

However, since $f(x|y)$ must be a pdf in x for each fixed y, we conclude that

$$f(x|y) = \frac{1}{\sqrt{2\pi}\gamma} e^{-\frac{(x-c(y))^2}{2\gamma^2}}.$$

(*Note that it was not necessary to find $f(y)$!*) Thus the conditional ($\equiv$ posterior) distribution of X given Y is

$$\begin{aligned} X|Y &\sim N\big(c(Y), \gamma^2\big) \\ &\equiv N\left(\frac{\frac{b(Y-a)}{\tau^2} + \frac{\mu}{\sigma^2}}{\frac{b^2}{\tau^2} + \frac{1}{\sigma^2}}, \frac{1}{\frac{b^2}{\tau^2} + \frac{1}{\sigma^2}}\right), \end{aligned} \tag{5.33}$$

so the best estimator $\equiv$ predictor of $X|Y$ is

$$\mathrm{E}[X|Y] = \frac{\frac{Y-a}{b} \cdot \frac{b^2}{\tau^2} + \mu \cdot \frac{1}{\sigma^2}}{\frac{b^2}{\tau^2} + \frac{1}{\sigma^2}}, \tag{5.34}$$

a linear function of Y. In fact, this is again a convex combination of the "unbiased" predictor $(Y-a)/b$ of X and the *a priori* predictor $\mu \equiv \mathrm{E}(X)$.

Note that the weights assigned to $(Y - a)/b$ and μ are proportional to b^2 and inversely proportional to τ^2 and σ^2, respectively [interpret!]. Also, since $\mathrm{E}[X|Y]$ is a linear function of Y, it is also the best linear predictor of $X|Y$ (recall Remark 5.3).

How good is $\mathrm{E}[X|Y]$? From (5.14) (with X and Y interchanged), its MSE is given by

$$\begin{aligned}
\mathrm{E}\big[(X - \mathrm{E}[X|Y])^2\big] &= \mathrm{Var}\big(X - \mathrm{E}[X|Y]\big) \\
&= \mathrm{E}\big(\mathrm{Var}[X|Y]\big) \\
&= \frac{1}{\frac{b^2}{\tau^2} + \frac{1}{\sigma^2}} \qquad (5.35) \\
&\approx \begin{cases} 0, & \text{if } \sigma^2 \approx 0 \text{ (i.e., if prior info very good);} \\ \frac{\tau^2}{b^2}, & \text{if } \sigma^2 \approx \infty \text{ (i.e., if prior info not good).} \end{cases}
\end{aligned}$$

The result (5.35) can also be derived in terms of $\rho_{X,Y}$. First,

$$\begin{aligned}
\mathrm{Var}\, X &= \sigma^2; && \text{[by (5.28)]} \\
\mathrm{Var}\, Y &= \mathrm{Var}\big(\mathrm{E}[Y|X]\big) + \mathrm{E}\big(\mathrm{Var}\,[Y|X]\big) && \text{[by (5.13)]} \\
&= b^2 \mathrm{Var}(X) + \mathrm{E}(\tau^2) && \text{[by (5.27)]} \\
&= b^2\sigma^2 + \tau^2; \\
\mathrm{Cov}(X,\, Y) &= \mathrm{Cov}(X,\, \mathrm{E}[Y|X]) && \text{[by (4.17)]} \\
&= b\,\mathrm{Cov}(X,\, X) \\
&= b\sigma^2.
\end{aligned}$$

Thus

$$1 - \rho_{X,Y}^2 = 1 - \frac{(b\sigma^2)^2}{\sigma^2(b^2\sigma^2 + \tau^2)} = \frac{\tau^2}{b^2\sigma^2 + \tau^2}, \qquad (5.36)$$

so by (5.22) (with X and Y interchanged), the MSE of the BLP $\mathrm{E}[X|Y]$ is

$$(1 - \rho_{X,Y}^2)\,\sigma^2 = \frac{\tau^2\sigma^2}{b^2\sigma^2 + \tau^2}, \qquad (5.37)$$

which agrees with (5.35). □

Remark 5.4. Consider a rvtr $(X_1, \ldots, X_k, Y)$ with $\mathrm{E}(Y^2) < \infty$. All the above results remain valid but matrix notation is required – see §8.2. For example, the best linear predictor $g(\mathbf{X}_k) \equiv a + b^t\mathbf{X}_k$ of Y based on $\mathbf{X}_k \equiv (X_1, \ldots, X_k)^t$ is given by (compare to (5.20) and (8.76))

$$\hat{a} + \hat{b}^t\mathbf{X}_k = \mathrm{E}\,Y + \mathrm{Cov}(Y, \mathbf{X}_k)[\mathrm{Cov}(\mathbf{X}_k)]^{-1}(\mathbf{X}_k - \mathrm{E}(\mathbf{X}_k)), \tag{5.38}$$

while (5.22)-(5.26) remain valid if $\rho^2_{X,Y}$ is replaced by the *multiple correlation coefficient*

$$\rho^2_{\mathbf{X}_k,Y} \equiv \mathrm{Cov}(Y, \mathbf{X}_k)[\mathrm{Cov}(\mathbf{X}_k)]^{-1}\mathrm{Cov}(\mathbf{X}_k, Y)/(\mathrm{Var}\,Y). \tag{5.39}$$

In particular $\mathrm{E}([Y - (\hat{a} + \hat{b}^t\mathbf{X}_k)]^2) = (1 - \rho^2_{\mathbf{X}_k,Y})\sigma^2_Y$ (cf. (5.22)), so

$$1 \geq \rho^2_{\mathbf{X}_k,Y} \geq \cdots \geq \rho^2_{\mathbf{X}_1,Y} \geq 0. \tag{5.40}$$

Remark 5.5. In practice, the population quantities μ_X, μ_Y, σ^2_X, σ^2_Y, and $\rho_{X,Y}$ that appear in the BLP $\hat{a} + \hat{b}X$ (5.21) are unknown, so must be estimated from a sample $(X_1, Y_1), \ldots, (X_n, Y_n)$. The usual estimators are:

$$\begin{aligned} &\hat{\mu}_X = \bar{X}_n, && \hat{\mu}_Y = \bar{Y}_n, \\ &\hat{\sigma}^2_X = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2, && \hat{\sigma}^2_Y = \frac{1}{n-1}\sum_{i=1}^n (Y_i - \bar{Y}_n)^2, \end{aligned} \tag{5.41}$$

$$\hat{\rho}_{X,Y} = \frac{\sum_{i=1}^n (X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\sum_{i=1}^n (X_i - \bar{X}_n)^2}\sqrt{\sum_{i=1}^n (Y_i - \bar{Y}_n)^2}}.$$

When (X, Y) has a bivariate normal distribution these estimators are the MLEs (CB Exercise 7.18), hence asymptotically optimal by Theorem 14.20.

Remark 5.6. If a linear predictor is not appropriate, i.e., if it is *not* the case that $Y \approx a + bX$ (+ error), it may be the case that a tranfsormation will convert the non-linear relation into a linear one. For example,

$$Y \approx ae^{bX} \quad \Rightarrow \quad Y' \equiv \log Y \approx (\log a) + bX \tag{5.42}$$

$$Y \approx aX^b \quad \Rightarrow \quad Y' \equiv \log Y \approx (\log a) + b(\log X), \tag{5.43}$$

$$Y \approx a + bX^2 \quad \Rightarrow \quad Y' \equiv Y \quad \approx a + b(X^2). \tag{5.44}$$

(Of course, one must assume that the error in Y', not Y, is additive.) □

Note: The examples (5.42) – (5.44) emphasize that the "linearity" in a "linear model" comes from the (approximate) linear dependence of the response Y on the unknown parameters a, b, $\log a$, etc., not on X! (See §8.5.)

5.4. Covariance and Regression.

Proposition 5.3. *If $E[Y|X]$ is a strictly increasing function of X, then* $\mathrm{Cov}(X, Y) > 0$.

Proof. This is an immediate consequence of (4.17) and the following lemma.

Lemma 5.1. (Chebyshev's Other Inequality) *If X is a non-degenerate rv and $g(X)$ and $h(X)$ are both strictly increasing in X, then*

$$\mathrm{Cov}(g(X), h(X)) > 0. \tag{5.45}$$

Proof. Let Y be another rv with the same distribution as X and independent of X. Then

$$[g(X) - g(Y)][h(X) - h(Y)] \geq 0,$$

with strict inequality whenever $X \neq Y$, which occurs with positive probability since X and Y are independent and non-degenerate [verify]. Thus

$$\begin{aligned} 0 &< \mathrm{E}\big([g(X) - g(Y)][h(X) - h(Y)]\big) \\ &= \mathrm{E}[g(X)h(X)] - \mathrm{E}[g(X)h(Y)] - \mathrm{E}[g(Y)h(X)] + \mathrm{E}[g(Y)h(Y)] \\ &= 2\big(\mathrm{E}[g(X)h(X)] - \mathrm{E}[g(X)]\mathrm{E}[h(X)]\big), \end{aligned}$$

since X and Y are i.i.d., which yields (5.45). □

6. Transforming Continuous Multivariate Distributions.

6.1. Two functions of two random variables.

Let (X, Y) be a continuous random vector with joint pdf $f_{X,Y}(x, y)$ and consider a transformation $(X, Y) \to (U, V)$, where

$$U = u(X, Y), \qquad V = v(X, Y). \tag{6.1}$$

The pdf $f_{U,V}(u, v)$ is obtained by differentiating the joint cdf $F_{U,V}(u, v)$:

$$\begin{aligned} f_{U,V}(u,v) &= \frac{\partial^2}{\partial u \partial v} F_{U,V}(u,v) \\ &= \frac{\partial^2}{\partial u \partial v} P[U \le u,\ V \le v] \\ &= \frac{\partial^2}{\partial u \partial v} \iint_{R(u,v)} f_{X,Y}(x,y) dx\, dy, \end{aligned} \tag{6.2}$$

where $R(u, v)$ is the region $\{(x, y) \mid u(x, y) \le u,\ v(x, y) \le v\}$. If the double integral can be evaluated[5] explicitly, then the derivatives can be taken to obtain $f_{U,V}(u, v)$. Three examples of this method[6] are now presented.

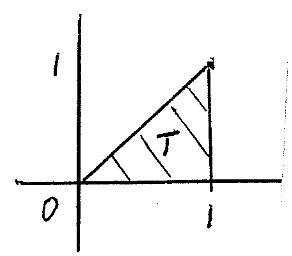
Example 6.1. Let (X, Y) be uniformly distributed on the unit square; set

$$U = \max(X, Y), \qquad V = \min(X, Y). \tag{6.3}$$

In Example 2.5 we found the marginal pdfs of U and V separately. Here we find the joint pdf of (U, V) by using (6.2).

First specify the range of (U, V): this is just the triangle

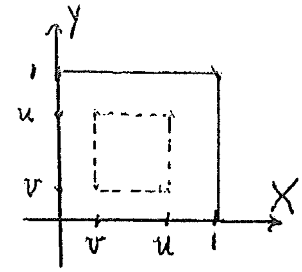
$$T \equiv \{(u, v) \mid 0 < u < 1,\ 0 < v < u\}$$



from Example 4.2. Then for $(u, v) \in T$,

[5] If the mapping $(X, Y) \to (U, V)$ given by (6.1) is differentiable and 1-1, then the integral in (6.2) need not be evaluated – instead, $f_{U,V}(u, v)$ can be obtained by simply multiplying $f_{X,Y}(x, y)$ by the *Jacobian* – see §6.2.

[6] Another example was given earlier – see (1.45).



$$
\begin{aligned}
F_{U,V}(u,v) &= P[U \le u] - P[U \le u,\, V > v] \\
&= P[X \le u,\, Y \le u] - P[v < X \le u,\, v < Y \le u] \\
&= u^2 - (u-v)^2, \qquad\qquad \text{[by independence]}
\end{aligned}
$$

$$
(6.4) \qquad f_{U,V}(u,v) = 2I_T(u,v),
$$

which is the same as (4.19). Thus (U,V) *is uniformly distributed on* T.

Note:

$$
\begin{aligned}
\mathrm{E}(UV) &= \mathrm{E}(XY) = (\mathrm{E}\,X)(\mathrm{E}\,Y) = \frac{1}{4}, \\
\mathrm{E}(U+V) &= \mathrm{E}(X+Y) = 1, \\
\mathrm{E}(V) &= \int_0^1 [1 - F_V(v)]dv = \int_0^1 (1-v)^2 dv = \frac{1}{3}, \qquad [\textit{CB Exer. 2.14}],
\end{aligned}
$$

so $\mathrm{E}(U) = \frac{2}{3}$, hence (recall Example 4.2)

$$
\mathrm{Cov}(U,\, V) = \frac{1}{4} - (\mathrm{E}\,U)(\mathrm{E}\,V) = \frac{1}{4} - \left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{1}{36}. \qquad \square
$$

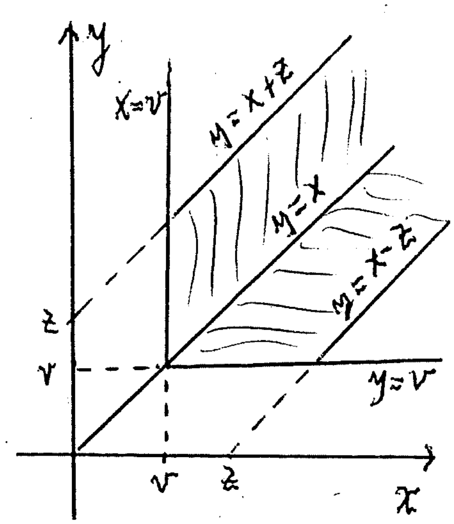
Example 6.2. Let X, Y be i.i.d. Exponential(1) rvs and set

$$
(6.5) \qquad V = \min(X,Y), \qquad Z = |X - Y| \equiv \max(X,Y) - \min(X,Y).
$$

In Example 2.6 we found the marginal pdfs of V and Z separately. Here we find the joint pdf of (V,Z) via (6.2).

The range of (V,Z) is $(0,\infty) \times (0,\infty)$. For $0 < v < \infty, 0 < z < \infty$,

$$
\begin{aligned}
P[V \le v,\, Z \le z] &= P[Z \le z] - P[V > v,\, Z \le z], \\
P[V > v,\, Z \le z] &= P[\,X > v,\, Y > v,\, |X - Y| \le z] \\
&= P[(X,Y) \text{ in shaded region}] \\
(6.6) \qquad\qquad &= 2P[(X,Y) \text{ in half of region}] \\
&= 2P[X > v,\, X \le Y \le X + z] \\
&= 2\int_v^\infty e^{-x}\Big(\int_x^{x+z} e^{-y} dy\Big) dx
\end{aligned}
$$



$$
\begin{aligned}
&= 2\int_v^\infty e^{-x}\left(e^{-x} - e^{-x-z}\right)dx \\
&= 2\left(1 - e^{-z}\right)\int_v^\infty e^{-2x}\,dx \\
&= e^{-2v}\left(1 - e^{-z}\right),
\end{aligned}
$$

where (6.6) follows by symmetry: $(X, Y) \sim (Y, X)$. Therefore

$$
\begin{aligned}
f_{V,Z}(v,z) &= -\frac{\partial^2}{\partial v \partial z}\left[e^{-2v}\left(1 - e^{-z}\right)\right] \\
&= \left(2e^{-2v}\right)\left(e^{-z}\right) \qquad (6.7) \\
&= f_V(v)\, f_Z(z), \qquad \text{[recall (2.17), (2.20)]}
\end{aligned}
$$

so V and Z are *independent*, with

$$
V \sim \text{Exponential}(2), \qquad Z \sim \text{Exponential}(1).
$$

Interpretation: Suppose that X, Y represent the lifetimes of two lightbulbs. Thus, $V \equiv \min(X, Y)$ is the time to the *first* burnout (either X or Y). Once the first burnout occurs, the time to the second burnout has the *original* exponential distribution Expo(1), *not* Expo(2). This is another memory-free property of the exponential distribution. It is stronger in that it shows that the process renews itself at the *random* time V. (The first memory-free property concerned any *fixed* renewal time t.) □

Exercise 6.1. Find $\text{Var}[\max(X, Y)]$. Find $\text{Var}[\max(X_1, \ldots, X_n)]$, where the X_i are i.i.d. Exponential(1) rvs (recall Exercise 2.4). □

Exercise 6.1.** *(Converse of Example 6.2: a second characterization of the exponential distribution (compare to Exercise 1.2).)* Let X, Y be positive i.i.d. rvs with common pdf f, assumed positive and continuous on $(0, \infty)$. Show that if $V \equiv \min(X, Y)$ and $Z \equiv |X - Y|$ are independent, then X and Y must be exponential rvs, i.e., $f(x) = \lambda e^{-\lambda x}$ for some $\lambda > 0$.

Hint: follow the approach of Example 6.2.

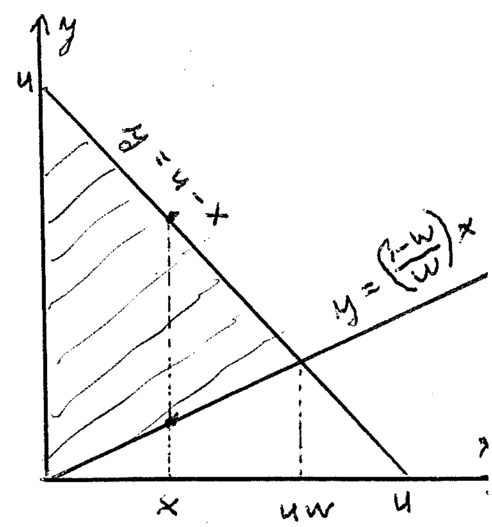
Example 6.3. Again let X, Y be i.i.d. Exponential(1) rvs and set

$$U = X + Y, \qquad W = \frac{X}{X+Y}. \tag{6.8}$$

In Example 2.6 we found the marginal pdfs of U and W separately. Here we find the joint pdf of (U, W) via (6.2).

The range of (U, W) is $(0, \infty) \times (0, 1)$. For $0 < u < \infty, 0 < w < 1$,

$$\begin{aligned} P[U \le u,\, W \le w] &= P[X + Y \le u,\, X \le w(X+Y)] \\ &= \int_0^{uw} e^{-x} \Big(\int_{\left(\frac{1-w}{w}\right)x}^{u-x} e^{-y} dy \Big) dx \\ &= \int_0^{uw} \left(e^{-\frac{x}{w}} - e^{-u} \right) dx \\ &= [1 - e^{-u} - ue^{-u}] \cdot w. \end{aligned} \tag{6.9}$$



Therefore

$$\begin{aligned} f_{U,W}(u, w) &= \frac{\partial^2}{\partial u \partial w} [1 - e^{-u} - ue^{-u}] \cdot w \\ &= \left(ue^{-u} \right) \cdot 1 \\ &= f_U(u)\, f_W(w), \qquad \text{[recall (2.16), (2.21)]} \end{aligned} \tag{6.10}$$

so U and W are *independent*, with

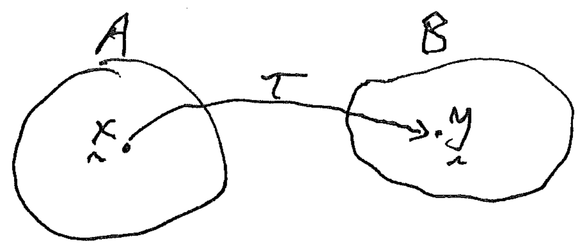
$$U \sim \text{Gamma}(2, 1), \qquad W \sim \text{Uniform}(0, 1).$$

Interpretation: As noted in Example 2.6, (6.9) and (6.10) can be viewed as a "backward" memory-free property of the exponential distribution: given $X+Y$, the location of X is uniformly distributed over the interval $(0, X+Y)$, i.e., over the "past". □

Exercise 6.2. (i) Let $U = \max(X, Y)$ and $V = \min(X, Y)$ in Example 6.2. Find the conditional distribution of $V \mid U$. Does this exhibit the "backward" memory-free property?

(ii) Repeat this question for Example 6.1, where X, Y are i.i.d uniform rvs.

(iii) Repeat this question for $f(x, y) = ce^{-\max(x,y)}$, $0 < x, y < \infty$. (Find c.)



6.2. The Jacobian method.

Let A, B be open sets in $\mathbf{R}^n$ and

$$T : A \to B \qquad (6.11)$$
$$x \equiv (x_1, \ldots, x_n) \mapsto y \equiv (y_1, \ldots, y_n)$$

a smooth bijective (1-1 and onto) mapping ($\equiv$ diffeomorphism). The *Jacobian matrix* of this mapping is given by

$$J_T(x) \equiv \left(\frac{\partial y}{\partial x}\right) := \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}, \qquad (6.12)$$

that is, $\left(\frac{\partial y}{\partial x}\right)_{ij} = \frac{\partial y_i}{\partial x_j}$. The *Jacobian* of the mapping is given by

$$|J_T(x)| \equiv \left|\frac{\partial y}{\partial x}\right| := \left|\det\left(\frac{\partial y}{\partial x}\right)\right| \geq 0.$$

Theorem 6.1. *(Substitution formula for multiple integrals.) Let A, B be open sets in $\mathbf{R}^n$ and $T : A \to B$ a diffeomorphism such that $|J_T(x)| > 0$ for "a.e." x. Let f be a real-valued integrable function on A. Then*

$$\int_A f(x)dx = \int_{B=T(A)} f(T^{-1}(y)) \left|\frac{\partial x}{\partial y}\right| dy. \qquad (6.13)$$

[explain]

[See (6.16) for the relation between $\left|\frac{\partial x}{\partial y}\right|$ and $\left|\frac{\partial y}{\partial x}\right|$.]

Corollary 6.2. *Let X be a random vector (rvtr) with pdf $f_X(x)$ (wrto Lebesgue measure) on $\mathbf{R}^n$. Suppose that $A := \{x \mid f_X(x) > 0\}$ is open and that $T : A \to B$ is a diffeomorphism with $|J_T(x)| > 0$ a.e. Then the pdf of $Y := T(X)$ is given by*

$$f_Y(y) = f_X(T^{-1}(y)) \cdot \left|\frac{\partial x}{\partial y}\right| \cdot I_B(y). \qquad (6.14)$$

Proof. For any (measurable) set $C \subseteq \mathbf{R}^n$,

$$\begin{aligned} P[Y \in C] &= P[X \in T^{-1}(C)] \\ &= \int_{T^{-1}(C)} f_X(x)dx \\ &= \int_C f(T^{-1}(y)) \left|\frac{\partial x}{\partial y}\right| dy, \end{aligned}$$

which confirms (6.14) [Why?] □

The calculation of Jacobians can be simplified by application of the following rules.

Chain Rule: Suppose that $x \mapsto y$ and $y \mapsto z$ are diffeomorphisms. Then $x \mapsto z$ is a diffeomorphism and

(6.15) $$\left|\frac{\partial z}{\partial x}\right| = \left|\frac{\partial z}{\partial y}\right|_{y=y(x)} \cdot \left|\frac{\partial y}{\partial x}\right|.$$

[This follows from the chain rule for partial derivatives:

$$\frac{\partial z_i(y_1(x_1, \ldots, x_n), \ldots, y_n(x_1, \ldots, x_n))}{\partial x_j} = \sum_k \frac{\partial z_i}{\partial y_k}\frac{\partial y_k}{\partial x_j} = \left[\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right)\right]_{ij}.$$

Therefore $\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right)$; now take determinants.]

Inverse Rule: Suppose that $x \mapsto y$ is a diffeomorphism. Then

$$\left|\frac{\partial x}{\partial y}\right|_{y=y(x)} = \left|\frac{\partial y}{\partial x}\right|^{-1}.$$

[Set $z = x$ in (6.15).] Reversing the roles of x and y we obtain

(6.16) $$\left|\frac{\partial y}{\partial x}\right|_{x=x(y)} = \left|\frac{\partial x}{\partial y}\right|^{-1}.$$

Combination Rule: Suppose that $x \mapsto u$ and $y \mapsto v$ are (unrelated) diffeomorphisms. Then

(6.17) $$\left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \left|\frac{\partial u}{\partial x}\right| \cdot \left|\frac{\partial v}{\partial y}\right|.$$

[The Jacobian matrix is given by

$$\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = \begin{pmatrix} \frac{\partial u}{\partial x} & 0 \\ 0 & \frac{\partial v}{\partial y} \end{pmatrix}. \quad]$$

Extended Combination Rule: Suppose that $(x, y) \mapsto (u, v)$ is a diffeomorphism of the form $u = u(x)$, $v = v(x, y)$. Then (6.17) remains valid.

[The Jacobian matrix is given by

$$\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = \begin{pmatrix} \frac{\partial u}{\partial x} & 0 \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}. \quad]$$

Jacobians of linear mappings. Let $A\colon p \times p$ and $B : n \times n$ be nonsingular matrices and c a nonzero scalar (A, B, L, M, U, V, c fixed.) Then:

(a) *vectors:* $y = cx$, $x, y\colon p \times 1$: $\left|\frac{\partial y}{\partial x}\right| = |c|^p$. [combination rule]

(b) *matrices:* $Y = cX$, $X, Y\colon p \times n$: $\left|\frac{\partial Y}{\partial X}\right| = |c|^{pn}$. [comb. rule]

(c) *symmetric matrices:* $Y = cX$, $X, Y\colon p \times p$, symmetric: $\left|\frac{\partial Y}{\partial X}\right| = |c|^{\frac{p(p+1)}{2}}$. [comb. rule]

(d) *vectors:* $y = Ax$, $x, y\colon p \times 1$, $A : p \times p$: $\left|\frac{\partial y}{\partial x}\right| = |\det A|$. [verify]

(e) *matrices:* $Y = AX$, $X, Y\colon p \times n$: $\left|\frac{\partial Y}{\partial X}\right| = |\det A|^n$. [comb. rule]

$Y = XB$, $X, Y\colon p \times n$: $\left|\frac{\partial Y}{\partial X}\right| = |\det B|^p$. [comb. rule]

$Y = AXB$, $X, Y\colon p \times n$: $\left|\frac{\partial Y}{\partial X}\right| = |\det A|^n |\det B|^p$. [chain rule]

Example 6.4. The Gamma(α, λ) distribution with shape parameter $\alpha > 0$ and intensity parameter $\lambda > 0$ has pdf

$$g(x; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} I_{(0,\infty)}(x). \tag{6.18}$$

Let $X \sim$ Gamma(α, λ) and $Y \sim$ Gamma(β, λ) be independent Gamma random variables with the same scale parameter and define

$$U = X + Y, \qquad W = \frac{X}{X + Y}. \tag{6.19}$$

Find the joint pdf of (U, W), find the marginal pdfs of U and W, and show that U and W are independent. (This extends Example 6.3.)

Solution. The ranges of (X, Y) and (U, W) are

$$A := \{(x, y) \mid x > 0, y > 0\} \equiv (0, \infty) \times (0, \infty),$$
$$B := \{(u, w) \mid u > 0, 0 < w < 1\} \equiv (0, \infty) \times (0, 1),$$

respectively. Notice that both A and B are Cartesian product sets. The transformation

$$\begin{aligned} T &: A \to B \\ (x, y) &\mapsto (x + y,\ x/(x + y)) \end{aligned} \tag{6.20}$$

is bijective, with inverse given by

$$\begin{aligned} T^{-1} &: B \to A \\ (u, w) &\mapsto (uw, u(1 - w)). \end{aligned} \tag{6.21}$$

Thus T^{-1} is continuously differentiable and bijective, and its Jacobian is given by

$$\left|\frac{\partial(x, y)}{\partial(u, w)}\right| = \begin{vmatrix} \frac{\partial(uw)}{\partial u} & \frac{\partial(uw)}{\partial w} \\ \frac{\partial(u(1-w))}{\partial u} & \frac{\partial(u(1-w))}{\partial w} \end{vmatrix} = \begin{vmatrix} w & u \\ 1 - w & -u \end{vmatrix} = u. \tag{6.22}$$

Because

$$f_{X,Y}(x, y) = \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} e^{-\lambda x} y^{\beta-1} e^{-\lambda y} I_A(x, y), \tag{6.23}$$

it follows from the transformation formula (6.14) that

$$\begin{aligned} f_{U,W}(u, w) =& \frac{\lambda^{\alpha+\beta}}{\Gamma(\alpha)\Gamma(\beta)} (uw)^{\alpha-1} e^{-\lambda uw} (u(1 - w))^{\beta-1} e^{-\lambda u(1-w)} \cdot u \cdot I_B(u, w) \\ =& \frac{\lambda^{\alpha+\beta} u^{\alpha+\beta-1} e^{-\lambda u}}{\Gamma(\alpha + \beta)} I_{(0,\infty)}(u) \cdot \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1} (1 - w)^{\beta-1} I_{(0,1)}(w) \\ \equiv& f_U(u) \cdot f_W(w). \end{aligned} \tag{6.24}$$

Thus $U \perp\!\!\!\perp W$, $U \sim \text{Gamma}(\alpha+\beta, \lambda)$, $W \sim \text{Beta}(\alpha, \beta)$. □

Remark 6.1. (The converse of Example 6.4 – a characterization of the Gamma distribution.) The Gamma family is the *only* family of distributions on $(0, \infty)$ with the property that $X+Y$ and $\frac{X}{X+Y}$ are independent. [References: Hogg (1951 *Ann. Math. Statist.*); Lukacs (1955 *Ann. Math. Statist.*); G. Marsaglia (Festschrift for I. Olkin); Kagan, Linnik, Rao (book).

Remark 6.2. Let U and W be independent with $U \sim \text{Gamma}(\alpha+\beta, \lambda)$ and $W \sim \text{Beta}(\alpha, \beta)$. It follows from Example 6.4 that $UW \sim \text{Gamma}(\alpha, \lambda)$, $U(1-W) \sim \text{Gamma}(\beta, \lambda)$, and $UW \perp\!\!\!\perp U(1-W)$.

Exercise 6.3. Let X, Y, Z be independent rvs with $X \sim \text{Gamma}(\alpha, \lambda)$, $Y \sim \text{Gamma}(\beta, \lambda)$, and $Z \sim \text{Gamma}(\gamma, \lambda)$. Set

$$U = \frac{X}{X+Y}, \qquad V = \frac{X+Y}{X+Y+Z}, \qquad W = X+Y+Z.$$

Show that $U \perp\!\!\!\perp V \perp\!\!\!\perp W$ and find the distributions of U, V, and W.

Example 6.5. Let X, Y be i.i.d. random variables each having an Exponential (λ) distribution on $(0, \infty)$ with pdf

$$f_\lambda(x) = \lambda e^{-\lambda x} I_{(0,\infty)}(x) \tag{6.25}$$

w.r.to Lebesgue measure. Find the joint pdf of (V, Z), where

$$V = \min(X, Y), \qquad Z = X - Y.$$

(Be sure to specify the range of (V, Z).) Find the marginal pdfs of V and Z, and show they are independent.

Solution. The ranges of (X, Y) and (V, Z) are

$$\begin{aligned} A &:= \{(x,y) \mid x > 0,\ y > 0\} \equiv (0,\infty) \times (0,\infty), \\ B &:= \{(v,z) \mid v > 0\} \equiv (0,\infty) \times (-\infty,\infty), \end{aligned} \tag{6.26}$$

respectively. Both A and B are Cartesian product sets. The transformation

$$\begin{aligned} &T : A \to B \\ &(x,y) \mapsto (\min(x,y),\ x-y) \equiv (v,z) \end{aligned} \tag{6.27}$$

is bijective, with inverse given by

$$(6.28)\qquad \begin{aligned} T^{-1} &: B \to A \\ (v,z) &\mapsto (v+z^+,\ v+z^-) \equiv (x,y), \end{aligned}$$

where $z^+ = \max(z,0)$, $z^- = -\min(z,0)$. Then T^{-1} is continuously differentiable on the open set $B^* := B \setminus N_1$, where $N_1 := \{(v,z) \mid z = 0\}$ is a (Lebesgue-) null set. The Jacobian of T^{-1} on B^* is given by [verify!]

$$(6.29)\qquad \left|\frac{\partial(x,y)}{\partial(v,z)}\right| = \begin{cases} +\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1, & \text{if } z > 0; \\ +\begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} = 1, & \text{if } z < 0. \end{cases}$$

Because

$$(6.30)\qquad f_{X,Y}(x,y) = \lambda^2 e^{-\lambda(x+y)} I_A(x,y),$$

it follows from the transformation formula (6.14) and (6.28) that

$$(6.31)\qquad \begin{aligned} f_{V,Z}(v,z) =& \lambda^2 e^{-\lambda(2v+|z|)} I_B(v,z) \\ =& 2\lambda e^{-2\lambda v} I_{(0,\infty)}(v) \cdot \frac{\lambda}{2} e^{-\lambda|z|} I_{(-\infty,\infty)}(z) \\ \equiv& f_V(v) \cdot f_Z(z). \end{aligned}$$

Thus, V and Z are independent, V has an Exponential(2λ) distribution on $(0,\infty)$, and Z has a "double exponential distribution" on $(-\infty,\infty)$. □

Exercise 6.4.** *(Converse of Example 6.5: another characterization of the exponential distribution (compare to Exercise 6.1**).)* Let X, Y be i.i.d. *positive* random variables, each having a positive and continuous pdf f on $(0,\infty)$. Show that if $V \equiv \min(X,Y)$ and $Z \equiv X - Y$ are independent, then each must have an Exponential(λ) distribution, i.e., $f = f_\lambda$ for some $\lambda > 0$.

Hint: Express the joint pdf $f_{V,Z}$ and the marginal pdfs f_V, f_Z in terms of f. By independence, $f_{V,Z} = f_V f_Z$. Deduce that

$$(6.32)\qquad f(v+|z|) = [1-F(v)]f(|z|) \quad \text{for } v > 0,\ -\infty < z < \infty,$$

where $F(v) = \int_0^v f(x)dx$. (To be rigorous, you have to beware of null sets, i.e., exceptional sets of measure 0.)

For extra credit, drop the assumption that f is positive; the conclusion is changed to $X, Y \sim$ Exponential(λ)+constant. For double super extra credit also drop the assumption that f is continuous. □

Exercise 6.5. (Continuation of Proposition 3.1.) Let $T_1, T_2, \ldots$ be the jump times in a homogeneous Poisson process with intensity parameter λ.

(i) Show that T_1 and $T_2 - T_1$ are independent Exponential(λ) rvs.

(ii) For any $n \geq 3$, show that T_1, $T_2 - T_1$, ..., $T_n - T_{n-1}$ are i.i.d. Exponential (λ) rvs.

Example 6.6. (Polar coordinates in $\mathbf{R}^2$). Let (X, Y) be a continuous bivariate rvtr with joint pdf $f_{X,Y}(x, y)$ on $\mathbf{R}^2$. Find the joint pdf $f_{R,\Theta}(r, \theta)$ of (R, Θ), where $(X, Y) \to (R, \Theta)$ is the 1-1 transformation [verify] whose inverse is given by

$$X = R\cos\Theta, \qquad Y = R\sin\Theta. \tag{6.33}$$

Solution. The range of (R, Θ) is $(0, \infty) \times [0, 2\pi)$, the Cartesian product of the ranges of R and Θ. The Jacobian of (6.33) is

$$\left|\frac{\partial(x, y)}{\partial(r, \theta)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r, \tag{6.34}$$

so from (6.14),

$$f_{R,\Theta}(r, \theta) = f_{X,Y}(r\cos\theta, r\sin\theta) \cdot r\ . \tag{6.35}$$

In the case where $f_{X,Y}(x, y)$ is *radial*, i.e.,

$$f_{X,Y}(x, y) = g(x^2 + y^2)$$

(recall (2.9),(2.10)), (6.35) becomes

$$\begin{aligned} f_{R,\Theta}(r, \theta) &= rg(r^2) \\ &= 2\pi rg(r^2) \cdot \frac{1}{2\pi}. \end{aligned} \tag{6.36}$$

This shows that:

(i) R and Θ are independent;

(ii) $f_R(r) = 2\pi\, r\, g(r^2)$;

(iii) $\Theta \sim \text{Uniform}[0, 2\pi)$.

A special case appeared in Example 1.12, where (X, Y) was uniformly distributed over the unit disk D in $\mathbf{R}^2$, i.e., (cf. (1.45))

$$f_{X,Y}(x, y) = \frac{1}{\pi} I_D(x, y) = \frac{1}{\pi} I_{(0,1)}(x^2 + y^2) \equiv g(x^2 + y^2).$$

Thus R and Θ are independent, $\Theta \sim \text{Uniform}[0, 2\pi)$, and (cf. (1.46a))

$$f_R(r) = 2\pi\, r\, g(r^2) = 2r I_{(0,1)}(r).$$

Another special case occurs when X, Y are i.i.d. $N(0, 1)$ rvs. Here

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \equiv g(x^2 + y^2),$$

so again R and Θ are independent, $\Theta \sim \text{Uniform}[0, 2\pi)$, and

$$f_R(r) = 2\pi\, r\, g(r^2) = r e^{-\frac{r^2}{2}}.$$

Finally, set $S = R^2$ $(= X^2 + Y^2)$. Then $\frac{ds}{dr} = 2r$, so

$$f_S(s) = f_R(r(s)) \cdot \frac{dr}{ds} = r e^{-\frac{r^2}{2}} \cdot \frac{1}{2r} = \frac{1}{2} e^{-\frac{s}{2}},$$

hence $S \sim \text{Expo}\left(\frac{1}{2}\right) \equiv \text{Gam}\left(\frac{2}{2}, \frac{1}{2}\right) \equiv \chi^2_2$ (see Remark 6.3). □

These results extend to *polar coordinates in* $\mathbf{R}^n$. Suppose that $X \equiv (X_1, \ldots, X_n)$ is a continuous rvtr with joint pdf $f(x_1, \ldots, x_n)$. Then X can be represented by polar coordinates $R, \Theta_1, \ldots, \Theta_{n-1}$ in several different ways, depending on how the angles Θ_i are defined. However, in each case $R = \sqrt{X_1^2 + \cdots X_n^2}$ and the Jacobian has the form

$$(6.37) \qquad \left| \frac{\partial(x_1, \ldots, x_n)}{\partial(r, \theta_1, \ldots, \theta_{n-1})} \right| = r^{n-1} \cdot h(\theta_1, \ldots, \theta_{n-1})$$

for some function h. Thus, if $f(x_1, \ldots, x_n) = g(x_1^2 + \cdots + x_n^2)$, i.e., if f is radial, then by (6.14) the joint pdf of $(R, \Theta_1, \ldots, \Theta_{n-1})$ again factors:

$$f_{R,\Theta_1,\ldots,\Theta_{n-1}}(r, \theta_1, \ldots, \theta_{n-1}) = r^{n-1} g(r^2) \cdot h(\theta_1, \ldots, \theta_{n-1}). \tag{6.38}$$

Thus R *is independent of* $(\Theta_1, \ldots, \Theta_{n-1})$ and has pdf of the form

$$f_R(r) = c_n \cdot r^{n-1} g(r^2), \tag{6.39}$$

where c_n does not depend on g.

To evaluate c_n, let $X_1, \ldots, X_n$ be i.i.d. $N(0,1)$ rvs, so

$$f_R(r) = \frac{c_n}{(2\pi)^{n/2}} \cdot r^{n-1} e^{-\frac{r^2}{2}}. \tag{6.40}$$

Again set $S = R^2$, so $dr/ds = 1/(2s^{1/2})$, hence

$$f_S(s) = \frac{c_n}{2\,(2\pi)^{n/2}} \cdot s^{\frac{n}{2}-1} e^{-\frac{s}{2}}, \tag{6.41}$$

from which we see that $c_n = 2(\pi^{n/2})/\Gamma(n/2)$ and

$$S \equiv R^2 \equiv X_1^2 + \cdots + X_n^2 \sim \text{Gamma}\left(\tfrac{n}{2}, \tfrac{1}{2}\right) \ (= \chi_n^2). \tag{6.42}$$

Remark 6.3. The *chi-square distribution with n degrees of freedom*, denoted by χ_n^2, is defined as in (6.42) to be the distribution of $X_1^2 + \cdots + X_n^2$ where $X_1, \ldots, X_n$ are i.i.d. standard normal $N(0,1)$ rvs. The pdf of χ_1^2 was shown directly in (2.7) of Example 2.4 to be that of the Gamma$\left(\frac{1}{2}, \frac{1}{2}\right)$ distribution, from which (6.42) also follows from Application 3.3.3 in §3.3 on moment generating functions, or from Example 6.4 in the present section.

Remark 6.4. The transformations in Examples 6.3 - 6.6 are 1-1, while those in Examples 6.1 and 6.2 are 2-1 [verify].

Exercise 6.6. (i) Show that $B(\alpha,\, \beta) \cdot B(\alpha+\beta,\, \delta) \stackrel{d}{=} B(\alpha,\, \beta+\delta)$, where the B's denote independent beta rvs.
[Hint: apply the result at the end of Example 6.4.]
(ii)* Show that $B(\alpha,\, \beta+\gamma) \cdot B(\alpha+\beta,\, \gamma+\delta) \stackrel{d}{=} B(\alpha,\, \beta+\gamma+\delta) \cdot B(\alpha+\beta,\, \gamma)$.

Exercise 6.7. Use c_n to find the volume of the unit ball in $\mathbf{R}^n$. □

7. The Multinomial Distribution.

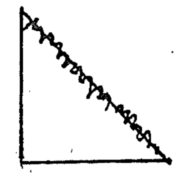
The multinomial experiment. Consider a random experiment with k possible outcomes (or "categories", or "cells") $C_1, \ldots, C_k$. Let $p_i = P(C_i)$. Suppose the experiment is repeated independently n times, i.e., n independent trials are carried out, resulting in X_i observations in cell C_i:

Cells :	C_1	C_2	$\ldots$	C_k
Probabilities	p_1	p_2	$\ldots$	p_k
Counts :	X_1	X_2	$\ldots$	X_k

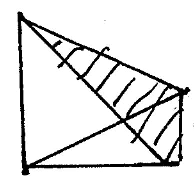
$$\sum_{i=1}^{k} p_i = 1,$$

$$\sum_{i=1}^{k} X_i = n.$$

$k=2$:



$k=3$:



Definition 7.1. The distribution of $(X_1, \ldots, X_k)$ is called the (complete) *multinomial distribution* for k cells, n trials, and cell probabilities $p_1, \ldots, p_k$. We write

$$(7.1) \qquad (X_1, \ldots, X_k) \sim M_k(n; p_1, \ldots, p_k). \qquad \square$$

The multinomial distribution is a *discrete multivariate distribution.* Note that the marginal distribution of each X_i is Binomial$(n; p_i)$. In fact, when $k = 2$, M_2 essentially reduces to the binomial distribution:

$$(7.2) \qquad X \sim \text{Binomial}(n; p) \iff (X, n - X) \sim M_2(n; p, 1 - p).$$

The multinomial pmf and mgf. For $x_1 \geq 0, \ldots, x_k \geq 0$, $\sum_{i=1}^k x_i = n$ (x_i's integers),

$$(7.3) \qquad P[(X_1, \ldots, X_k) = (x_1, \ldots, x_k)] = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k}.$$

[Draw picture of range; discuss complete vs. incomplete multinomial dist'n.]

Here the *multinomial coefficient*

$$\frac{n!}{x_1! \cdots x_k!} = \binom{n}{x_1, \ldots, x_k}$$

is the number of ways that the labels "$C_1, \ldots, C_k$" can be assigned to $1, \ldots n$ such that label C_i occurs x_i times, $i = 1, \ldots, k$.
[First, the n *distinguishable* labels $C_1^1, \ldots, C_1^{x_1}, \ldots, C_k^1, \ldots, C_k^{x_k}$ can be assigned to $1, \ldots, n$ in $n!$ ways. But there are $x_i!$ permutations of $C_i^1, \ldots, C_i^{x_i}$.]

The fact that the probabilities in (7.3) sum to 1 follows either from their interpretation as probabilities or from the *multinomial expansion*

$$(7.4) \qquad (p_1 + \cdots + p_k)^n = \sum_{x_i \geq 0, \sum x_i = n} \binom{n}{x_1, \ldots, x_k} p_1^{x_1} \cdots p_k^{x_k}.$$

If we replace p_i by $p_i e^{t_i}$ in (7.4) we obtain the multinomial mgf (7.5):

$$\begin{aligned} m_{X_1, \ldots, X_k}(t_1, \ldots, t_k) &\equiv \mathrm{E}\left(e^{t_1 X_1 + \cdots + t_k X_k}\right) \\ (7.5) \qquad &= \left(p_1 e^{t_1} + \cdots + p_k e^{t_k}\right)^n \quad [p_i \mapsto p_i e^{t_i} \text{ in } (7.4)]. \end{aligned}$$

Also, the mgf of the incomplete multinomial rv $(X_1, \ldots, X_{k-1})$ is

$$\begin{aligned} m_{X_1, \ldots, X_{k-1}}(t_1, \ldots, t_{k-1}) &= m_{X_1, \ldots, X_{k-1}, X_k}(t_1, \ldots, t_{k-1}, 0) \\ &= \left(p_1 e^{t_1} + \cdots + p_{k-1} e^{t_{k-1}} + p_k e^0\right)^n \\ (7.6) \qquad &= \left[p_1(e^{t_1} - 1) + \cdots + p_{k-1}(e^{t_{k-1}} - 1) + 1\right]^n. \end{aligned}$$

Additional trials. Let

$$\begin{aligned} (X_1, \ldots, X_k) &\sim M_k(m; p_1, \ldots, p_k), \\ (Y_1, \ldots, Y_k) &\sim M_k(n; p_1, \ldots, p_k), \end{aligned}$$

denote the cell counts based on m and n *independent* multinomial trials with the same k and same p_i's. Then obviously

$$(7.7) \qquad (X_1 + Y_1, \ldots, X_k + Y_k) \sim M_k(m + n; p_1, \ldots, p_k).$$

Combining cells. Suppose $(X_1, \ldots, X_k) \sim M_k(n; p_1, \ldots, p_k)$ and define new "combined cells" $D_1, D_2, \ldots, D_r$ as follows:

$$(7.8) \qquad \overbrace{C_1, \ldots, C_{k_1}}^{D_1}, \overbrace{C_{k_1+1}, \ldots, C_{k_2}}^{D_2}, \ldots, \overbrace{C_{k_{r-1}+1}, \ldots, C_{k_r}}^{D_r},$$

where $1 \le r < k$ and $1 \le k_1 < k_2 < \ldots < k_{r-1} < k_r \equiv k$. Define the *combined cell counts* to be

$$Y_1 = X_1 + \cdots + X_{k_1},\ Y_2 = X_{k_1+1} + \cdots + X_{k_2}, \ldots,\ Y_r = X_{k_{r-1}+1} + \cdots + X_{k_r}$$

and the *combined cell probabilities* to be

$$q_1 = p_1 + \cdots + p_{k_1},\ q_2 = p_{k_1+1} + \cdots + p_{k_2}, \ldots,\ q_r = p_{k_{r-1}+1} + \cdots + p_{k_r}.$$

Then obviously

$$(Y_1, \ldots, Y_r) \sim M_r(n;\, q_1, \ldots, q_r). \qquad \text{[same } n\text{]} \tag{7.9}$$

In particular, for any l with $1 \le l < k$,

$$\big(X_1, \ldots, X_l,\ n - (X_1 + \cdots + X_l)\big) \sim M_{l+1}\big(n;\, p_1, \ldots, p_l,\ 1 - (p_1 + \cdots + p_l)\big). \tag{7.10}$$

Conditional distributions. First consider $k = 4$:

$$(X_1, X_2, X_3, X_4) \sim M_4(n;\, p_1, p_2, p_3, p_4).$$

In (7.8) let $r = 2$, $k_1 = 2$, $k_2 = 4$, so

$$Y_1 = X_1 + X_2, \qquad Y_2 = X_3 + X_4,$$
$$q_1 = p_1 + p_2, \qquad q_2 = p_3 + p_4,$$

hence by (7.9),

$$(Y_1, Y_2) \sim M_2(n;\, q_1, q_2).$$

Therefore the conditional distribution of (X_1, X_2, X_3, X_4) given (Y_1, Y_2) is as follows: for integers x_1, x_2, x_3, x_4 and y_1, y_2 such that $x_1 + x_2 = y_1$, $x_3 + x_4 = y_2$, and $y_1 + y_2 = n$,

$$P\big[(X_1, X_2, X_3, X_4) = (x_1, x_2, x_3, x_4) \mid (Y_1, Y_2) = (y_1, y_2)\big]$$

$$= \frac{P\big[(X_1, X_2, X_3, X_4) = (x_1, x_2, x_3, x_4)\big]}{P\big[(Y_1, Y_2) = (y_1, y_2)\big]}$$

$$= \frac{\frac{n!}{x_1!x_2!x_3!x_4!} p_1^{x_1} p_2^{x_2} p_3^{x_3} p_4^{x_4}}{\frac{n!}{y_1!y_2!} q_1^{y_1} q_2^{y_2}} \qquad \text{[by (7.3) and (7.9)]}$$

$$= \frac{y_1!}{x_1!x_2!} \left(\frac{p_1}{q_1}\right)^{x_1} \left(\frac{p_2}{q_1}\right)^{x_2} \cdot \frac{y_2!}{x_3!x_4!} \left(\frac{p_3}{q_2}\right)^{x_3} \left(\frac{p_4}{q_2}\right)^{x_4}. \tag{7.11}$$

This shows that (X_1, X_2) *and* (X_3, X_4) *are conditionally independent given* $X_1 + X_2$ *and* $X_3 + X_4$ ($\equiv n - (X_1 + X_2)$), and that [discuss]

$$(7.12)\qquad \begin{aligned}(X_1, X_2) \mid X_1 + X_2 &\sim M_2\Big(X_1 + X_2;\ \frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\Big)\\ (X_3, X_4) \mid X_3 + X_4 &\sim M_2\Big(X_3 + X_4;\ \frac{p_3}{p_3+p_4}, \frac{p_4}{p_3+p_4}\Big).\end{aligned}$$

In particular,

$$(7.13)\qquad X_1 \mid X_1 + X_2 \sim \text{Binomial}\Big(X_1 + X_2,\ \frac{p_1}{p_1+p_2}\Big),$$

so

$$(7.14)\qquad \text{E}[X_1 \mid X_1 + X_2] = (X_1 + X_2)\Big(\frac{p_1}{p_1+p_2}\Big).$$

Verify:

$$np_1 = \text{E}(X_1) = \text{E}\big(\text{E}[X_1 \mid X_1 + X_2]\big) = \text{E}\Big[(X_1 + X_2)\Big(\frac{p_1}{p_1+p_2}\Big)\Big] = np_1.$$

It also follows from (7.12) that

$$(7.15)\qquad (X_1, X_2) \mid X_3 + X_4 \sim M_2\Big(n - (X_3 + X_4);\ \frac{p_1}{p_1+p_2}, \frac{p_2}{p_1+p_2}\Big),$$

which shows the *negative* (linear!) relation between (X_1, X_2) and (X_3, X_4).

Now consider the general case, as in (7.8). Then a similar argument shows that

$$(X_1, \ldots, X_{k_1}),\ (X_{k_1+1},\ \ldots,\ X_{k_2}),\ \ldots,\ (X_{k_{r-1}+1}, \ldots, X_{k_r})$$

are conditionally independent given

$$X_1 + \cdots + X_{k_1},\ X_{k_1+1} + \cdots + X_{k_2},\ \ldots,\ X_{k_{r-1}+1} + \cdots + X_{k_r},$$

with

$$(7.16)\qquad (X_{k_{i-1}+1}, \ldots, X_{k_i}) \mid X_{k_{i-1}+1} + \cdots + X_{k_i}$$

$$\sim M_{k_i - k_{i-1}}\Big(X_{k_{i-1}+1} + \cdots + X_{k_i};\ \frac{p_{k_{i-1}+1}}{p_{k_{i-1}+1} + \cdots + p_{k_i}}, \ldots, \frac{p_{k_i}}{p_{k_{i-1}+1} + \cdots + p_{k_i}}\Big)$$

for $i = 1, \ldots, r$, where $k_0 \equiv 0$.

In particular, for $2 \le l < k$,

$$(X_1, \ldots, X_l) \mid X_{l+1}, \ldots, X_k \tag{7.17}$$

$$\sim M_l\Big(n - (X_{l+1} + \cdots + X_k); \frac{p_1}{p_1 + \cdots + p_l}, \ldots, \frac{p_l}{p_1 + \cdots + p_l}\Big).$$

Because $X_1 + \cdots + X_k = n$, we know there is a negative linear relation between any X_i and X_j. Let's verify this explicitly. For $i \ne j$,

$$X_i \mid X_j \sim \text{Binomial}\Big(n - X_j; \frac{p_i}{1 - p_j}\Big), \tag{7.18}$$

$$\text{E}[X_i \mid X_j] = (n - X_j)\Big(\frac{p_i}{1 - p_j}\Big), \tag{7.19}$$

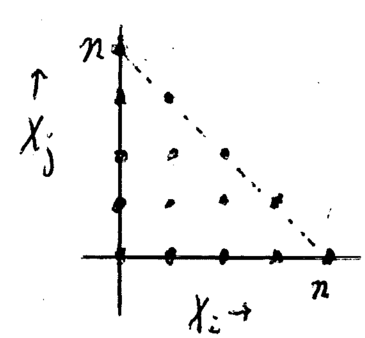
$$\begin{aligned}
\text{Cov}(X_i, X_j) &= \text{Cov}\big(\text{E}[X_i \mid X_j], X_j\big) \\
&= \text{Cov}\Big((n - X_j)\Big(\frac{p_i}{1 - p_j}\Big), X_j\Big) \\
&= -\Big(\frac{p_i}{1 - p_j}\Big)\text{Var}(X_j) \\
&= -\Big(\frac{p_i}{1 - p_j}\Big) n p_j (1 - p_j) \\
&= -n p_i p_j \le 0. \qquad (7.20)
\end{aligned}$$

An alternative derivation is based on the variance formula (3.13):

$$\begin{aligned}
2\,\text{Cov}(X_i, X_j) &= \text{Var}(X_i + X_j) - \text{Var}(X_i) - \text{Var}(X_j) \\
&= n(p_i + p_j)(1 - p_i - p_j) - n p_i(1 - p_i) - n p_j(1 - p_j) \\
&= -n(p_i + p_j)^2 + n p_i^2 + n p_j^2 \\
&= -2 n p_i p_j,
\end{aligned}$$

which yields (7.20). (The second line follows from the fact that $X_i + X_j \sim \text{Binomial}(n; p_i + p_j)$.)

Remark 7.1. For $k \ge 3$ the range of (X_i, X_j) shown here, indicates the negative relation between X_i and X_j:



Exercise 7.1. (Representation of a multinomial rvtr in terms of independent Poisson rvs.) Let $Y_1, \ldots, Y_k$ be independent Poisson rvs with $Y_i \sim \text{Poisson}(\lambda_i)$, $i = 1, \ldots, k$. Show that

$$(Y_1, \ldots, Y_k)|\{Y_1 + \cdots + Y_k = n\} \sim M_k\Big(n; \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_k}, \ldots, \frac{\lambda_k}{\lambda_1 + \cdots + \lambda_k}\Big).$$

Exponential family. From (7.3), the multinomial pmf can be written as an exponential family (see Example 11.11). When $x_1 + \cdots + x_k = n$,

(7.21)

$$\begin{aligned}
&\frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \cdots p_k^{x_k} \\
=&\frac{n!}{x_1! \cdots x_k!} e^{x_1 \log p_1 + \cdots + x_{k-1} \log p_{k-1} + x_k \log p_k} \\
=&\frac{n!}{x_1! \cdots x_k!} e^{n \log(1 - p_1 - \cdots - p_{k-1})} e^{x_1 \log \frac{p_1}{1 - p_1 - \cdots - p_{k-1}} + \cdots + x_{k-1} \log \frac{p_{k-1}}{1 - p_1 - \cdots - p_{k-1}}}.
\end{aligned}$$

This is a $(k-1)$-parameter exponential family with natural parameters

(7.22) $$\theta_i = \log \frac{p_i}{1 - p_1 - \cdots - p_{k-1}}, \qquad i = 1, \ldots, k-1.$$

(Note that the Binomial(n, p) family is a one-parameter exponential family (see Example 11.10) with natural parameter $\log \frac{p}{1-p}$.)

Maximum likelihood estimates (MLE). The MLE of $(p_1, \ldots, p_k)$ is $(\hat{p}_1, \ldots, \hat{p}_k) = \big(\frac{X_1}{n}, \ldots, \frac{X_k}{n}\big)$. [See Example 14.24.]

Representation of a multinomial rvtr as a sum of i.i.d. Bernoulli (0-1) rvtrs. First recall that a binomial rv $X \sim \text{Bin}(n, p)$ can be represented as the sum of n i.i.d. Bernoulli rvs:

(7.23) $$X = U_1 + \cdots + U_n,$$

$$\text{where} \qquad U_j = \begin{cases} 1, & \text{if Success on trial } j; \\ 0, & \text{if Failure on trial } j. \end{cases}$$

Now extend this to multinomial trials. Consider a multinomial experiment as in §7.1 with n i.i.d. trials and k possible outcomes (cells) $C_1, \ldots, C_k$. Again let p_i be the probability of cell C_i and X_i be the total number of outcomes in cell C_i. Form the column vectors

$$\text{(7.24)} \qquad \mathbf{X} \equiv \begin{pmatrix} X_1 \\ \vdots \\ X_k \end{pmatrix}, \qquad \mathbf{p} \equiv \begin{pmatrix} p_1 \\ \vdots \\ p_k \end{pmatrix},$$

so we can write $\mathbf{X} \sim M_k(n; \mathbf{p})$. Then the multinomial rvtr $\mathbf{X}$ can be represented as

$$\text{(7.25)} \qquad \mathbf{X} = \mathbf{U}_1 + \cdots + \mathbf{U}_n,$$

$$\text{where} \qquad \mathbf{U}_j = \begin{pmatrix} U_{1j} \\ \vdots \\ U_{kj} \end{pmatrix} : k \times 1$$

$$\text{with} \qquad U_{ij} = \begin{cases} 1, & \text{if cell } C_i \text{ occurs on trial } j; \\ 0, & \text{if cell } C_i \text{ does not occur on trial } j. \end{cases}$$

Note that each $\mathbf{U}_j$ is a *Bernoulli rvtr:* it has exactly one 1 and $k-1$ 0's. Clearly (7.25) generalizes (7.23).

The representations (7.23) and(7.25) are very convenient for finding moments and applying the Central Limit Theorem to obtain normal approximations to the binomial and multinomial distributions. For example, since $\mathrm{E}(U_j) = p$ and $V(U_j) = p(1-p)$ in (7.23), it follows that in the binomial case, $\mathrm{E}(X) = np$ and $\mathrm{Var}(X) = np(1-p)$, and that as $n \to \infty$,

$$\text{(7.26)} \qquad \frac{X - np}{\sqrt{np(1-p)}} \xrightarrow{d} N(0,1) \qquad \text{if } 0 < p < 1,$$

$$\text{(7.27)} \quad \text{or equivalently,} \qquad \frac{X - np}{\sqrt{n}} \xrightarrow{d} N(0, p(1-p)).$$

For the multinomial, the mean vector and covariance matrix of $\mathbf{U}_j$ are

$$\mathrm{E}(\mathbf{U}_j) = \begin{pmatrix} \mathrm{E}(U_{1j}) \\ \vdots \\ \mathrm{E}(U_{kj}) \end{pmatrix} = \begin{pmatrix} p_1 \\ \vdots \\ p_k \end{pmatrix} = \mathbf{p},$$

$$\operatorname{Cov}(\mathbf{U}_j) = \begin{pmatrix} \operatorname{Var}(U_{1j}) & \operatorname{Cov}(U_{1j}, U_{2j}) & \cdots & \operatorname{Cov}(U_{1j}, U_{kj}) \\ \operatorname{Cov}(U_{2j}, U_{1j}) & \operatorname{Var}(U_{2j}) & \cdots & \operatorname{Cov}(U_{2j}, U_{kj}) \\ \vdots & \vdots & & \vdots \\ \operatorname{Cov}(U_{kj}, U_{1j}) & \operatorname{Cov}(U_{kj}, U_{2j}) & \cdots & \operatorname{Var}(U_{kj}) \end{pmatrix}$$

$$= \begin{pmatrix} p_1(1-p_1) & -p_1p_2 & \cdots & -p_1p_k \\ -p_2p_1 & p_2(1-p_2) & \cdots & -p_2p_k \\ \vdots & \vdots & & \vdots \\ -p_kp_1 & -p_kp_2 & \cdots & p_k(1-p_k) \end{pmatrix} \qquad \text{[verify]}$$

$$\equiv D_{\mathbf{p}} - \mathbf{p}\mathbf{p}',$$

where $D_{\mathbf{p}} = \operatorname{diag}(p_1, \ldots, p_k)$. (Note that $D_{\mathbf{p}} - \mathbf{p}\mathbf{p}'$ is a *singular* matrix of rank $k-1$, so it has no inverse.) Thus by (7.25) and the independence of $\mathbf{U}_1, \ldots, \mathbf{U}_k$,

$$\text{(7.28)} \qquad \mathrm{E}(\mathbf{X}) = n\mathbf{p}, \qquad \operatorname{Cov}(\mathbf{X}) = n(D_{\mathbf{p}} - \mathbf{p}\mathbf{p}').$$

Therefore, it follows from the multivariate Central Limit Theorem that

$$\text{(7.29)} \qquad \frac{\mathbf{X} - n\mathbf{p}}{\sqrt{n}} \xrightarrow{d} N_k\left(0, D_{\mathbf{p}} - \mathbf{p}\mathbf{p}'\right).$$

Now suppose that $p_1 > 0, \ldots, p_k > 0$. Then $D_{\mathbf{p}}$ is nonsingular, so by the continuity of convergence in distribution (§10.2),

$$\text{(7.30)} \qquad D_{\mathbf{p}}^{-\frac{1}{2}} \left(\frac{\mathbf{X} - n\mathbf{p}}{\sqrt{n}}\right) \xrightarrow{d} N_k\left(0, I_k - \mathbf{u}\mathbf{u}'\right),$$

where $D_{\mathbf{p}}^{-\frac{1}{2}} = \operatorname{diag}\left(p_1^{-\frac{1}{2}}, \ldots, p_k^{-\frac{1}{2}}\right)$ and $\mathbf{u} \equiv D_{\mathbf{p}}^{-\frac{1}{2}}\mathbf{p}$ is a unit vector, i.e., $\mathbf{u}'\mathbf{u} = 1$. Again by the continuity of convergence in distribution,

$$\text{(7.31)} \qquad \left\| D_{\mathbf{p}}^{-\frac{1}{2}} \left(\frac{\mathbf{X} - n\mathbf{p}}{\sqrt{n}}\right) \right\|^2 \xrightarrow{d} \left\| N_k\left(0, I_k - \mathbf{u}\mathbf{u}'\right) \right\|^2 \sim \chi^2_{k-1},$$

since $I_k - \mathbf{u}\mathbf{u}'$ is a *projection matrix* of rank $k-1$ (apply Fact 8.5).

But

$$\left\| D_{\mathbf{p}}^{-\frac{1}{2}} \left(\frac{\mathbf{X} - n\mathbf{p}}{\sqrt{n}} \right) \right\|^2 = \sum_{i=1}^{k} \frac{(X_i - np_i)^2}{np_i}$$

$$\equiv \sum_{i=1}^{k} \frac{(\text{Observed}_i - \text{Expected}_i)^2}{\text{Expected}_i} \equiv \chi^2, \tag{7.32}$$

which is (Karl) Pearson's classical *chi-square goodness-of-fit statistic* for testing the simple null hypothesis $\mathbf{p}$. Thus we have derived Pearson's classic result that $\chi^2 \xrightarrow{d} \chi^2_{k-1}$.

(However, Pearson got the degrees of freedom wrong! He first asserted that $\chi^2 \xrightarrow{d} \chi^2_k$, but was corrected by Fisher, which Pearson did not entirely appreciate!)

Remark 7.2. Note that (7.29) is an extension of (7.27), not (7.26), which has no extension to the multinomial case since $D_{\mathbf{p}} - \mathbf{p}\mathbf{p}'$ is singular, hence has no inverse. However, if we reduce $\mathbf{X}$ to $\tilde{\mathbf{X}} \equiv (X_1, \ldots, X_{k-1})'$, then

$$\mathrm{Cov}(\tilde{\mathbf{X}}) \equiv n(\tilde{D}_{\mathbf{p}} - \tilde{\mathbf{p}}\tilde{\mathbf{p}}') \tag{7.33}$$

is nonsingular provided that $p_1 > 0, \ldots, p_k > 0$ (see (*)), where $\tilde{D}_{\mathbf{p}} = \mathrm{diag}(p_1, \ldots, p_{k-1})$ and $\tilde{\mathbf{p}} = (p_1, \ldots, p_{k-1})'$. Then (7.26) can be extended:

$$\left(\tilde{D}_{\mathbf{p}} - \tilde{\mathbf{p}}\tilde{\mathbf{p}}' \right)^{-\frac{1}{2}} \left(\frac{\tilde{\mathbf{X}} - n\tilde{\mathbf{p}}}{\sqrt{n}} \right) \xrightarrow{d} N_{k-1}(0, I_{k-1}), \tag{7.34}$$

from which (7.31) can also be obtained (but with more algebra).

(*) For $a \equiv (a_1, \ldots, a_{k-1})' \neq 0$, use the variance inequality [how?]:

$$a' \left(\tilde{D}_{\mathbf{p}} - \tilde{\mathbf{p}}\tilde{\mathbf{p}}' \right) a = \sum\nolimits_1^{k-1} a_i^2 p_i - \left(\sum\nolimits_1^{k-1} a_i p_i \right)^2$$

$$> \left(\sum\nolimits_1^{k-1} a_i^2 p_i \right) \left(\sum\nolimits_1^{k-1} p_i \right) - \left(\sum\nolimits_1^{k-1} a_i p_i \right)^2 \geq 0.$$

8. Linear Models and the Multivariate Normal Distribution.

8.1. Review of vectors and matrices. (The results are stated for vectors and matrices with real entries but also hold for complex entries.)

An $m \times n$ matrix $A \equiv \{a_{ij}\}$ is an array of mn numbers:

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}.$$

This matrix represents the *linear mapping* ($\equiv$ *linear transformation*)

$$\begin{aligned} A : \mathbf{R}^n &\to \mathbf{R}^m \\ x &\mapsto Ax, \end{aligned} \tag{8.1}$$

where $x \in \mathbf{R}^n$ is written as an $n \times 1$ column vector and

$$Ax \equiv \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \equiv \begin{pmatrix} \sum_{j=1}^n a_{1j} x_j \\ \vdots \\ \sum_{j=1}^n a_{mj} x_j \end{pmatrix} \in \mathbf{R}^m.$$

Thus the jth column vector of A is the image $A\mathbf{u}_j$ of the jth coordinate column vector $\mathbf{u}_j$ (see (8.19)). The mapping (8.1) is clearly *linear:*

$$A(bx + cy) = bAx + cAy.$$

Matrix addition: If $A \equiv \{a_{ij}\}$ and $B \equiv \{b_{ij}\}$ are $m \times n$ matrices, then

$$(A + B)_{ij} = a_{ij} + b_{ij}.$$

Matrix multiplication: If A is $m \times n$ and B is $n \times p$, then the *matrix product* AB is the $m \times p$ matrix AB whose ij-th element is

$$(AB)_{ij} = \sum_{k=1}^n a_{ik} b_{kj}. \tag{8.2}$$

Then AB is the matrix of the composition $\mathbf{R}^p \xrightarrow{B} \mathbf{R}^n \xrightarrow{A} \mathbf{R}^m$ of the two linear mappings determined by A and B [verify]:

$$(AB)x = A(Bx) \qquad \forall x \in \mathbf{R}^p.$$

Rank of a matrix: The *row (column) rank* of a matrix $A : m \times n$ is the dimension of the linear space spanned by its rows (columns). The *rank* of A is the dimension r of the largest nonzero *minor* (= $r \times r$ subdeterminant) of A. Then [verify]

$$\begin{aligned}
\text{row rank}(A) &\leq \min(m, n), \\
\text{column rank}(A) &\leq \min(m, n), \\
\text{rank}(A) &\leq \min(m, n), \\
\text{row rank}(A) &= \text{column rank}(A) \\
= \text{rank}(A) &= \text{rank}(A') \\
= \text{rank}(AA') &= \text{rank}(A'A).
\end{aligned}$$

Furthermore, for $A : m \times n$ and $B : n \times p$,

$$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)).$$

Inverse matrix: If $A : n \times n$ is a square matrix, its *inverse* A^{-1} (if it exists) is the unique matrix that satisfies

$$AA^{-1} = A^{-1}A = I,$$

where $I \equiv I_n$ is the $n \times n$ *identity matrix*[7] $\text{diag}(1, \ldots, 1)$. If A^{-1} exists then A is called *nonsingular* (or *regular*). The following are equivalent:

(a) A is nonsingular.

(b) The n columns of A are linearly independent (i.e., column $\text{rank}(A) = n$).
Equivalently, $Ax \neq 0$ for every nonzero $x \in \mathbf{R}^n$.

(c) The n rows of A are linearly independent (i.e., row $\text{rank}(A) = n$).
Equivalently, $x'A \neq 0$ for every nonzero $x \in \mathbf{R}^n$.

(d) The determinant $|A| \neq 0$ (i.e., $\text{rank}(A) = n$). [Define det geometrically.]

- If A is nonsingular then A^{-1} is nonsingular and $(A^{-1})^{-1} = A$.

[7] It is called the "identity" matrix since $Ix = x \; \forall x \in R^n$.

- If $A : m \times m$ and $C : n \times n$ are nonsingular and B is $m \times n$, then

$$\operatorname{rank}(AB) = \operatorname{rank}(B) = \operatorname{rank}(BC).$$

- If $A : n \times n$ and $B : n \times n$ are nonsingular then so is AB, and

$$(AB)^{-1} = B^{-1}A^{-1}. \tag{8.3}$$

- If $A \equiv \operatorname{diag}(d_1, \ldots, d_n)$ with all $d_i \neq 0$ then $A^{-1} = \operatorname{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

Transpose matrix: If $A \equiv \{a_{ij}\}$ is $m \times n$, its *transpose* is the $n \times m$ matrix A' (also denoted by A^t) whose ij-th element is a_{ji}. That is, the m row vectors (n column vectors) of A are the m column vectors (n row vectors) of A'. Note that [verify]

$$(A+B)' = A' + B'; \tag{8.4}$$

$$(AB)' = B'A' \qquad (A : m \times n,\ B : n \times p); \tag{8.5}$$

$$(A^{-1})' = (A')^{-1} \qquad (A : n \times n,\ \text{nonsingular}). \tag{8.6}$$

Trace: For a square matrix $A \equiv \{a_{ij}\} : n \times n$, the *trace* of A is

$$\operatorname{tr}(A) = \sum\nolimits_{i=1}^{n} a_{ii}, \tag{8.7}$$

the sum of the diagonal entries of A. Then

$$\operatorname{tr}(aA + bB) = a\operatorname{tr}(A) + b\operatorname{tr}(B); \tag{8.8}$$

$$\operatorname{tr}(AB) = \operatorname{tr}(BA); \qquad (A : m \times n,\ B : n \times m) \tag{8.9}$$

$$\operatorname{tr}(A') = \operatorname{tr}(A). \qquad (A : n \times n) \tag{8.10}$$

Proof of (8.9):

$$\begin{aligned}\operatorname{tr}(AB) &= \sum\nolimits_{i=1}^{m} (AB)_{ii} = \sum\nolimits_{i=1}^{m} \Big(\sum\nolimits_{k=1}^{n} a_{ik}b_{ki}\Big) \\ &= \sum\nolimits_{k=1}^{n} \Big(\sum\nolimits_{i=1}^{m} b_{ki}a_{ik}\Big) = \sum\nolimits_{k=1}^{n} (BA)_{kk} = \operatorname{tr}(BA).\end{aligned}$$

Determinant: For a square matrix $A \equiv \{a_{ij}\} : n \times n$, its *determinant* is

$$
\begin{aligned}
|A| &= \sum\nolimits_{\pi} \epsilon(\pi) \prod\nolimits_{i=1}^{n} a_{i\pi(i)} \\
&= \pm \text{Volume}(A([0,1]^n)),
\end{aligned}
$$

where π ranges over all $n!$ permutations of $1, \ldots, n$ and $\epsilon(\pi) = \pm 1$ according to whether π is an even or odd permutation. Then

(8.11) $\quad |AB| = |A| \cdot |B| \qquad (A, B : n \times n);$

(8.12) $\quad |A^{-1}| = |A|^{-1}$

(8.13) $\quad |A'| = |A|;$

(8.14) $\quad |A| = \prod\nolimits_{i=1}^{n} a_{ii}$ if A is triangular (or diagonal).

Orthogonal matrix. An $n \times n$ matrix U is *orthogonal* if

(8.15) $$UU' = I.$$

This is equivalent to the fact that the n row vectors of U form an orthonormal basis for $\mathbf{R}^n$. Note that (8.15) implies that $U' = U^{-1}$, hence also $U'U = I$, which is equivalent to the fact that the n column vectors of U also form an orthonormal basis for $\mathbf{R}^n$.

Note that U preserves angles and lengths, i.e., preserves the usual inner product and norm in $\mathbf{R}^n$: for $x, y \in \mathbf{R}^n$,

$$(Ux,\, Uy) \equiv (Ux)'(Uy) = x'U'Uy = x'y \equiv (x,\, y),$$

so

$$\|Ux\|^2 \equiv (Ux,\, Ux) = (x,\, x) \equiv \|x\|^2.$$

In fact, any orthogonal transformation is a product of rotations and reflections. Also, from (8.13) and (8.15), $|U|^2 = 1$, so $|U| = \pm 1$.

Complex numbers and matrices. For any complex number $c \equiv a+ib \in \mathbf{C}$, let $\bar{c} \equiv a - ib$ denote the *complex conjugate* of c. Note that $\bar{\bar{c}} = c$ and

$$\begin{aligned} c\bar{c} &= a^2 + b^2 \equiv |c|^2, \\ \overline{cd} &= \bar{c}\bar{d}. \end{aligned}$$

For any complex matrix $C \equiv \{c_{ij}\}$, let $\bar{C} = \{\bar{c}_{ij}\}$ and define $C^* = \bar{C}'$. Then

$$(CD)^* = D^* C^*. \tag{8.16}$$

The characteristic roots $\equiv$ eigenvalues of a real $n \times n$ matrix A are the roots $l_1, \dots, l_n$ (with multiplicities) of the n-th degree polynomial equation

$$|A - l\,I| = 0. \tag{8.17}$$

These roots may be real or complex; the complex roots occur in conjugate pairs. Note that the eigenvalues of a triangular or diagonal matrix are just its diagonal elements.

By the equivalence of (b) and (d) for the (possibly complex) matrix $A - lI$, for each eigenvalue l there exists some nonzero (possibly complex) vector $u \in \mathbf{C}^n$ s.t.

$$(A - l\,I)u = 0;$$

equivalently,

$$Au = lu. \tag{8.18}$$

The vector u is called a *characteristic vector* $\equiv$ *eigenvector* for the eigenvalue l. Since any nonzero multiple cu is also an eigenvector for l, we will usually normalize u to be a unit vector, i.e., $\|u\|^2 \equiv u^* u = 1$.

For example, if A is a diagonal matrix, say

$$A = \text{diag}(d_1, \dots, d_n) \equiv \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix},$$

then its eigenvalues are just $d_1, \ldots, d_n$, with corresponding eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$, where

$$\mathbf{u}_i \equiv (0, \ldots, 0, \underbrace{1}_{i}, 0, \ldots, 0)' \tag{8.19}$$

is the i-th unit vector.

Note, however, that in general, eigenvalues need not be distinct and eigenvectors need not be unique. For example, if A is the identity matrix I, then its eigenvalues are $1, \ldots, 1$ and *every* unit vector $u \in \mathbf{R}^n$ is an eigenvector for the eigenvalue 1: $Iu = 1 \cdot u$.

However, eigenvectors u, v associated with two *distinct* eigenvalues l, m cannot be colinear: if $u = cv$ then

$$lu = Au = cAv = cmv = mu,$$

which contradicts the assumption that $l \neq m$.

Symmetric matrix. An $n \times n$ matrix $S \equiv \{s_{ij}\}$ is *symmetric* if $S = S'$, i.e., if $s_{ij} = s_{ji}\ \forall i, j$.

Fact 8.1. *Let S be a real symmetric $n \times n$ matrix.*

(a) Each eigenvalue l of S is real and has a real eigenvector $u \in \mathbf{R}^n$.

(b) If $l \neq m$ are distinct eigenvalues of S with corresponding real eigenvectors u and v, then $u \perp v$, i.e., $u'v = 0$. Thus if all the eigenvalues of S are distinct, each eigenvalue l has exactly one real eigenvector u.

(c) If S^{-1} exists, it is also symmetric.

Proof. (a) Let l be an eigenvalue of S with eigenvector $u \neq 0$. Then

$$Su = lu \quad \Rightarrow \quad u^*Su = lu^*u = l.$$

But S is real and symmetric, so $S^* = S$, hence

$$\overline{u^*Su} = (u^*Su)^* = u^*S^*(u^*)^* = u^*Su.$$

Thus u^*Su is real, hence l is real. Since $S - l\,I$ is real, the existence of a real eigenvector u for l now follows from (b), p.93.

(b) We have $Su = lu$ and $Sv = mv$, hence

$$l(u'v) = (lu)'v = (Su)'v = u'Sv = u'(mv) = m(u'v),$$

so $u'v = 0$ since $l \neq m$.

(c) $I = SS^{-1} = (SS^{-1})' = (S^{-1})'S'$, so $(S^{-1})' = (S')^{-1} = S^{-1}$. □

Fact 8.2. *(Spectral decomposition of a real symmetric matrix.) Let S be a real symmetric $n \times n$ matrix with eigenvalues $l_1, \ldots, l_n$ (necessarily real). Then there exists a real orthogonal matrix U such that*

$$S = U\,D_l\,U', \tag{8.20}$$

where $D_l = \text{diag}(l_1, \ldots, l_n)$. Since $SU = UD_l$, the i-th column vector u_i of U is a real eigenvector for l_i.

Proof. For simplicity we suppose that $l_1, \ldots, l_n$ are distinct. Let $u_1, \ldots, u_n$ be the corresponding unique real unit eigenvectors (apply Fact 8.1b). Since $u_1, \ldots, u_n$ is an orthonormal basis for $\mathbf{R}^n$, the matrix

$$U \equiv (\,u_1 \quad \cdots \quad u_n\,) \quad : n \times n \tag{8.21}$$

satisfies $U'U = I$, i.e., U is an orthogonal matrix. Since each u_i is an eigenvector for l_i, $SU = UD_l$ [verify], which is equivalent to (8.20).

(The case where the eigenvalues are not distinct can be established by a "perturbation" argument. Perturb S slightly so that its eigenvalues become distinct [non-trivial] and apply the first case. Now use a limiting argument based on the compactness of the set of all $n \times n$ orthogonal matrices.) □

Fact 8.3. *If S is a real symmetric matrix with eigenvalues $l_1, \ldots, l_n$,,*

$$\text{tr}(S) = \sum_{i=1}^{n} l_i\ ; \tag{8.22}$$

$$|S| = \prod_{i=1}^{n} l_i\ . \tag{8.23}$$

Proof. This is immediate from the spectral decomposition (8.20) of S. □

Positive definite matrix. A real symmetric $n \times n$ matrix S is *positive semi-definite (psd)* if its quadratic form is nonnegative:

$$(8.24) \qquad x'Sx \geq 0 \qquad \forall\, x \in \mathbf{R}^n;$$

S is *positive definite (pd)* if its quadratic form is positive:

$$(8.25) \qquad x'Sx > 0 \qquad \forall \text{ nonzero } x \in \mathbf{R}^n.$$

- The identity matrix is pd: $x'Ix = \|x\|^2 > 0$ if $x \neq 0$.
- A diagonal matrix $\text{diag}(d_1, \ldots, d_n)$ is psd (pd) iff each $d_i \geq 0$ (> 0).
- If $S : n \times n$ is psd, then ASA' is psd for any $A : m \times n$.
- If $S : n \times n$ is pd, then ASA' is pd for any $A : m \times n$ of full rank $m \leq n$.
- AA' is psd for any $A : m \times n$.
- AA' is pd for any $A : m \times n$ of full rank $m \leq n$.

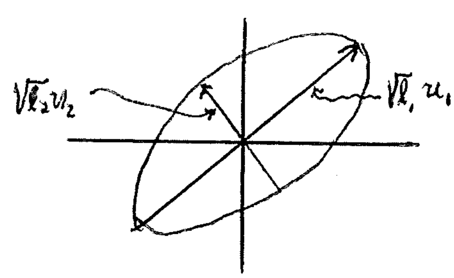
Note: This shows that the proper way to "square" a matrix A is to form AA' (or $A'A$), not A^2 (which need not even be symmetric).

- S pd $\Rightarrow$ S has full rank $\Rightarrow$ S^{-1} exists $\Rightarrow$ $S^{-1} \equiv (S^{-1})S(S^{-1})'$ is pd.

Fact 8.4. *(a) A real symmetric $n \times n$ matrix S with eigenvalues $l_1, \ldots, l_n$ is psd (pd) iff each $l_i \geq 0$ (> 0). In particular, $|S| \geq 0$ (> 0) if S is psd (pd), so a pd matrix is nonsingular.*

(b) Suppose S is pd with distinct eigenvalues $l_1 > \cdots > l_n > 0$ and corresponding unique real unit eigenvectors $u_1, \ldots, u_n$. Then the set

$$(8.26) \qquad \mathcal{E} \equiv \{x \in \mathbf{R}^n \mid x'S^{-1}x = 1\}$$



is the ellipsoid with principle axes $\sqrt{l_1}u_1, \ldots, \sqrt{l_n}u_n$.

Proof. (a) Apply the above results and the spectral decomposition (8.20).

(b) From (8.20), $S = UD_lU'$ with $U = (u_1 \cdots u_n)$, so $S^{-1} = UD_l^{-1}U'$ and,

$$\begin{aligned}\mathcal{E} &= \{x \in \mathbf{R}^n \mid (U'x)'D_l^{-1}(U'x) = 1\} \\ &= U\{y \in \mathbf{R}^n \mid y'D_l^{-1}y = 1\} \qquad\qquad (y = U'x) \\ &= U\left\{y \equiv (y_1, \ldots, y_n)' \;\middle|\; \frac{y_1^2}{l_1} + \cdots + \frac{y_n^2}{l_n} = 1\right\} \\ &\equiv U\mathcal{E}_0.\end{aligned}$$

But $\mathcal{E}_0$ is the ellipsoid with principle axes $\sqrt{l_1}\mathbf{u}_1, \ldots, \sqrt{l_n}\mathbf{u}_n$ (recall (8.19)) and $U\mathbf{u}_i = u_i$, so $\mathcal{E}$ is the ellipsoid with principle axes $\sqrt{l_1}u_1, \ldots, \sqrt{l_n}u_n$.

Square root of a pd matrix. Let S be an $n \times n$ pd matrix. Any $n \times n$ matrix A such that $AA' = S$ is called a *square root* of S, denoted by $S^{\frac{1}{2}}$. From the spectral decomposition $S = UD_lU'$, one version of $S^{\frac{1}{2}}$ is

(8.27) $$S^{\frac{1}{2}} = U\mathrm{diag}(l_1^{\frac{1}{2}}, \ldots, l_n^{\frac{1}{2}})U' \equiv UD_l^{\frac{1}{2}}U'.$$

this is a *symmetric square root* of S. Any square root $S^{\frac{1}{2}}$ is nonsingular, for

(8.28) $$|S^{\frac{1}{2}}| = |S|^{\frac{1}{2}} \neq 0.$$

Partitioned pd matrix. Partition the pd matrix $S : n \times n$ as

(8.29) $$S = \begin{matrix} & n_1 & n_2 \\ n_1 & \multicolumn{2}{c}{\multirow{2}{*}{}} \\ n_2 \end{matrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

where $n_1 + n_2 = n$. Then both S_{11} and S_{22} are symmetric pd [why?], $S_{12} = S_{21}'$, and [verify!]

(8.30)
$$\begin{pmatrix} I_{n_1} & -S_{12}S_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ -S_{22}^{-1}S_{21} & I_{n_2} \end{pmatrix} = \begin{pmatrix} S_{11\cdot 2} & 0 \\ 0 & S_{22} \end{pmatrix},$$

where

(8.31) $$S_{11\cdot 2} \equiv S_{11} - S_{12}S_{22}^{-1}S_{21}$$

is necessarily pd [why?] This in turn implies the two fundamental identities

$$(8.32)\qquad \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix} = \begin{pmatrix} I_{n_1} & S_{12}S_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} S_{11\cdot 2} & 0 \\ 0 & S_{22} \end{pmatrix} \begin{pmatrix} I_{n_1} & 0 \\ S_{22}^{-1}S_{21} & I_{n_2} \end{pmatrix},$$

$$(8.33)\qquad \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}^{-1} = \begin{pmatrix} I_{n_1} & 0 \\ -S_{22}^{-1}S_{21} & I_{n_2} \end{pmatrix} \begin{pmatrix} S_{11\cdot 2}^{-1} & 0 \\ 0 & S_{22}^{-1} \end{pmatrix} \begin{pmatrix} I_{n_1} & -S_{12}S_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix},$$

The following three consequences of (8.32) and (8.33) are immediate:

(8.34) *S is pd* $\iff$ *$S_{11\cdot 2}$ and S_{22} are pd* $\iff$ *$S_{22\cdot 1}$ and S_{11} are pd;*

$$(8.35)\qquad |S| = |S_{11\cdot 2}| \cdot |S_{22}| = |S_{22\cdot 1}| \cdot |S_{11}|\,;$$

for $x \equiv \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbf{R}^n$, the quadratic form $x'S^{-1}x$ can be decomposed as

$$(8.36)\qquad x'S^{-1}x = (x_1 - S_{12}S_{22}^{-1}x_2)'S_{11\cdot 2}^{-1}(x_1 - S_{12}S_{22}^{-1}x_2) + x_2'S_{22}^{-1}x_2.$$

Projection matrix. An $n \times n$ matrix P is a *projection* matrix if it is symmetric and *idempotent:* $P^2 = P$.

Fact 8.5. P is a projection matrix iff it has the form

$$(8.37)\qquad P = U \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} U'$$

for some orthogonal matrix $U : n \times n$ and some $m \leq n$. In this case, $\operatorname{rank}(P) = m = \operatorname{tr}(P)$.

Proof. Since P is symmetric, $P = UD_lU'$ by its spectral decomposition. But the idempotence of P implies that each $l_i = 0$ or 1. (A permutation of the rows and columns, which is also an orthogonal transformation, may be necessary to obtain the form (8.37).) □

Interpretation of (8.37): Partition U as

$$(8.38)\qquad U = n \overset{\begin{matrix} m & \quad n-m \end{matrix}}{\begin{pmatrix} U_1 & \quad U_2 \end{pmatrix}},$$

so (8.37) becomes

$$P = U_1 U_1'. \tag{8.39}$$

But U is orthogonal so $U'U = I$, hence

$$\begin{pmatrix} I_m & 0 \\ 0 & I_{n-m} \end{pmatrix} = U'U = \begin{pmatrix} U_1'U_1 & U_1'U_2 \\ U_2'U_1 & U_2'U_2 \end{pmatrix}. \tag{8.40}$$

Thus from (8.39) and (8.40),

$$\begin{aligned} PU_1 &= (U_1U_1')\,U_1 = U_1, \\ PU_2 &= (U_1U_1')\,U_2 = 0. \end{aligned}$$

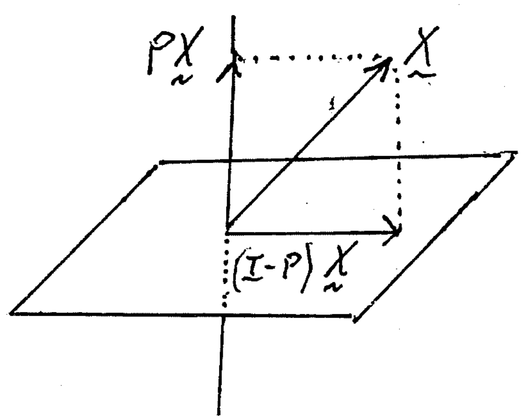
This shows that P *represents the linear transformation that projects* $\mathbf{R}^n$ *orthogonally onto the column space of* U_1*, which has dimension* $m = \mathrm{tr}(P)$.

Furthermore, $I - P$ is also symmetric and idempotent [verify] with $\mathrm{rank}(I - P) = n - m$. In fact,

$$I - P = UU' - P = (U_1U_1' + U_2U_2') - U_1U_1' = U_2U_2',$$

so $I - P$ *represents the linear transformation that projects* $\mathbf{R}^n$ *orthogonally onto the column space of* U_2*; the dimension of this space is* $n - m = \mathrm{tr}(I - P)$.

Note that the column spaces of U_1 and U_2 are perpendicular, since $U_1'U_2 = 0$. Equivalently, $P(I - P) = (I - P)P = 0$, i.e., applying P and $I - P$ successively sends any $x \in \mathbf{R}^n$ to 0.



8.2. Random vectors and covariance matrices. Let $X \equiv \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix}$ be a rvtr in $\mathbf{R}^n$. The *expected value* of X is the vector

$$\mathrm{E}(X) \equiv \begin{pmatrix} \mathrm{E}(X_1) \\ \vdots \\ \mathrm{E}(X_n) \end{pmatrix}.$$

which is the center of gravity of the probability distribution of X in $\mathbf{R}^n$. Note that expectation is linear: for rvtrs X, Y and constant matrices A, B,

$$\mathrm{E}(AX + BY) = A\,\mathrm{E}(X) + B\,\mathrm{E}(Y). \tag{8.41}$$

Similarly, if $Z \equiv \begin{pmatrix} Z_{11} & \cdots & Z_{1n} \\ \vdots & & \vdots \\ Z_{m1} & \cdots & Z_{mn} \end{pmatrix}$ is a random matrix in $\mathbf{R}^{m\times n}$, $\mathrm{E}(Z)$ is also defined component-wise:

$$\mathrm{E}(Z) = \begin{pmatrix} \mathrm{E}(Z_{11}) & \cdots & \mathrm{E}(Z_{1n}) \\ \vdots & & \vdots \\ \mathrm{E}(Z_{m1}) & \cdots & \mathrm{E}(Z_{mn}) \end{pmatrix}.$$

Then for constant matrices $A : k \times m$ and $B : n \times p$,

$$\mathrm{E}(AZB) = A\,\mathrm{E}(Z)\,B. \tag{8.42}$$

The covariance matrix of X ($\equiv$ the *variance-covariance matrix*), is

$$\begin{aligned} \mathrm{Cov}(X) &= \mathrm{E}\left[(X - \mathrm{E}X)(X - \mathrm{E}X)'\right] \\ &= \begin{pmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) & \cdots & \mathrm{Cov}(X_1, X_n) \\ \mathrm{Cov}(X_2, X_1) & \mathrm{Var}(X_2) & \cdots & \mathrm{Cov}(X_2, X_n) \\ \vdots & \vdots & & \vdots \\ \mathrm{Cov}(X_n, X_1) & \mathrm{Cov}(X_n, X_2) & \cdots & \mathrm{Var}(X_n) \end{pmatrix}. \end{aligned}$$

The following formulas are essential: for $X : n \times 1, \;\; A : m \times n, \;\; a : n \times 1,$

$$\text{(8.43)} \qquad \operatorname{Cov}(X) = \mathrm{E}(XX') - (\mathrm{E}X)(\mathrm{E}X)';$$

$$\text{(8.44)} \qquad \operatorname{Cov}(AX + b) = A \operatorname{Cov}(X)\, A';$$

$$\text{(8.45)} \qquad \operatorname{Var}(a'X + b) = a' \operatorname{Cov}(X)\, a.$$

Fact 8.6. *Let $X \equiv (X_1, \ldots, X_n)'$ be a rvtr in $\mathbf{R}^n$.*

(a) $\operatorname{Cov}(X)$ *is psd.*

(b) $\operatorname{Cov}(X)$ *is pd unless $\exists$ a nonzero $a \equiv (a_1, \ldots, a_n)' \in \mathbf{R}^n$ s.t. the linear combination*

$$a'X \equiv a_1 X_1 + \cdots + a_n X_n = c \quad \text{(a constant)}.$$

Thus the support of X is contained in a hyperplane of dimension $\leq n - 1$.

Proof. (a) This follows from (8.45) since $\operatorname{Var}(\cdot) \geq 0$.

(b) If $\operatorname{Cov}(X)$ is not pd, then $\exists$ a nonzero $a \in \mathbf{R}^n$ s.t.

$$0 = a' \operatorname{Cov}(X)\, a = \operatorname{Var}(a'X).$$

But this implies that $a'X =$ constant. □

For rvtrs $X : m \times 1$ and $Y : n \times 1$, define

$$\operatorname{Cov}(X, Y) = \mathrm{E}\left[(X - \mathrm{E}X)(Y - \mathrm{E}Y)'\right]$$

$$= \begin{pmatrix} \operatorname{Cov}(X_1, Y_1) & \operatorname{Cov}(X_1, Y_2) & \cdots & \operatorname{Cov}(X_1, Y_n) \\ \operatorname{Cov}(X_2, Y_1) & \operatorname{Cov}(X_2, Y_2) & \cdots & \operatorname{Cov}(X_2, Y_n) \\ \vdots & \vdots & & \vdots \\ \operatorname{Cov}(X_m, Y_1) & \operatorname{Cov}(X_m, Y_2) & \cdots & \operatorname{Cov}(X_m, Y_n) \end{pmatrix} : m \times n.$$

Clearly $\operatorname{Cov}(X, Y) = [\operatorname{Cov}(Y, X)]'$. Thus, if $m = n$ then [verify]

$$\text{(8.46)} \qquad \operatorname{Cov}(X \pm Y) = \operatorname{Cov}(X) + \operatorname{Cov}(Y) \pm \operatorname{Cov}(X, Y) \pm \operatorname{Cov}(Y, X).$$

and [verify]

$$\begin{aligned} &X \perp\!\!\!\perp Y \Rightarrow \mathrm{Cov}(X,Y)=0 \\ (8.47)\qquad &\qquad\Rightarrow \mathrm{Cov}(X \pm Y) = \mathrm{Cov}(X) + \mathrm{Cov}(Y). \end{aligned}$$

Variance of sample average (sample mean) of rvtrs: Let $X_1, \ldots, X_n$ be i.i.d. rvtrs in $\mathbf{R}^p$, each with mean vector μ and covariance matrix Σ. Set

$$\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n).$$

Then $\mathrm{E}(\bar{X}_n) = \mu$ and, by (8.47),

$$(8.48)\qquad \mathrm{Cov}(\bar{X}_n) = \frac{1}{n^2}\mathrm{Cov}(X_1 + \cdots + X_n) = \frac{1}{n}\Sigma.$$

Exercise 8.1. Verify the *Weak Law of Large Numbers (WLLN) for rvtrs: $\bar{X}_n$ converges to μ in probability ($X_n \xrightarrow{p} \mu$), that is, for each $\epsilon > 0$,*

$$P[\|\bar{X}_n - \mu\| \le \epsilon] \to 1 \quad \textit{as } n \to \infty.$$

Example 8.1a. (Extension of (3.18) to identically distributed but correlated rvs.) Let $X_1, \ldots, X_n$ be rvs with common mean μ and variance σ^2. Suppose they are *equicorrelated*, i.e., $\mathrm{Cor}(X_i,\, X_j) = \rho\ \forall i \ne j$. Let

$$(8.49)\qquad \bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n), \qquad s_n^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X}_n)^2,$$

the *sample mean* and *sample variance*, respectively. Then

$$\begin{aligned} (8.50)\qquad \mathrm{E}(\bar{X}_n) &= \mu \qquad\qquad (\text{so } \bar{X}_n \text{ is } \textit{unbiased} \text{ for } \mu);. \\ \mathrm{Var}(\bar{X}_n) &= \frac{1}{n^2}\mathrm{Var}(X_1 + \cdots + X_n) \\ &= \frac{1}{n^2}[n\sigma^2 + n(n-1)\rho\sigma^2] \qquad [\text{why?}] \\ (8.51)\qquad &= \frac{\sigma^2}{n}[1 + (n-1)\rho]. \end{aligned}$$

When $X_1, \ldots, X_n$ are uncorrelated ($\rho = 0$), in particular when they are independent, then (8.51) reduces to $\frac{\sigma^2}{n}$, which $\to 0$ as $n \to \infty$. But when $\rho > 0$, $\operatorname{Var}(\bar{X}_n) \to \sigma^2\rho \neq 0$, so *the WLLN fails for equicorrelated i.d. rvs.* This argument does not apply for $\rho < 0$ since (8.51) imposes the constraint

$$-\frac{1}{n-1} \leq \rho \leq 1. \tag{8.52}$$

Next, using (8.51),

$$\begin{aligned}
\mathrm{E}(s_n^2) &= \left(\frac{1}{n-1}\right) \mathrm{E}\left(\sum_{i=1}^{n} X_i^2 - n(\bar{X}_n)^2\right) \\
&= \left(\frac{1}{n-1}\right) \left[n(\sigma^2 + \mu^2) - n\left(\frac{\sigma^2}{n}[1 + (n-1)\rho] + \mu^2\right)\right] \\
&= (1-\rho)\sigma^2. \qquad (8.53)
\end{aligned}$$

Thus s_n^2 is unbiased for σ^2 if $\rho = 0$ but not otherwise! □

Example 8.1b. We now re-derive (8.51) and (8.53) via covariance matrices, using properties (8.44) and (8.45). Set $X = (X_1, \ldots, X_n)'$, so

$$\mathrm{E}(X) = \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} \equiv \mu\,\mathbf{e}_n, \qquad \text{where } \mathbf{e}_n = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} : n \times 1, \tag{8.54}$$

$$\begin{aligned}
\operatorname{Cov}(X) &= \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \rho \\ \rho & \cdots & \rho & 1 \end{pmatrix} \\
&\equiv \sigma^2[(1-\rho)I_n + \rho\,\mathbf{e}_n\mathbf{e}_n']. \qquad (8.55)
\end{aligned}$$

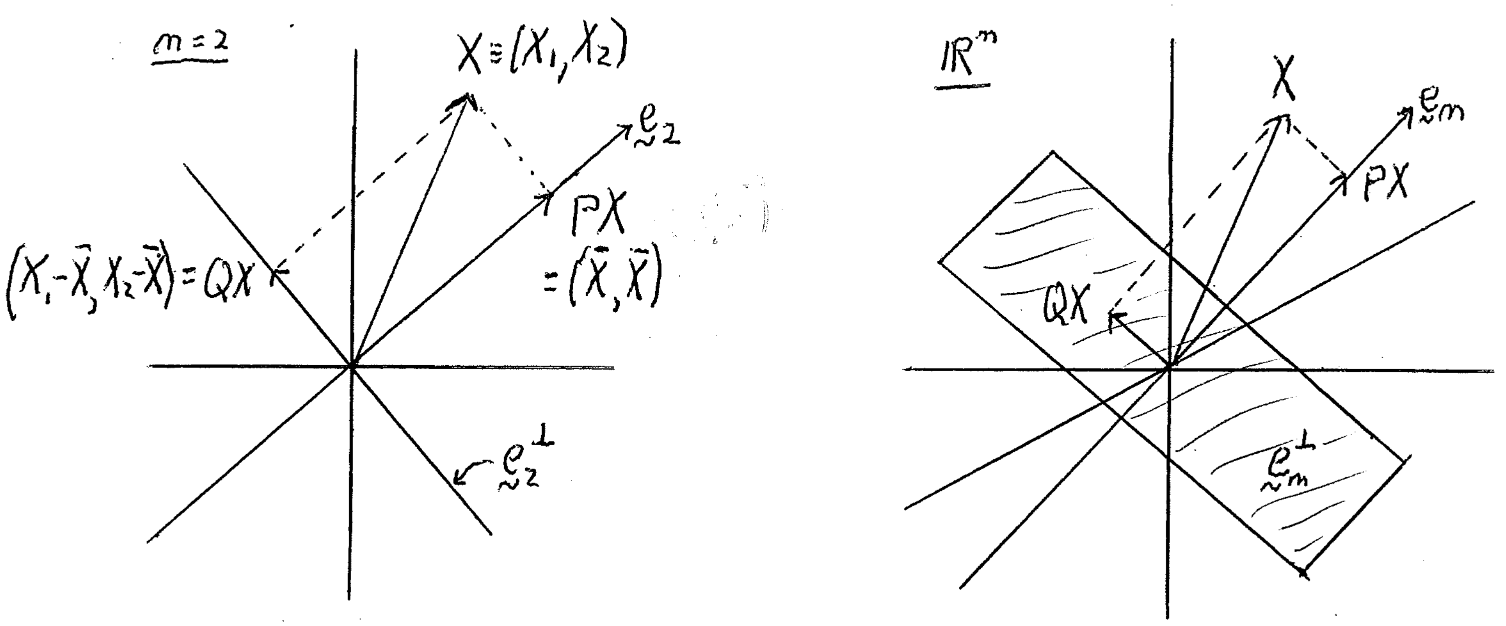
Then $\bar{X}_n = \frac{1}{n}\mathbf{e}_n' X$, so by (8.45),

$$\begin{aligned}
\operatorname{Var}(\bar{X}_n) &= \frac{\sigma^2}{n^2}\mathbf{e}_n'[(1-\rho)I_n + \rho\,\mathbf{e}_n\mathbf{e}_n']\mathbf{e}_n \\
&= \frac{\sigma^2}{n^2}[(1-\rho)n + \rho n^2] \qquad [\text{since } \mathbf{e}_n'\mathbf{e}_n = n] \\
&= \frac{\sigma^2}{n}[1 + (n-1)\rho],
\end{aligned}$$

which agrees with (8.51). Next, to find $\mathrm{E}(s_n^2)$, write

$$\begin{aligned}\sum_{i=1}^{n}(X_i-\bar{X}_n)^2 &= \sum_{i=1}^{n}X_i^2 - n(\bar{X}_n)^2\\ &= X'X - \frac{1}{n}(\mathbf{e}_n'X)^2\\ &= X'X - \frac{1}{n}(X'\mathbf{e}_n)(\mathbf{e}_n'X)\\ &\equiv X'\left(I_n - P\right)X\\ &\equiv X'QX, \qquad (8.56)\end{aligned}$$

where $P \equiv \frac{1}{n}\mathbf{e}_n\mathbf{e}_n'$ is the projection matrix of rank $1 \equiv \mathrm{tr}(\frac{1}{n}\mathbf{e}_n\mathbf{e}_n')$ that projects $\mathbf{R}^n$ orthogonally onto the 1-dimensional subspace spanned by $\mathbf{e}_n$, and $Q \equiv I_n - \frac{1}{n}\mathbf{e}_n\mathbf{e}_n'$ is the projection matrix of rank $n-1 \equiv \mathrm{tr}\,Q$ that projects $\mathbf{R}^n$ orthogonally onto the $(n-1)$-dimensional subspace $\mathbf{e}_n^{\perp}$ (see figure). Now complete Exercise 8.2:



Exercise 8.2. Prove Fact 8.7 below, and use it to show that

$$\mathrm{E}(X'QX) = (n-1)(1-\rho)\sigma^2, \tag{8.57}$$

which is equivalent to (8.53).

Fact 8.7. Let $X : n \times 1$ be a rvtr with $\mathrm{E}(X) = \theta$ and $\mathrm{Cov}(X) = \Sigma$. Then for any $n \times n$ symmetric matrix A,

$$\mathrm{E}(X'AX) = \mathrm{tr}(A\Sigma) + \theta' A \theta. \tag{8.58}$$

(This generalizes the relation $\mathrm{E}(X^2) = \mathrm{Var}(X) + (\mathrm{E}\,X)^2$.)

Exercise 8.3. Show that $\mathrm{Cov}(X) \equiv \sigma^2[(1-\rho)I_n + \rho \mathbf{e}_n \mathbf{e}_n']$ in (8.55) has one eigenvalue $= \sigma^2[1+(n-1)\rho]$ with eigenvector $\mathbf{e}_n$, and $n-1$ eigenvalues $= \sigma^2(1-\rho)$.

Example 8.1c. Eqn. (8.53) also can be obtained from the properties of the projection matrix Q. First note that

$$Q\mathbf{e}_n = 0. \tag{8.59}$$

Define

$$Y \equiv \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = QX \quad : n \times 1, \tag{8.60}$$

so

$$\mathrm{E}(Y) = Q\,\mathrm{E}(X) = \mu\, Q\mathbf{e}_n = 0, \tag{8.61}$$

$$\begin{aligned} \mathrm{Cov}(Y) &= \sigma^2 Q[(1-\rho)I_n + \rho \mathbf{e}_n \mathbf{e}_n']Q \\ &= \sigma^2(1-\rho)Q. \end{aligned} \tag{8.62}$$

Thus, since Q is idempotent ($Q^2 = Q$),

$$\begin{aligned} \mathrm{E}(X'QX) &= \mathrm{E}(Y'Y) = \mathrm{E}[\mathrm{tr}\,(Y'Y)] \\ &= \mathrm{E}[\mathrm{tr}\,(YY')] = \mathrm{tr}\,[\mathrm{E}(YY')] = \mathrm{tr}\,[\mathrm{Cov}(Y)] \\ &= \sigma^2(1-\rho)\mathrm{tr}\,(Q) = \sigma^2(1-\rho)(n-1), \end{aligned}$$

which again is equivalent to (8.53).

8.3. The multivariate normal distribution. [This section will revisit some results from §4, 5, 6.] As in Example 3.5, first let $Z_1, \ldots, Z_n$ be i.i.d. standard normal $N(0,1)$ rvs. Then the rvtr $Z \equiv (Z_1, \ldots, Z_n)'$ has pdf

$$
\begin{aligned}
f_Z(z) &= \prod_{i=1}^{n} \frac{1}{(2\pi)^{1/2}} e^{-\frac{1}{2}z_i^2} \\
&= \frac{1}{(2\pi)^{n/2}} e^{-\frac{1}{2}z'z}, \qquad (8.63)
\end{aligned}
$$

where $z = (z_1, \ldots, z_n)'$. This is an extension of the univariate standard normal pdf to $\mathbf{R}^n$ with $E\,Z = 0$ and $\operatorname{Cov} Z = I_n$, so we write $Z \sim N_n(0, I_n)$.

Now let $X = AZ + \mu$, with $A : n \times n$ *nonsingular* and $\mu : n \times 1$. This is a linear (actually, affine) transformation with inverse given by

$$Z = A^{-1}(X - \mu)$$

and Jacobian $|A|$, so by the Jacobian method for transformations,

$$
\begin{aligned}
f_X(x) &= \frac{1}{(2\pi)^{n/2}|A|} e^{-\frac{1}{2}(x-\mu)'(A^{-1})'A^{-1}(x-\mu)} \\
&= \frac{1}{(2\pi)^{n/2}|AA'|^{1/2}} e^{-\frac{1}{2}(x-\mu)'(AA')^{-1}(x-\mu)} \\
&= \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}, \qquad (8.64)
\end{aligned}
$$

where $\Sigma = AA' = \operatorname{Cov}(AZ + \mu) = \operatorname{Cov} X$. Thus the distribution of X depends only on the first two multivariate moments $(\mu, \Sigma) = (\mathrm{E}\,X, \operatorname{Cov} X)$, so we write

$$X \sim N_n(\mu, \Sigma),$$

the (nonsingular, n-dimensional) *multivariate normal distribution (MVND) with mean vector μ and covariance matrix Σ*. (Note that $\Sigma \equiv AA'$ is pd.)

We can use the representation $X = AZ + \mu$ to derive the basic linearity property of the nonsingular MVND:

8.3.1. Linearity of $N_n(\mu, \Sigma)$: If $X \sim N_n(\mu, \Sigma)$ then for $C : n \times n$ and $d : n \times 1$ with C nonsingular,

$$
\begin{aligned}
Y \equiv CX + d &= (CA)Z + (C\mu + d) \\
&\sim N_n\big(C\mu + d, (CA)(CA)'\big) \\
&= N_n(C\mu + d,\ C\Sigma C'). \qquad (8.65)
\end{aligned}
$$

Remark 8.1. It is important to remember that the *general* (possibly singular) MVND was already defined in Example 3.5 via its moment generating function, where it was shown (recall (3.59)) that the linearity property (8.65) holds for *any* $C : m \times n$. Thus the following results are valid for *general* MVNDs.

8.3.2. Marginal distributions are normal. Let

$$
(8.66) \qquad \begin{matrix} n_1 \\ n_2 \end{matrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_{n_1+n_2}\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \right).
$$

Then by the linearity property (8.65) (actually (3.59)) with $C = (\, I_{n_1} \quad 0 \,)$,

$$
(8.67) \qquad X_1 = (\, I_{n_1} \quad 0 \,) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_{n_1}(\mu_1, \Sigma_{11}),
$$

and similarly

$$
(8.68) \qquad X_2 = (\, 0 \quad I_{n_2} \,) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_{n_2}(\mu_2, \Sigma_{22}).
$$

8.3.3. Linear combinations are normal. If X_1, X_2 satisfy (8.66) then for $A_1 : m \times n_1$ and $A_2 : m \times n_2$, (8.65) (actually (3.59)) implies that

$$
\begin{aligned}
&A_1 X_1 + A_2 X_2 = (\, A_1 \quad A_2 \,) \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\
(8.69) &\sim N_m(A_1\mu_1 + A_2\mu_2,\ A_1\Sigma_{11}A_1' + A_1\Sigma_{12}A_2' + A_2\Sigma_{21}A_1' + A_2\Sigma_{22}A_2').
\end{aligned}
$$

8.3.4. Independence $\Longleftrightarrow$ Uncorrelation. If X_1, X_2 satisfy (8.66) then

$$
(8.70) \qquad X_1 \perp\!\!\!\perp X_2 \iff \operatorname{Cov}(X_1, X_2) \equiv \Sigma_{12} = 0.
$$

Proof. $\Rightarrow$ is obvious. Next, suppose that $\Sigma_{12} = 0$. Then $\Leftarrow$ is established for a general MVND via its mgf (recall (3.58)):

$$\begin{aligned} m_{X_1,X_2}(t_1,t_2) &= e^{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}' \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}} e^{\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}' \begin{pmatrix} \Sigma_{11} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} /2} \\ &= e^{t_1'\mu_1} e^{t_1'\Sigma_{11}t_1/2} \cdot e^{t_2'\mu_2} e^{t_2'\Sigma_{22}t_2/2} \\ &= m_{X_1}(t_1) \cdot m_{X_2}(t_2). \end{aligned}$$

Since this mgf factors, $X_1 \perp\!\!\!\perp X_2$ [by (8.67), (8.68), and the uniqueness property of mgfs.]

Exercise 8.4. Prove (8.70) for a *nonsingular* MVND using the pdf (8.64).

Exercise 8.5. Extend (8.70) as follows. If $X \sim N_n(\mu, \Sigma)$ then for any two matrices $A : l \times n$ and $B : m \times n$,

$$AX \perp\!\!\!\perp BX \iff A\Sigma B' = 0. \tag{8.71}$$

Hint: Consider $\begin{pmatrix} A \\ B \end{pmatrix} X$ and apply (8.65).

8.3.5. Conditional distributions are normal. If X_1, X_2 satisfy (8.66) and Σ_{22} is pd (in particular if Σ itself is pd), then

$$X_1 \mid X_2 \sim N_{n_1}\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2),\ \Sigma_{11\cdot 2}\right). \tag{8.72}$$

Proof. Again apply linearity ((8.65), actually (3.59)) and the identity (8.30):

$$\begin{aligned} &\begin{pmatrix} X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 \\ X_2 \end{pmatrix} \\ &= \begin{pmatrix} I_{n_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\ &\sim N_{n_1+n_2}\left(\begin{pmatrix} I_{n_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \right. \\ &\qquad \left. \begin{pmatrix} I_{n_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix} \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} \begin{pmatrix} I_{n_1} & -\Sigma_{12}\Sigma_{22}^{-1} \\ 0 & I_{n_2} \end{pmatrix}' \right) \\ &= N_{n_1+n_2}\left(\begin{pmatrix} \mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11\cdot 2} & 0 \\ 0 & \Sigma_{22} \end{pmatrix} \right). \end{aligned}$$

By (8.67) and (8.70), this implies that

$$X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 \sim N_{n_1}\left(\mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2,\, \Sigma_{11\cdot 2}\right) \tag{8.73}$$

and

$$X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 \perp\!\!\!\perp X_2. \tag{8.74}$$

Thus (8.73) holds conditionally:

$$X_1 - \Sigma_{12}\Sigma_{22}^{-1}X_2 \mid X_2 \sim N_{n_1}\left(\mu_1 - \Sigma_{12}\Sigma_{22}^{-1}\mu_2,\, \Sigma_{11\cdot 2}\right), \tag{8.75}$$

which implies (8.72) [verify]. □

Exercise 8.6. Prove (8.72) for a *nonsingular* MVND using the pdf (8.64). That is, apply the formula $f(x_1|x_2) = \frac{f(x_1,x_2)}{f(x_2)}$.

Hint: Apply (8.35) and (8.36) for Σ.

8.3.6. Regression is linear, covariance is constant (≡ homoscedastic). It follows immediately from (8.72) that

$$\mathrm{E}(X_1 \mid X_2) = \mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2 - \mu_2), \tag{8.76}$$

$$\mathrm{Cov}(X_1 \mid X_2) = \Sigma_{11\cdot 2}. \tag{8.77}$$

(Compare (8.76) with (5.38) in Remark 5.4.)

8.3.7. The bivariate case. Take $n_1 = n_2 = 1$ and consider

$$\begin{aligned}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &\sim N_2\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}\right] \\ &\equiv N_2\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right].\end{aligned} \tag{8.78}$$

Note that

$$\Sigma \equiv \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}, \tag{8.79}$$

so

$$|\Sigma| = \sigma_1^2\sigma_2^2(1-\rho^2), \tag{8.80}$$

$$\Sigma^{-1} = \frac{1}{1-\rho^2}\begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}\begin{pmatrix} \sigma_1^{-1} & 0 \\ 0 & \sigma_2^{-1} \end{pmatrix}. \tag{8.81}$$

Thus from (8.64) the pdf of (X_1, X_2) is

$$\text{(8.82)} \qquad \frac{1}{2\pi\sigma_1\sigma_2(1-\rho^2)^{1/2}} e^{-\frac{1}{2(1-\rho^2)}\left[(\frac{x_1-\mu_1}{\sigma_1})^2+(\frac{x_2-\mu_2}{\sigma_2})^2-2\rho(\frac{x_1-\mu_1}{\sigma_1})(\frac{x_2-\mu_2}{\sigma_2})\right]},$$

while (8.76) and (8.77) become

$$\text{(8.83)} \qquad \mathrm{E}(X_1 \mid X_2) = \mu_1 + \rho\left(\frac{\sigma_1}{\sigma_2}\right)(X_2 - \mu_2),$$

$$\text{(8.84)} \qquad \mathrm{Var}(X_1 \mid X_2) = (1-\rho^2)\,\sigma_1^2.$$

Note that (8.83) agrees with the best linear predictor $\hat{X}_1$ given by (5.21), and (8.84) agrees with (5.27), the variance of the linear residual $X_1 - \hat{X}_1$.

Remark 8.2. In the special case where $\sigma_1^2 = \sigma_2^2 \equiv \sigma^2$ then

$$\begin{aligned}
&\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \sim N_2\left(\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \sigma^2\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right) \\
\Rightarrow &\begin{pmatrix} X_1 + X_2 \\ X_1 - X_2 \end{pmatrix} \equiv \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \\
&\qquad \sim N_2\left(\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \sigma^2\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\right) \\
&\qquad \sim N_2\left(\begin{pmatrix} \mu_1 + \mu_2 \\ \mu_1 - \mu_2 \end{pmatrix}, 2\sigma^2\begin{pmatrix} 1+\rho & 0 \\ 0 & 1-\rho \end{pmatrix}\right)
\end{aligned}$$

In particular, this implies that $(X_1 + X_2) \perp\!\!\!\perp (X_1 - X_2)$ in this case. (This extends CB Exercise 4.27, where it was assumed that $\rho = 0$.)

Exercise 8.7. (cf. Example 5.1.) Define the joint distribution of X, Y via the hierarchy

$$\begin{aligned} Y \mid X &\sim N_1(\beta X,\, \tau^2), \\ X &\sim N_1(0,\, \sigma^2). \end{aligned}$$

Show that the joint distribution of X, Y is $N_2(\mu,\, \Sigma)$ and find μ and Σ. Find the conditional distribution of $X|Y$ and the marginal distribution of Y.

Exercise 8.8. Clearly $X_1, \dots, X_n$ i.i.d. $\Rightarrow X_1, \dots, X_n$ are exchangeable.

(i) Find a trivariate normal example showing that $\Leftarrow$ need not hold.

(ii) Clearly $X_1, \dots, X_n$ exchangeable $\Rightarrow$ $X_1, \dots, X_n$ are identically distributed. Find a trivariate normal example showing that $X_1, \dots, X_n$ identically distributed $\not\Rightarrow$ $X_1, \dots, X_n$ exchangeable. Show, however, that $\Rightarrow$ holds for bivariate ($n = 2$) normal and bivariate binary distributions.

(iii) Find a non-normal bivariate distribution where X_1 and X_2 are identically distributed but X_1 and X_2 are not exchangeable.

8.4. The MVND and the chi-square distribution.

In Remark 6.3, the *chi-square distribution* χ^2_n *with n degrees of freedom* was defined to be the distribution of

$$Z_1^2 + \cdots + Z_n^2 \equiv Z'Z \equiv \|Z\|^2,$$

where $Z \equiv (Z_1, \dots, Z_n)' \sim N_n(0, I_n)$. (That is, $Z_1, \dots, Z_n$ are i.i.d. standard $N(0,1)$ rvs.) Recall from Example 2.4 that

$$\chi^2_n \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right), \tag{8.85}$$

$$\text{E}(\chi^2_n) = n, \tag{8.86}$$

$$\text{Var}(\chi^2_n) = 2n. \tag{8.87}$$

Now consider $X \sim N_n(\mu, \Sigma)$ with Σ pd. Then

$$Z \equiv \Sigma^{-1/2}(X - \mu) \sim N_n(0, I_n), \tag{8.88}$$

$$(X - \mu)'\Sigma^{-1}(X - \mu) = Z'Z \sim \chi^2_n. \tag{8.89}$$

Suppose, however, that we omit Σ^{-1} in (8.89) and seek the distribution of $(X - \mu)'(X - \mu)$. Then this will *not* have a chi-square distribution in general. Instead, by the spectral decomposition $\Sigma = UD_\lambda U'$, (8.88) yields

$$\begin{aligned}(X - \mu)'(X - \mu) = Z'\Sigma Z &= (U'Z)'D_\lambda(U'Z) \\ &\equiv V'D_\lambda V = \lambda_1 V_1^2 + \cdots \lambda_n V_n^2,\end{aligned} \tag{8.90}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of Σ and $V \equiv U'Z \sim N_n(0, I_n)$. Thus the distribution of $(X-\mu)'(X-\mu)$ is a *positive linear combination of independent* χ_1^2 *rvs,* which is not (proportional to) a χ_n^2 rv. [Check via mgfs!]

8.4.1. Quadratic forms and projection matrices. Let $X \sim N_n(\xi, \sigma^2 I_n)$ and let P be an $n \times n$ projection matrix with $\text{rank}(P) = \text{tr}(P) \equiv m$. Then the quadratic form determined by $X - \xi$ and P satisfies

$$(X-\xi)'P(X-\xi) \sim \sigma^2 \chi_m^2. \tag{8.91}$$

Proof. By Fact 8.5, there exists an orthogonal matrix $U : n \times n$ s.t.

$$P = U \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} U'.$$

Then $Y \equiv U'(X-\xi) \sim N_n(0, \sigma^2 I_n)$, so with $Y = (Y_1, \ldots, Y_n)'$,

$$(X-\xi)'P(X-\xi) = Y' \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix} Y = Y_1^2 + \cdots Y_m^2 \sim \sigma^2 \chi_m^2.$$

8.4.2. Joint distribution of $\bar{X}_n$ and s_n^2. Let $X_1, \ldots, X_n$ be a random (i.i.d) sample from the univariate normal distribution $N_1(\mu, \sigma^2)$ and let

$$\bar{X}_n = \frac{1}{n}(X_1 + \cdots + X_n), \qquad s_n^2 = \frac{1}{n-1}\sum\nolimits_{i=1}^n (X_i - \bar{X}_n)^2.$$

(Recall (3.18), also Example 8.1a,b,c.) Then:

$$\bar{X}_n \textit{ and } s_n^2 \textit{ are independent;} \tag{8.92}$$

$$\bar{X}_n \sim N\left(\mu, \tfrac{\sigma^2}{n}\right); \tag{8.93}$$

$$s_n^2 \sim \tfrac{\sigma^2}{n-1}\chi_{n-1}^2. \tag{8.94}$$

Proof. As in Example 8.1b, let $P = \frac{1}{n}\mathbf{e}_n\mathbf{e}_n'$, $Q = I_n - \frac{1}{n}\mathbf{e}_n\mathbf{e}_n'$, and

$$X = \begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} \sim N_n(\mu\mathbf{e}_n, \sigma^2 I_n). \tag{8.95}$$

[Figure p.107]. Then $P(\sigma^2 I_n)Q' = \sigma^2 PQ = 0$ so $PX \perp\!\!\!\perp QX$ by (8.71). But

$$\begin{pmatrix} \bar{X}_n \\ \vdots \\ \bar{X}_n \end{pmatrix} = \bar{X}_n \mathbf{e}_n = PX \sim N_n(\mu P \mathbf{e}_n,\, \sigma^2 P) = N_n\left(\mu \mathbf{e}_n,\, \frac{\sigma^2}{n} \mathbf{e}_n \mathbf{e}_n'\right),$$

$$\begin{pmatrix} X_1 - \bar{X}_n \\ \vdots \\ X_n - \bar{X}_n \end{pmatrix} = X - \bar{X}_n \mathbf{e}_n = (I - P)X = QX,$$

so this implies (8.92) and (8.93). Finally,

$$\sum\nolimits_{i=1}^{n} (X_i - \bar{X})^2 = (QX)'(QX) = (X - \mu \mathbf{e}_n)' Q (X - \mu \mathbf{e}_n) \quad \text{[verify]}$$

and $\text{rank}(Q) = \text{tr}(Q) = \text{tr}(I - P) = n - 1$, so (8.94) follows from (8.91). □

Geometrical interpretation of (8.95): The i.i.d. normal model (8.95) is the simplest example of the *univariate normal linear model.* If we let L denote the 1-dimensional linear subspace spanned by $\mathbf{e}_n$, then (8.95) can be expressed as follows:

(8.96) $$X \sim N_n(\xi,\, \sigma^2 I_n) \quad \text{with} \quad \xi \in L.$$

If we let $P \equiv P_L \equiv \frac{1}{n}\mathbf{e}_n \mathbf{e}_n'$ denote projection onto L, then $Q \equiv Q_{L^\perp} = I_n - P_L$ is the projection onto the "residual" subspace $L^\perp$ [Figure p.107], and Pythagoras gives us the following *Analysis of Variance (ANOVA):*

(8.97)
$$\begin{aligned} I_n &= P_L + Q_L, \\ X &= P_L X + Q_L X, \\ \|X\|^2 &= \|P_L X\|^2 + \|Q_L X\|^2, \\ X_1^2 + \cdots + X_n^2 &= n(\bar{X}_n)^2 + \sum\nolimits_{i=1}^{n} (X_i - \bar{X})^2, \\ \text{Total Sum of Squares} &= \text{SS}(L) \quad + \quad \text{SS}(L^\perp). \end{aligned}$$

[$\text{SS}(L^\perp)$ is often called the "residual sum of squares".] Then we can see similarly that under the normal linear model (8.96),

(8.98) $$\text{SS}(L) \perp\!\!\!\perp \text{SS}(\text{L}^\perp) \quad \text{and} \quad \text{SS}(\text{L}^\perp) \sim \sigma^2 \chi^2_{\dim(L^\perp)}.$$

Furthermore it follows from (8.111) below that for $\xi \in L$,

$$\|P_L X\|^2 \equiv \mathrm{SS}(L) \sim \sigma^2 \chi^2_{\dim(L)}\left(\frac{\|\xi\|^2}{\sigma^2}\right), \tag{8.99}$$

a *noncentral chi-square distribution with noncentrality parameter* $\frac{\|\xi\|^2}{\sigma^2}$ (see §8.4.3). Note that for the model (8.95), $\xi = \mu \mathbf{e}_n$, so

$$\frac{\|\xi\|^2}{\sigma^2} = \frac{n\mu^2}{\sigma^2} \geq 0,$$

hence the noncentrality parameter $= 0$ iff $\mu = 0$. Thus the "null hypothesis" $\mu = 0$ can be tested by means of the *F-statistic* $\equiv$ *F-ratio*

$$F \equiv \frac{\mathrm{SS}(L)}{\mathrm{SS}(L^\perp)} \sim \frac{\chi^2_{\dim(L)}\left(\frac{\|\xi\|^2}{\sigma^2}\right)}{\chi^2_{\dim(L^\perp)}} = \frac{\chi^2_1\left(\frac{n\mu^2}{\sigma^2}\right)}{\chi^2_{n-1}}. \tag{8.100}$$

The null hypothesis $\mu = 0$ is rejected in favor of the alternative hypothesis $\mu \neq 0$ for sufficiently large values of F. [See §8.4.3 for the noncentral chi-square distribution, also Remark 8.4 and Exercise 18.27.]

Exercise 8.9. Extend (8.92) - (8.94) to the case where $X_1, \ldots, X_n$ are equicorrelated, as in Examples 8.1a,b,c. That is, if as in Example 8.1b, $X \sim N_n(\mu \mathbf{e}_n, \sigma^2[(1-\rho)I_n + \rho \mathbf{e}_n \mathbf{e}_n'])$, show that

(8.101) $\bar{X}_n$ *and* s_n^2 *are independent;*

$$\bar{X}_n \sim N_1\left(\mu, \tfrac{\sigma^2}{n}[1 + (n-1)\rho]\right); \tag{8.102}$$

$$s_n^2 \sim \tfrac{\sigma^2(1-\rho)}{n-1}\chi^2_{n-1}. \tag{8.103}$$

Hint: Follow the matrix formulation in Example 8.1b.

8.4.3. The noncentral chi-square distribution. Extend the results of §8.4 as follows: First let $Z \equiv (Z_1, \ldots, Z_n)' \sim N_n(\mu, I_n)$, where $\mu \equiv (\mu_1, \ldots, \mu_n)' \in \mathbf{R}^n$. The distribution of

$$Z_1^2 + \cdots + Z_n^2 \equiv Z'Z \equiv \|Z\|^2$$

is called the *noncentral chi-square distribution with* n *degrees of freedom and noncentrality parameter* $\|\mu\|^2$, denoted by $\chi_n^2(\|\mu\|^2)$. Note that as in §8.4, $Z_1, \ldots, Z_n$ are independent, each with variance = 1, but now $\mathrm{E}(Z_i) = \mu_i$.

To show that the distribution of $\|Z\|^2$ depends on μ only through its (squared) length $\|\mu\|^2$, choose[8] an orthogonal (rotation) matrix $U : n \times n$ s.t. $U\mu = (\|\mu\|, 0, \ldots, 0)'$, i.e., U rotates μ into $(\|\mu\|, 0, \ldots, 0)'$, and set

$$Y = UZ \sim N_n(U\mu,\, UU') = N_n\left((\|\mu\|, 0, \ldots, 0)',\, I_n\right).$$

Then the desired result follows since

$$\begin{aligned} \|Z\|^2 = \|Y\|^2 &\equiv Y_1^2 + Y_2^2 + \cdots + Y_n^2 \\ &\sim [N_1(\|\mu\|,\, 1)]^2 + [N_1(0,\, 1)]^2 + \cdots + [N_1(0,\, 1)]^2 \\ &\equiv \chi_1^2(\|\mu\|^2) + \chi_1^2 + \cdots + \chi_1^2 \\ &\equiv \chi_1^2(\|\mu\|^2) + \chi_{n-1}^2, \end{aligned} \tag{8.104}$$

where the chi-square variates in each line are mutually independent.

To find the pdf of $V \equiv Y_1^2 \sim \chi_1^2(\delta) \sim [N_1(\sqrt{\delta},\, 1)]^2$, where $\delta = \|\mu\|^2$, recall the method of Example 2.3:

$$\begin{aligned} f_V(v) &= \frac{d}{dv} P[Y_1^2 \le v] = \frac{d}{dv} P[-\sqrt{v} \le Y_1 \le \sqrt{v}] \\ &= \frac{d}{dv} \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{v}}^{\sqrt{v}} e^{-\frac{1}{2}(t-\sqrt{\delta})^2} dt \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{\delta}{2}} \frac{d}{dv} \int_{-\sqrt{v}}^{\sqrt{v}} e^{t\sqrt{\delta}} e^{-\frac{t^2}{2}} dt \end{aligned}$$

---

[8] Let the first row of U be $\tilde{\mu} \equiv \frac{\mu}{\|\mu\|}$ and let the remaining $n-1$ rows be any orthonormal basis for $L^{\perp}$.

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\delta}{2}} \frac{d}{dv} \int_{-\sqrt{v}}^{\sqrt{v}} \left[\sum_{k=0}^{\infty} \frac{t^k \delta^{\frac{k}{2}}}{k!} \right] e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\delta}{2}} \sum_{k=0}^{\infty} \frac{\delta^{\frac{k}{2}}}{k!} \frac{d}{dv} \int_{-\sqrt{v}}^{\sqrt{v}} t^k e^{-\frac{t^2}{2}} dt$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\delta}{2}} \sum_{k=0}^{\infty} \frac{\delta^k}{(2k)!} \frac{d}{dv} \int_{-\sqrt{v}}^{\sqrt{v}} t^{2k} e^{-\frac{t^2}{2}} dt \qquad \text{[why?]}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\delta}{2}} \sum_{k=0}^{\infty} \frac{\delta^k}{(2k)!} v^{k-\frac{1}{2}} e^{-\frac{v}{2}} \qquad \text{[verify]}$$

(8.105)
$$= \underbrace{e^{-\frac{\delta}{2}} \sum_{k=0}^{\infty} \frac{\left(\frac{\delta}{2}\right)^k}{k!}}_{\text{Poisson}\left(\frac{\delta}{2}\right) \text{ weights}} \underbrace{\left[\frac{v^{\frac{1+2k}{2}-1} e^{-\frac{v}{2}}}{2^{\frac{1+2k}{2}} \Gamma\left(\frac{1+2k}{2}\right)} \right]}_{\text{pdf of } \chi^2_{1+2k}} \cdot c_k,$$

where

$$c_k = \frac{2^k k! 2^{\frac{1+2k}{2}} \Gamma\left(\frac{1+2k}{2}\right)}{(2k)!\sqrt{2\pi}} = 1$$

by the Legendre "duplication formula" for the Gamma function! Thus we have represented the pdf of a $\chi^2_1(\delta)$ rv as a mixture (weighted average) of central chi-square pdfs with Poisson weights. This can be written as follows:

(8.106)
$$\chi^2_1(\delta) \mid K \sim \chi^2_{1+2K} \qquad \text{where} \qquad K \sim \text{Poisson}\,(\delta/2)\,.$$

(Compare this to the result of Example 2.4.) Thus by (8.104) this implies that $Z'Z \equiv \|Z\|^2 \sim \chi^2_n(\delta)$ satisfies

(8.107)
$$\chi^2_n(\delta) \mid K \sim \chi^2_{n+2K} \qquad \text{where} \qquad K \sim \text{Poisson}\,(\delta/2)\,.$$

That is, *the pdf of the noncentral chi-square rv* $V \equiv \chi^2_n(\delta)$ *is a Poisson*($\delta/2$)*-mixture of the pdfs of central chi-square rvs with* $n + 2k$ *d.f.*, $k = 0, 1, \ldots$.

The representation (8.107) can be used to obtain the mean and variance of $\chi^2_n(\delta)$:

$$
\begin{aligned}
\mathrm{E}[\chi^2_n(\delta)] &= \mathrm{E}\{\mathrm{E}[\chi^2_{n+2K} \mid K]\} \\
&= \mathrm{E}(n+2K) \\
&= n + 2(\delta/2) \\
&= n + \delta; \qquad (8.108) \\
\mathrm{Var}[\chi^2_n(\delta)] &= \mathrm{E}[\mathrm{Var}(\chi^2_{n+2K} \mid K)] + \mathrm{Var}[\mathrm{E}(\chi^2_{n+2K} \mid K)] \\
&= \mathrm{E}[2(n+2K)] + \mathrm{Var}(n+2K) \\
&= [2n + 4(\delta/2)] + 4(\delta/2) \\
&= 2n + 4\delta. \qquad (8.109)
\end{aligned}
$$

Exercise 8.10*. Show that the noncentral chi-square distribution $\chi^2_n(\delta)$ is stochastically increasing in both n and δ. □

Next, consider $X \sim N_n(\mu, \Sigma)$ with a general pd Σ. Then

$$X'\Sigma^{-1}X = (\Sigma^{-\frac{1}{2}}X)'(\Sigma^{-\frac{1}{2}}X) \sim \chi^2_n(\mu'\Sigma^{-1}\mu), \qquad (8.110)$$

since $Z \equiv \Sigma^{-\frac{1}{2}}X \sim N_n(\Sigma^{-\frac{1}{2}}\mu, I_n)$ and $\|\Sigma^{-\frac{1}{2}}\mu\|^2 = \mu'\Sigma^{-1}\mu$. Thus, by Exercise 8.10, the distribution of $X'\Sigma^{-1}X$ in (8.110) is stochastically increasing in $\mu'\Sigma^{-1}\mu$.

Finally, let $Y \sim N_n(\xi, \sigma^2 I_n)$ and let P be a projection matrix with $\mathrm{rank}(P) = m$. Then $P = U_1U_1'$ where $U_1'U_1 = I_m$ (cf. (8.38) - (8.40)), so

$$\|PY\|^2 = \|U_1U_1'Y\|^2 = (U_1U_1'Y)'(U_1U_1'Y) = Y'U_1U_1'Y = \|U_1'Y\|^2.$$

But

$$U_1'Y \sim N_m(U_1'\xi, \sigma^2 U_1'U_1) = N_m(U_1'\xi, \sigma^2 I_m),$$

so by (8.110) with $X = U_1'Y$, $\mu = U_1'\xi$, and $\Sigma = \sigma^2 I_m$,

$$\frac{\|PY\|^2}{\sigma^2} = \frac{(U_1'Y)'(U_1'Y)}{\sigma^2} \sim \chi^2_m\left(\frac{\xi'U_1U_1'\xi}{\sigma^2}\right) = \chi^2_m\left(\frac{\|P\xi\|^2}{\sigma^2}\right),$$

so

$$\|PY\|^2 \sim \sigma^2\chi^2_m\left(\frac{\|P\xi\|^2}{\sigma^2}\right). \qquad (8.111)$$

Remark 8.4. By taking $P = P_L$ for an m-dimensional linear subspace L of $\mathbf{R}^n$, this confirms (8.99). Furthermore, under the general univariate linear model (8.96) it is assumed that $\xi \in L$, so $P_L\xi = \xi$ and $\|P_L\xi\|^2 = \|\xi\|^2$. In view of Exercise 8.10*, this shows why the F-ratio

$$F \equiv \frac{\mathrm{SS}(L)}{\mathrm{SS}(L^\perp)} \equiv \frac{\|P_L Y\|^2}{\|Q_L Y\|^2} \sim \frac{\chi^2_{\dim(L)}\left(\frac{\|\xi\|^2}{\sigma^2}\right)}{\chi^2_{\dim(L^\perp)}} \tag{8.112}$$

in (8.100) will tend to take larger values when $\xi \neq 0$ than when $\xi = 0$, hence why this F-ratio is a reasonable statistic for testing $\xi = 0$ vs. $\xi \neq 0$ ($\xi \in L$). (See Exercise 18.27 for a generalization.)

8.5. Further examples of univariate normal linear models.

As indicated by (8.96), the *univariate normal linear model* has the following form: observe

$$X \sim N_n(\xi,\, \sigma^2 I_n) \quad \text{with} \quad \xi \in L, \tag{8.113}$$

where L is a *d-dimensional linear subspace of* $\mathbf{R}^n$ ($0 < d < n$). The components $X_1, \ldots, X_n$ of X are independent[9] normal rvs with common unknown variance σ^2. Thus the model (8.113) imposes the linear constraint that $\xi \equiv \mathrm{E}(X) \in L$. Our goal is to estimate ξ and σ^2 subject to this constraint.

Let P_L denote projection onto L, so $Q_L \equiv I_n - P_L$ is the projection onto the "residual" subspace $L^\perp$ (recall the figure in Example 8.1b). Then it can be shown (compare to §8.4.2) that

(8.114) $\hat{\xi} \equiv P_L X$ *is the best linear unbiased estimator (BLUE) and maximum likelihood estimator (MLE) of* ξ *(see §14.1);*

(8.115) $\hat{\sigma}^2 \equiv \frac{\|Q_L X\|^2}{n}$ *is the MLE of* σ^2; $\tilde{\sigma}^2 \equiv \frac{\|Q_L X\|^2}{n-d}$ *is unbiased for* σ^2;

(8.116) $(\hat{\xi},\, \tilde{\sigma}^2)$ *is a complete sufficient statistic for* $(\xi,\, \sigma^2)$ *(see §11, §12);*

(8.117) $\hat{\xi}$ *and* $\tilde{\sigma}^2$ *are independent;*

[9] More generally, we may assume that $X \sim N_n(\xi,\, \sigma^2\Sigma_0)$ where Σ_0 is a known p.d. matrix. This can be reduced to the form (8.113) by the transformation $Y = \Sigma_0^{-1/2}X$.

(8.118) $\hat{\xi} \sim N_n(\xi, \sigma^2 P_L)$;

(8.119) $\tilde{\sigma}^2 \sim \frac{\sigma^2}{n-d}\chi^2_{n-d}$.

This leaves only one task: the algebraic problem of finding the matrices P_L and Q_L and thereby calculating $\hat{\xi}$ and $\tilde{\sigma}^2$. Typically the linear subspace L is specified as the subspace spanned by d linearly independent $n \times 1$ vectors $z_1, \ldots, z_d \in \mathbf{R}^n$ $(d < n)$. That is,

$$\begin{aligned} L &= \{\beta_1 z_1 + \cdots + \beta_d z_d \mid \beta_1, \ldots \beta_d \in \mathbf{R}^1\} \\ &\equiv \{Z\beta \mid \beta : d \times 1 \in \mathbf{R}^d\}, \end{aligned} \tag{8.120}$$

where

$$Z = (\, z_1 \quad \cdots \quad z_d \,), \qquad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_d \end{pmatrix}.$$

The matrix $Z : n \times d$, called the *design matrix*, is a matrix of full rank d, so $Z'Z$ is nonsingular. The linear model (8.113) can now be written as

$$X \sim N_n(Z\beta, \sigma^2 I_n) \quad \text{with} \quad \beta \in \mathbf{R}^d, \tag{8.121}$$

and our goal becomes that of estimating β and σ^2.

For this we establish the following relation between P_L and Z:

$$P_L = Z(Z'Z)^{-1}Z'. \tag{8.122}$$

To see this, simply note that $Z(Z'Z)^{-1}Z'$ is symmetric and idempotent, $Z(Z'Z)^{-1}Z'(\mathbf{R}^n) \subseteq Z(\mathbf{R}^d) = L$ by (8.120), and

$$\text{rank}[Z(Z'Z)^{-1}Z'] = \text{tr}[Z(Z'Z)^{-1}Z'] = \text{tr}(I_d) = d,$$

so $Z(Z'Z)^{-1}Z'(\mathbf{R}^n) = L$, which establishes (8.122). Now by (8.114) and (8.122),

$$\begin{aligned} Z\hat{\beta} \equiv \hat{\xi} &= Z(Z'Z)^{-1}Z'X, \\ Z'Z\hat{\beta} &= Z'X, \end{aligned}$$

so

(8.123) $$\hat{\beta} = (Z'Z)^{-1}Z'X.$$

Finally,

$$Q_L X = (I_n - P_L)X = [I_n - Z(Z'Z)^{-1}Z']X,$$

so

(8.124) $$\tilde{\sigma}^2 = \frac{1}{n-d}\|Q_L X\|^2 = \frac{1}{n-d}X'[I_n - Z(Z'Z)^{-1}Z']X.$$

An alternative expression for $\tilde{\sigma}^2$ is

(8.125) $$\tilde{\sigma}^2 = \frac{1}{n-d}\|X - P_L X\|^2 = \frac{1}{n-d}\|X - \hat{\xi}\|^2.$$

It follows from (8.121) and (8.123) that

(8.126) $$\hat{\beta} \sim N_d[\beta,\, \sigma^2(Z'Z)^{-1}],$$

so

(8.127) $$(\hat{\beta} - \beta)'(Z'Z)(\hat{\beta} - \beta) \sim \sigma^2\chi^2_d.$$

Thus by (8.117), (8.119), and (8.127),

(8.128) $$\frac{(\hat{\beta} - \beta)'(Z'Z)(\hat{\beta} - \beta)}{d\,\tilde{\sigma}^2} \sim F_{d,n-d},$$

from which a $(1-\alpha)$-confidence ellipsoid for β easily can be obtained:

(8.129) $$(1-\alpha) = P\left[(\hat{\beta} - \beta)'(Z'Z)(\hat{\beta} - \beta) \le d\,\tilde{\sigma}^2 F_{d,n-d;\,\alpha}\right].$$

Example 8.2. (The one-sample model.) As in §8.4.2, let $X_1, \ldots, X_n$ be a random (i.i.d) sample from the univariate normal distribution $N_1(\mu, \sigma^2)$, so that

$$X \equiv (X_1, \ldots, X_n)' \sim N_n(\mathbf{e}_n\mu,\, \sigma^2 I_n)$$

satisfies (8.121) with $Z = \mathbf{e}_n$, $d = 1$, $\beta = \mu$. Then from (8.123) and (8.124),

$$\hat{\mu} = (\mathbf{e}_n'\mathbf{e}_n)^{-1}\mathbf{e}_n'X = \frac{1}{n}\sum\nolimits_{i=1}^{n} X_i = \bar{X}_n,$$
$$\tilde{\sigma}^2 = \frac{1}{n-1}X'[I_n - \mathbf{e}_n(\mathbf{e}_n'\mathbf{e}_n)^{-1}\mathbf{e}_n']X = \frac{1}{n-1}\sum\nolimits_{i=1}^{n}(X_i - \bar{X})^2 = s_n^2,$$

the sample mean and sample variance as before. The $(1-\alpha)$-confidence ellipsoid (8.129) becomes the usual Student-t confidence interval

$$\hat{\mu} \pm \frac{s_n}{\sqrt{n}} t_{n-1;\,\alpha/2}. \tag{8.130}$$

Example 8.3. (Simple linear regression.) Let $X_1, \ldots, X_n$ be independent rvs that depend linearly on known *regressor variables* $t_1, \ldots, t_n$, that is,

$$X_i = a + bt_i + \epsilon_i, \qquad i = 1, \ldots, n, \tag{8.131}$$

where a and b are unknown parameters and $\epsilon_1, \ldots, \epsilon_n$ are i.i.d unobservable random errors with $\epsilon_i \sim N_1(0, \sigma^2)$ (σ^2 unknown). We can write (8.131) in the vector form (8.121) as follows:

$$\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} = \begin{pmatrix} 1 & t_1 \\ \vdots & \vdots \\ 1 & t_n \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix},$$

or equivalently,

$$X = Z\beta + \epsilon. \tag{8.132}$$

Since $\epsilon \sim N_n(0, \sigma^2 I_n)$, (8.126) is a special case of the univariate normal linear model (8.121). Here the design matrix Z is of full rank $d = 2$ iff $(t_1, \ldots, t_n)$ and $(1, \ldots, 1)$ are linearly independent, i.e., iff at least two t_i's are different. (That is, we can't fit a straight line through $(t_1, X_1), \ldots, (t_n, X_n)$ if all the t_i's are the same.) In this case

$$Z'Z = \begin{pmatrix} n & \sum t_i \\ \sum t_i & \sum t_i^2 \end{pmatrix}, \tag{8.133}$$

so (8.123) and (8.125) become [verify]

$$\hat{\beta} \equiv \begin{pmatrix} \hat{a} \\ \hat{b} \end{pmatrix} = \begin{pmatrix} n & \sum t_i \\ \sum t_i & \sum t_i^2 \end{pmatrix}^{-1} \begin{pmatrix} \sum X_i \\ \sum t_i X_i \end{pmatrix} = \begin{pmatrix} \bar{X}_n - \hat{b}\bar{t}_n \\ \frac{\sum (t_i - \bar{t}_n)(X_i - \bar{X}_n)}{\sum (t_i - \bar{t}_n)^2} \end{pmatrix},$$

$$\tilde{\sigma}^2 = \frac{1}{n-2} \sum [X_i - (\hat{a} + \hat{b}t_i)]^2 = \frac{1}{n-2} \sum [(X_i - \bar{X}_n) - \hat{b}(t_i - \bar{t}_n)]^2.$$

(The expressions for $\hat{a}$ and $\hat{b}$ should be compared to (5.17) and (5.18), where (X, Y) corresponds to (t, X).)

Note that $\hat{b}$ can be expressed as

$$\hat{b} = \frac{\sum (t_i - \bar{t}_n)^2 \left(\frac{X_i - \bar{X}_n}{t_i - \bar{t}_n}\right)}{\sum (t_i - \bar{t}_n)^2}, \tag{8.134}$$

a weighted average of the slopes $\frac{X_i - \bar{X}_n}{t_i - \bar{t}_n}$ determined by the individual data points (t_i, X_i), where the weights are proportional to the squared distances $(t_i - \bar{t}_n)^2$. Furthermore, it follows from (8.126) and(8.133) that [verify]

$$\hat{b} \sim N_1\left(b, \frac{\sigma^2}{\sum (t_i - \bar{t}_n)^2}\right), \tag{8.135}$$

so a $(1 - \alpha)$-confidence interval for the slope b is given by [verify]

$$\hat{b} \pm \frac{\tilde{\sigma}}{\sqrt{\sum (t_i - \bar{t}_n)^2}}\, t_{n-2;\,\alpha/2}. \tag{8.136}$$

Note that this confidence interval can be made narrower by increasing the dispersion of $t_1, \ldots, t_n$. In fact, the most accurate (narrowest) confidence interval for the slope b is obtained by placing half the t_i's at each extreme of their range. (Of course, this precludes the possibility of detecting departures from the assumed linear regression model.)

Exercise 8.11. (Quadratic regression.) Replace the simple linear regression model (8.131) by the quadratic regression model

$$X_i = a + bt_i + ct_i^2 + \epsilon_i, \qquad i = 1, \ldots, n, \tag{8.137}$$

where a, b, c are unknown parameters and the ϵ_i's are as in Example 8.3. Then (8.137) can be written in the vector form (8.121) with

$$Z \equiv \begin{pmatrix} 1 & t_1 & t_1^2 \\ \vdots & \vdots & \vdots \\ 1 & t_n & t_n^2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} a \\ b \\ c \end{pmatrix}. \tag{8.138}$$

Assume that the design matrix Z has full rank $d = 3$. (Note that although the regression is not linear in the (known) t_i's, this qualifies as a linear model because $\mathrm{E}(X) \equiv Z\beta$ is linear in the unknown parameters a, b, c.)

Find the MLEs $\hat{b}$ and $\hat{c}$, find $\tilde{\sigma}^2$, and find (individual) confidence intervals for b and c. Express your answers in terms of the t_i's and the X_i's.

Exercise 8.12 (The two-sample model.) Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be independent random (i.i.d) samples from the univariate normal distributions $N_1(\mu, \sigma^2)$ and $N_1(\nu, \sigma^2)$, respectively, where $m \geq 1$ and $n \geq 1$. Express this as a univariate normal linear model (8.121) – what are Z, d, and β, and what if any additional condition(s) on m and n are needed for the existence of $\tilde{\sigma}^2$? Find the MLEs $\hat{\mu}$ and $\hat{\nu}$, find $\tilde{\sigma}^2$, and find a confidence interval for $\mu - \nu$. Express your answers in terms of the X_i's and Y_j's. (Also see Exercise 18.26.)

9. Order Statistics from a Continuous Univariate Distribution.

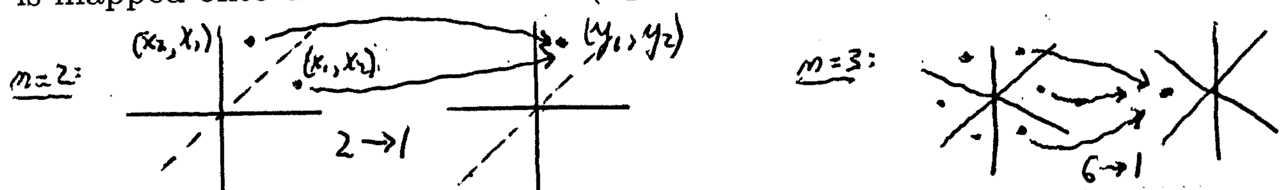
Let $X_1, \ldots, X_n$ be an i.i.d. random sample from a *continuous* distribution with pdf $f_X(x)$ and cdf $F_X(x)$ on $(-\infty, \infty)$. The *order statistics* $Y_1 < Y_2 < \cdots < Y_n$ are the ordered values of $X_1, \ldots, X_n$. Often the notation $X_{(1)} < \cdots < X_{(n)}$ is used instead. Thus, for example,

$$Y_1 = X_{(1)} = \min(X_1, \ldots, X_n),$$
$$Y_n = X_{(n)} = \max(X_1, \ldots, X_n).$$

The mapping

$$(X_1, \ldots, X_n) \rightarrow (Y_1 < Y_2 < \cdots < Y_n)$$

is not 1-1 but rather is $(n!)$ - 1: Each of the $n!$ permutations of $(X_1, \ldots, X_n)$ is mapped onto the same value of $(Y_1 < Y_2 < \cdots < Y_n)$:



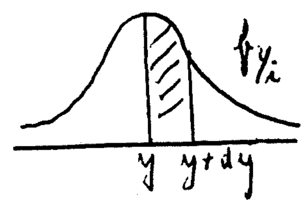
We now present approximate (but valid) derivations of the pdfs of:

(i) a single order statistic Y_i ;

(ii) a pair of order statistics Y_i, Y_j ;

(iii) the entire set of order statistics $Y_1, \ldots, Y_n$.

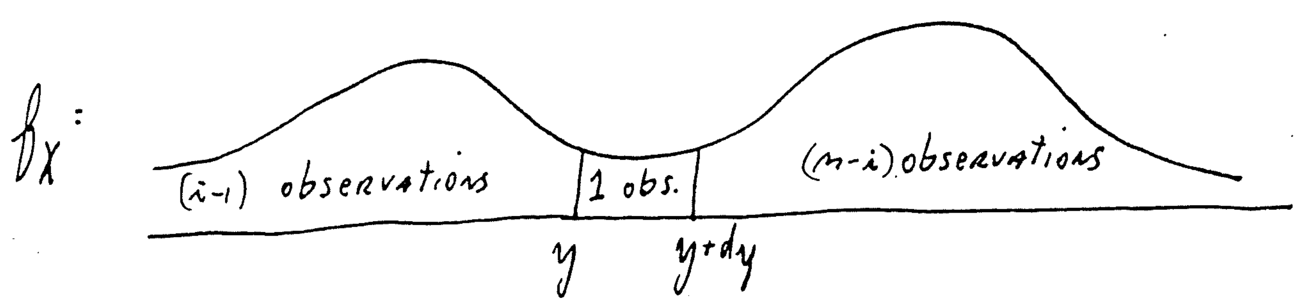
(These derivations are based on the multinomial distribution).



(i) Approximately, $f_{Y_i}(y)dy \approx P[y < Y_i < y + dy]$:

But the event $\{y < Y_i < y + dy\}$ is approximately the same as the event

$$\{(i-1)\ X'\text{s} \in (-\infty, y),\ 1\ X \in (y,\ y + dy),\ (n-i)\ X'\text{s} \in (y + dy,\ \infty).\}$$

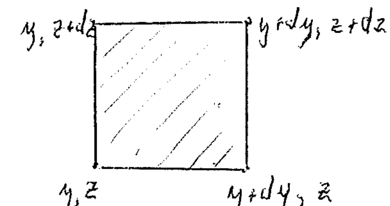


The probability of this event is approximated by the trinomial probability

$$\frac{n!}{(i-1)!\,1!\,(n-i)!}\,[F_X(y)]^{i-1}\,[f_X(y)dy]^1\,[1-F_X(y)]^{n-i}.$$

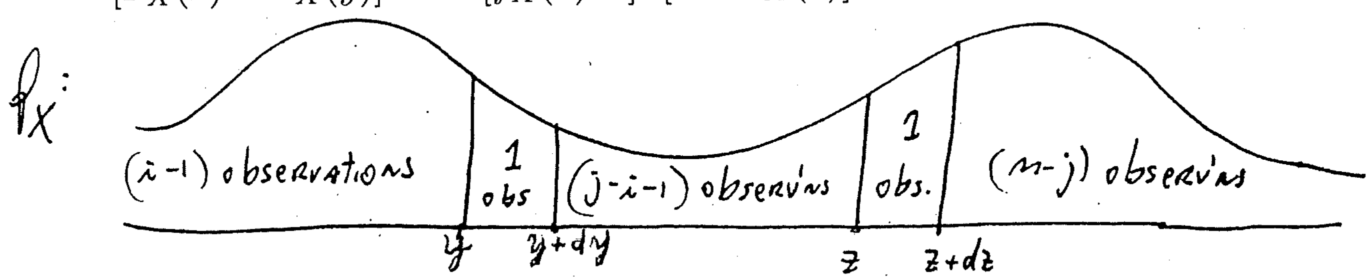
Thus, cancelling the dy's we find that

$$f_{Y_i}(y) = \frac{n!}{(i-1)!(n-i)!}\,[F_X(y)]^{i-1}\,[1-F_X(y)]^{n-i}\,f_X(y). \tag{9.1}$$



(ii) Similarly, for $i<j$ and $y<z$,

$$\begin{aligned}
&f_{Y_i,Y_j}(y,z)dydz\\
\approx& P[y<Y_i<y+dy,\ z<Y_j<z+dz]\\
\approx& P[(i-1)\ X'\text{s} \in (-\infty,y), 1\,X\in(y,y+dy),\ (j-i-1)\ X'\text{s}\in(y+dy,z),\\
&\quad 1\,X\in(z,\,z+dz),\ (n-j)\ X'\text{s}\in(z+dz,\,\infty)] \qquad \text{[see figure]}\\
\approx& \frac{n!}{(i-1)!\,1!\,(j-i-1)!\,1!\,(n-j)!}\,[F_X(y)]^{i-1}\,[f_X(y)dy]^1\\
&\cdot [F_X(z)-F_X(y)]^{j-i-1}\,[f_X(z)dz]^1\,[1-F_X(z)]^{n-j}.
\end{aligned}$$

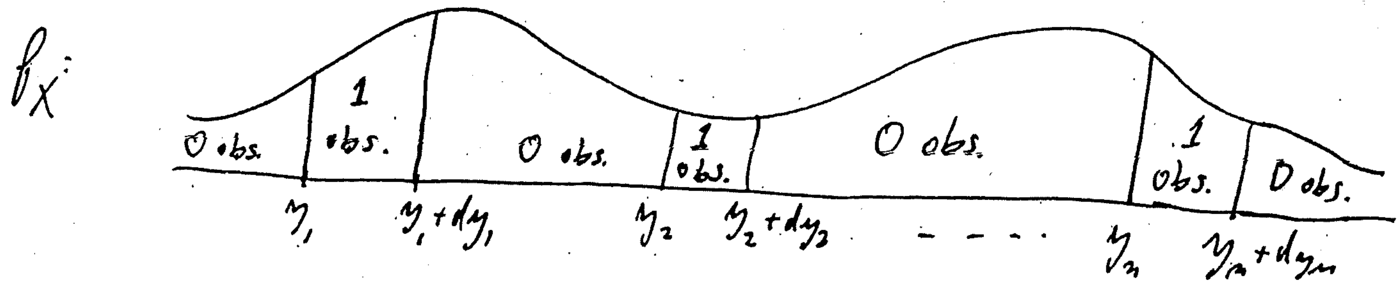


Thus, cancelling $dydz$ we obtain

$$\begin{aligned}
f_{Y_i,Y_j}(y,z) =& \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\,[F_X(y)]^{i-1}\,[F_X(z)-F_X(y)]^{j-i-1}\\
&\cdot [1-F_X(z)]^{n-j}\,f_X(y)f_X(z).
\end{aligned} \tag{9.2}$$

(iii) Finally, for $y_1<\cdots<y_n$,

$$\begin{aligned}
&f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n)dy_1\cdots dy_n\\
\approx& P[y_1<Y_1<y_1+dy_1,\ \ldots,\ y_n<Y_n<y_n+dy_n]\\
\approx& \frac{n!}{1!\cdots 1!}\,[f_X(y_1)dy_1]^1\cdots[f_X(y_n)dy_n]^1.
\end{aligned}$$



Thus, cancelling $dy_1 \cdots dy_n$ we obtain

$$f_{Y_1,\ldots,Y_n}(y_1,\ldots,y_n) = n! f_X(y_1)\cdots f_X(y_n). \tag{9.3}$$

The factor $n!$ occurs because $(X_1,\ldots,X_n) \to (Y_1 < Y_2 < \cdots < Y_n)$ is an $(n!)$ - 1 mapping.

Exercise 9.1. Extend (9.3) to the case of *exchangeable* rvs: Let $X \equiv (X_1,\ldots,X_n)$ have pdf $f_X(x_1,\ldots,x_n)$ with f_X *symmetric* $\equiv$ *permutation-invariant*, that is,

$$f_X(x_1,\ldots,x_n) = f_X(x_{\pi(1)},\ldots,x_{\pi(n)}) \qquad \text{for all permutations } \pi. \tag{9.4}$$

Show that the order statistics $Y \equiv (Y_1 < \cdots < Y_n)$ have joint pdf given by

$$f_Y(y_1,\ldots,y_n) = n! f_X(y_1,\ldots,y_n). \tag{9.5}$$

Example 9.1. For the rest of this section assume that $X_1,\ldots,X_n$ are i.i.d Uniform$(0,1)$ rvs with order statistics $Y_1 < \cdots < Y_n$. Here

$$f_X(x) = 1 \quad \text{and} \quad F_X(x) = x \qquad \text{for } 0 < x < 1, \tag{9.6}$$

so (9.1) becomes

$$f_{Y_i}(y) = \frac{n!}{(i-1)!(n-i)!} y^{i-1}(1-y)^{n-i}, \qquad 0 < y < 1. \tag{9.7}$$

Thus we see that the i-th order statistic has a Beta distribution:

$$Y_i \equiv X_{(i)} \sim \text{Beta}\,(i,\, n-i+1), \tag{9.8}$$

$$\text{E}(Y_i) = \text{E}(X_{(i)}) = \frac{i}{n+1}, \tag{9.9}$$

$$\text{Var}(Y_i) = \text{Var}(X_{(i)}) = \frac{i(n-i+1)}{(n+1)^2(n+2)}. \tag{9.10}$$

Note that if n is odd, $\text{Var}(X_{(i)})$ is maximized when $i = \frac{n+1}{2}$, i.e., when $X_{(i)}$ is the sample median. In this case,

$$\text{Var}(X_{(\frac{n+1}{2})}) = \frac{1}{4(n+2)} = O\left(\tfrac{1}{n}\right). \tag{9.11}$$

On the other hand, the variance is minimized by the extreme order statistics:

$$\text{Var}(X_{(1)}) = \text{Var}(X_{(n)}) = \frac{n}{(n+1)^2(n+2)} = O\left(\tfrac{1}{n^2}\right), \tag{9.12}$$

a smaller order of magnitude. The asymptotic distributions of the sample median and sample extremes are also different: the median is asymptotically normal (§10.6), the extremes are asymptotically exponential (Exercise 9.4.).

Relation between the Beta and Binomial distributions: It follows from (9.7) that for any $0 < y < 1$,

$$\begin{aligned}\frac{n!}{(i-1)!(n-i)!}\int_y^1 u^{i-1}(1-u)^{n-i}du &= P[Y_i > y] \\ &= P[(i-1) \text{ or fewer } X'\text{s} \le y] = \sum_{k=0}^{i-1}\binom{n}{k}y^k(1-y)^{n-k}.\end{aligned}$$

Now let $j = i-1$ and $t = 1-u$ to obtain the following relation between the Beta and Binomial cdfs:

$$\frac{n!}{(n-j-1)!j!}\int_0^{1-y} t^{n-j-1}(1-t)^j\,dt = \sum_{k=0}^{j}\binom{n}{k}y^k(1-y)^{n-k}. \tag{9.13}$$

Joint distribution of the sample spacings from Uniform$(0,1)$ rvs. Define the *sample spacings* $W_1, \ldots, W_n, W_{n+1}$ by

$$\begin{aligned} W_1 &= Y_1 \\ W_2 &= Y_2 - Y_1 \\ &\;\vdots \\ W_n &= Y_n - Y_{n-1} \\ W_{n+1} &= 1 - Y_n. \end{aligned} \tag{9.14}$$

Note that $0 < W_i < 1$ and $W_1 + \cdots + W_n + W_{n+1} = 1$. Furthermore,

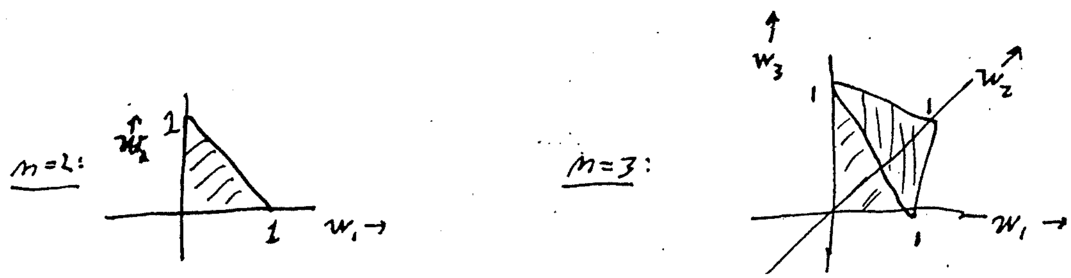
$$
(9.15) \qquad
\begin{aligned}
Y_1 &= W_1 \\
Y_2 &= W_1 + W_2 \\
&\vdots \\
Y_n &= W_1 + \cdots + W_n,
\end{aligned}
$$

From (9.3) and (9.6) the joint pdf of $Y_1, \ldots, Y_n$ is

$$
(9.16) \qquad f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_n) = n!\,, \quad 0 < y_1 < \cdots < y_n < 1,
$$

and the Jacobian of the mapping (9.15) is $\left|\frac{\partial Y}{\partial W}\right| = 1$ [verify], so[10]

$$
(9.17) \quad f_{W_1,\ldots,W_n}(w_1, \ldots, w_n) = n!\,, \quad 0 < w_i < 1,\ 0 < w_1 + \cdots + w_n < 1.
$$



Clearly both $f_{W_1,\ldots,W_n}(w_1, \ldots, w_n)$ and its range are invariant under all permutations of $w_1, \ldots, w_n$, so $W_1, \ldots, W_n$ are exchangeable:

$$
(9.18) \qquad (W_1, \ldots, W_n) \stackrel{\text{distn}}{=} (W_{\pi(1)}, \ldots, W_{\pi(n)})
$$

for all permutations $(\pi(1), \ldots, \pi(n))$ of $(1, \ldots, n)$. Thus, from (9.8)-(9.10),

$$
\begin{aligned}
&(9.19) \quad W_i \stackrel{\text{distn}}{=} W_1 = Y_1 \sim \text{Beta}(1,\, n), && 1 \le i \le n, \\
&(9.20) \quad \text{E}(W_i) = \frac{1}{n+1}, && 1 \le i \le n, \\
&(9.21) \quad \text{Var}(W_i) = \frac{n}{(n+1)^2(n+2)}, && 1 \le i \le n.
\end{aligned}
$$

---

[10] Note that (9.17) is a special case of the (incomplete) *Dirichlet distribution* – cf. CB Exercise 4.40 with $a = b = c = 1$.

Also, for $1 \le i < j \le n$, $(W_i, W_j) \overset{\text{distn}}{=} (W_1, W_2)$, so

$$\begin{aligned}\operatorname{Cov}(W_i, W_j) &= \operatorname{Cov}(W_1, W_2)\\ &= \frac{1}{2}[\operatorname{Var}(W_1 + W_2) - \operatorname{Var}(W_1) - \operatorname{Var}(W_2)]\\ &= \frac{1}{2}[\operatorname{Var}(Y_2) - 2\operatorname{Var}(Y_1)]\\ &= \frac{1}{2}\left[\frac{2(n-1)}{(n+1)^2(n+2)} - \frac{2n}{(n+1)^2(n+2)}\right]\\ &= -\frac{1}{(n+1)^2(n+2)} < 0. \qquad (9.22)\end{aligned}$$

Exercise 9.2*. Extend the above results to $(W_1, \ldots, W_n, W_{n+1})$, that is, show that $W_1, \ldots, W_n, W_{n+1}$ are exchangeable. Thus (9.19) - (9.22) remain valid for $i = n+1$.

Note: since $W_1 + \cdots + W_n + W_{n+1} = 1$, $W_1, \ldots, W_n, W_{n+1}$ do *not* have a joint pdf on $\mathbf{R}^{n+1}$.

Exercise 9.3. Find $\operatorname{Cov}(Y_i, Y_j) \equiv \operatorname{Cov}(X_{(i)}, X_{(j)})$ for $1 \le i < j \le n$.

Exercise 9.4. Show that $nW_i \xrightarrow{d} \text{Exponential}(\lambda = 1)$ as $n \to \infty$.

Hint: One method is to show that $P[nW_i > w] \to e^{-w}$ for $0 < w < \infty$. Another is to use (9.19), the representation of a Beta rv in Example 6.4, and Slutsky's Theorem 10.6 (as applied in Example 10.3).

10. Asymptotic (Large Sample) Distribution Theory.

10.1. (Nonparametric) Estimation of a cdf and its functionals.

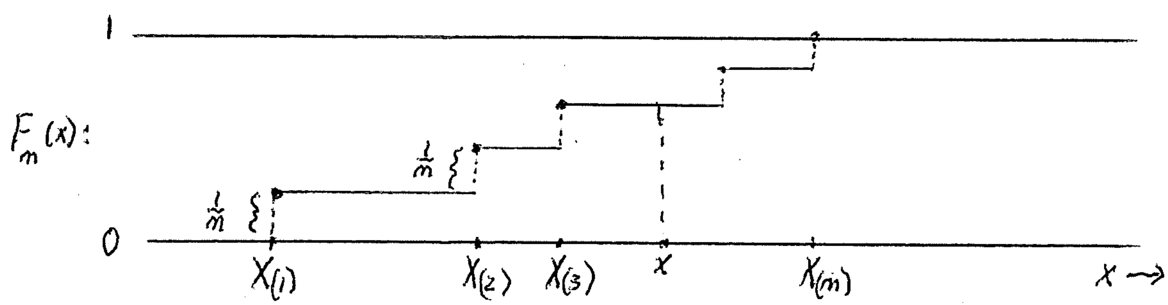
Let $X_1, \dots, X_n$ be a random (i.i.d.) sample from an unknown probability distribution P with cdf F on $(-\infty, \infty)$. This distribution may be discrete, continuous, or a mixture. As it stands this is a *nonparametric* statistical model[11] because no parametric form is assumed for F. Our goal is to estimate F based on the data $X_1, \dots, X_n$.

Definition 10.1. The *empirical (≡ sample) distribution* P_n is the (discrete!) probability distribution that assigns probability $\frac{1}{n}$ to each observation X_i. Equivalently, P_n assigns probability $\frac{1}{n}$ to each order statistic $X_{(i)}$, so P_n depends on the data only through the order statistics. The *empirical cdf* F_n is the cdf associated with P_n, i.e.,

$$F_n(x) = \frac{1}{n}\#\{X_i \le x\}$$

(10.1) $$= \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}(X_i):$$

(10.2) $$= \frac{1}{n}\sum_{i=1}^{n} I_{(-\infty,x]}(X_{(i)}):$$



(Note that F_n is a *random* function.) Eqn. (10.2) shows that F_n depends only on the order statistics, while (10.1) shows that for each fixed x,

(10.3) $$nF_n(x) \equiv \sum_{i=1}^{n} I_{(-\infty,x]}(X_i) \sim \text{Binomial}(n,\, p \equiv F(x)).$$

[11] Some parametric models: $\{N(\mu, \sigma^2)\}$, $\{\text{Exponential}(\lambda)\}$, $\{\text{Poisson}(\lambda)\}$, etc.

Thus by the LLN and the CLT, for each fixed x, $\hat{p}_n \equiv F_n(x)$ is a *consistent, asymptotically normal estimator of* $p \equiv F(x)$: as $n \to \infty$,

$$F_n(x) \to F(x), \tag{10.4}$$

$$\sqrt{n}[F_n(x) - F(x)] \xrightarrow{d} N_1[0,\, F(x)(1-F(x))]. \tag{10.5}$$

In fact, (10.4) and (10.5) can be greatly strengthened by considering the asymptotic behavior of the *entire random function* $F_n(\cdot)$ on $(-\infty, \infty)$:

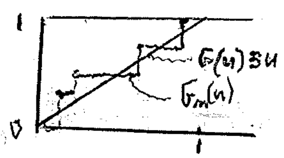
The Glivenko-Cantelli Theorem: If F is continuous, then $F_n(x) \to F(x)$ *uniformly in* x:

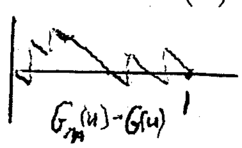
$$\sup_{-\infty<x<\infty} |F_n(x) - F(x)| \to 0 \qquad \text{as } n \to \infty. \tag{10.6}$$

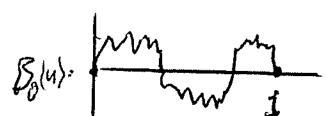
The Brownian Bridge: First consider the case where $U_1, \dots, U_n$ are i.i.d. Uniform(0,1) rvs with cdf $G(u) = u$ for $0 \le u \le 1$ and empirical cdf G_n. Here $G_n(u) - G(u)$ is a random function of $u \in [0,1]$ that is "tied down" to 0 at each endpoint. Then

$$\sqrt{n}\,[G_n(u) - G(u)] \xrightarrow{d} B_0 \quad \text{as random functions on } [0,1] \text{ as } n \to \infty, \tag{10.7}$$

where B_0 denotes the *Brownian Bridge* stochastic process. This is a random function on $[0,1]$ that can be thought of as the conditional distribution of a standard Brownian motion $B(u)$ starting at 0, given that $B(1) = 0$.







Now let $X_1, \dots, X_n$ be i.i.d. $\sim$ any (continuous) F on $(-\infty, \infty)$ with empirical cdf F_n. Then $F(X_i) \stackrel{d}{=} U_i$, so for $0 \le u \le 1$,

$$nF_n(F^{-1}(u)) = \#\{X_i \le F^{-1}(u)\} = \#\{F(X_i) \le u\} \stackrel{d}{=} \#\{U_i \le u\} = nG_n(u).$$

Because $u = G(u)$, it follows from (10.7) that

$$\sqrt{n}\,[F_n(F^{-1}(u)) - F(F^{-1}(u))] \stackrel{d}{=} \sqrt{n}\,[G_n(u) - G(u)] \xrightarrow{d} B_0 \text{ on } [0,1]. \tag{10.8}$$

This gives the asymptotic distribution of the *Kolmogorov-Smirnov goodness-of-fit statistic* for testing a specified null hypothesis F: with $x = F^{-1}(u)$,

$$\sup_{-\infty<x<\infty} \sqrt{n}\,|F_n(x) - F(x)| \xrightarrow{d} \sup_{0\le u\le 1} |B_0(u)| \qquad \text{as } n \to \infty. \tag{10.9}$$

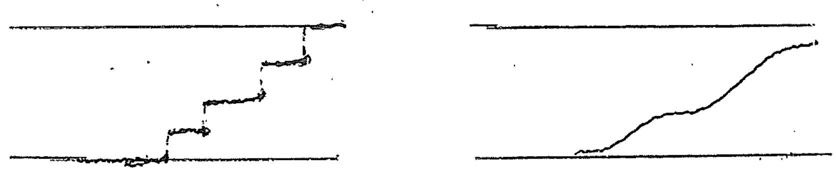
Theoretical (=population) quantity $\phi(P)$ or $\phi(F)$		Empirical (= sample) estimator $\phi(P_n)$ or $\phi(F_n)$

probability distribution:

$$P \quad (\text{range} = (-\infty, \infty)) \qquad \leftrightarrow \qquad P_n \quad (\text{range} = \{X_{(1)}, \ldots, X_{(n)}\})$$

cumulative distribution function (cdf):

$$F \qquad \leftrightarrow \qquad F_n$$



probability of a set A:

$$P(A) = \int_A dP \qquad \leftrightarrow \qquad P_n(A) = \frac{\sum I_A(X_{(i)})}{n} \equiv \int_A dP_n$$

probability of $A \equiv (-\infty, x]$:

$$F(x) = \int_{-\infty}^{x} dF \qquad \leftrightarrow \qquad F_n(x) = \frac{\sum I_{(-\infty,x]}(X_{(i)})}{n} \equiv \int_{-\infty}^{x} dF_n$$

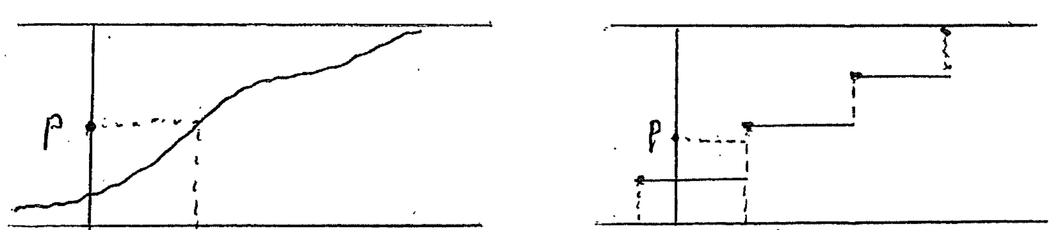
mean:

$$\mathrm{E}_F(X) = \int x dF \qquad \leftrightarrow \qquad \bar{X}_n = \frac{\sum_{i=1}^{n} X_{(i)}}{n} \equiv \int x dF_n$$

variance:

$$\mathrm{Var}_F(X) = \int (x - \mathrm{E}_F(X))^2 dF \quad \leftrightarrow \tilde{s}_n^2 = \frac{\sum_{i=1}^{n} (X_{(i)} - \bar{X}_n)^2}{n} \equiv \int (x - \bar{X}_n)^2 dF_n$$

p-th quantile ($0 < p < 1$):



$F^{-1}(p)$

$F_n^{-1}(p) = X_{([np]+1)}$ if np is not an integer.

Basic method of nonparametric estimation: To estimate a "functional" $\phi(F)$ of F, use $\phi(F_n)$.[12,13] (Equivalently, to estimate a "functional" $\phi(P)$ of P, use $\phi(P_n)$.) Simple examples are given on the preceding page. Since F_n is a *consistent and asymptotically normal (CAN)* estimator of F (both for fixed x and as a random function), we would like to conclude that $\phi(F_n)$ is a CAN estimator of $\phi(F)$. For this we need to show:

- The functional ϕ of interest is continuous, in fact differentiable, in $F_n(!)$;
- Consistency and asymptotic normality are preserved by such a ϕ.

These results require the general theory of "weak convergence" ($\equiv$ convergence in distribution) of stochastic processes. We will not prove them in general, but only for the above examples. We will need the following:

(i) Definition and properties: convergence in probability ($X_n \xrightarrow{p} X$) in $\mathbf{R}^k$,
convergence in distribution ($X_n \xrightarrow{d} X$) in $\mathbf{R}^k$,

(ii) Perturbation $\equiv$ Slutsky-type results ("c" represents a constant in $\mathbf{R}^k$):

(10.10) $X_n \xrightarrow{d} X, \quad Y_n \xrightarrow{p} 0 \quad \Rightarrow \quad X_n + Y_n \xrightarrow{d} X$;

(10.11) $X_n \xrightarrow{d} X, \quad Y_n \xrightarrow{p} c \quad \Rightarrow \quad h(X_n, Y_n) \xrightarrow{d} h(X, c)$ if h is continuous.

(iii) Central Limit Theorem (CLT) (for $\mathbf{R}^1$, see CB Theorems 5.5.14-15): Let $X_1, X_2, \ldots$ i.i.d. rvtrs in $\mathbf{R}^k$, $\mathrm{E}(X_i) = \mu$, $\mathrm{Cov}(X_i) = \Sigma$. Then

(10.12) $$\sqrt{n}\,(\bar{X}_n - \mu) \xrightarrow{d} N_k(0, \Sigma).$$

(iv) Propagation of error $\equiv$ δ-method $\equiv$ Taylor approximation: Let $\{Y_n\}$ be a sequence of rvtrs in $\mathbf{R}^k$ such that

(10.13) $$\sqrt{n}(Y_n - \mu) \xrightarrow{d} N_k(0, \Sigma).$$

---

[12] Sometimes we prefer to adjust $\phi(F_n)$ slightly to obtain an unbiased estimator. For example, the unbiased sample variance is $s_n^2 = \frac{n}{n-1}\tilde{s}_n^2$. This remains a CAN estimator.

[13] If the statistical model is parametric, a nonparametric estimator need not be efficient. In the Poisson(λ) model, for example, the optimal estimator of $E(X) \equiv \lambda$ is $\bar{X}_n$ (see Example 12.13), so the nonparametric estimator s_n^2 of $Var(X) \equiv \lambda$ is inefficient.

If $g(y_1, \ldots, y_k)$ is differentiable at $\mu \equiv (\mu_1, \ldots, \mu_k)$, then

$$\sqrt{n}\,[g(Y_n) - g(\mu)] \xrightarrow{d} N_1\big(0,\, (\nabla g)_\mu\, \Sigma\, (\nabla g)^t_\mu\big), \tag{10.14}$$

where $(\nabla g)_\mu = \left(\frac{\partial g}{\partial y_1}, \ldots, \frac{\partial g}{\partial y_k}\right)_\mu$ is the gradient of g with the partial derivatives evaluated at $y = \mu$, provided that at least one $\frac{\partial g}{\partial y_i} \neq 0$. In the univariate case ($k = 1$) this can be written as

$$\sqrt{n}\,[g(Y_n) - g(\mu)] \xrightarrow{d} N_1\left(0,\, [g'(\mu)]^2\sigma^2\right) \quad (\text{if } g'(\mu) \neq 0). \tag{10.15}$$

10.2. Modes of convergence.

Let X, $X_1, X_2, \ldots$ be a sequence of rvs or rvtrs with range $\mathcal{X}$, all occurring in the *same* random experiment, so we can meaningfully write $X_n - X$, $X_1 + \cdots + X_n$, etc. (These rvs/rvtrs are *not* necessarily independent.) We treat the case that $\mathcal{X} = \mathbf{R}^k$ (although most of these concepts and results are valid when $\mathcal{X}$ is any complete separable metric space).

Definition 10.2. *X_n converges to X in probability* ($X_n \xrightarrow{p} X$) if $\forall\, \epsilon > 0$,

$$P[\, \|X_n - X\| \geq \epsilon\,] \to 0 \qquad \text{as } n \to \infty. \tag{10.16}$$

X_n converges to X almost surely ($X_n \xrightarrow{a.s.} X$) if

$$P[\lim_{n\to\infty} X_n = X\,] = 1, \tag{10.17}$$

or equivalently, if $\forall\, \epsilon > 0$,

$$P[\, \|X_{n+k} - X\| \geq \epsilon \text{ for at least one } k \geq 0\,] \to 0 \qquad \text{as } n \to \infty. \tag{10.18}$$

Note that convergence in probability is weaker than a.s. convergence: the former only involves the joint distributions of all pairs (X_n, X), while the latter involves the joint distribution of the entire infinite sequence $X, X_1, X_2, \ldots$. Furthermore, if $X \equiv c$, then $X_n \xrightarrow{p} c$ only involves the marginal distribution of each X_n. (We will see that $X_n \xrightarrow{p} c \iff X_n \xrightarrow{d} c$.)

Example 10.1. The *Weak (Strong) Law of Large Numbers* states that if $X_1, X_2, \ldots$ is a sequence of *i.i.d.* rvs in $\mathbf{R}^1$ with finite mean $\mathrm{E}(X_i) \equiv \mu$, then

$$\bar{X}_n \xrightarrow{p} \mu \quad (\bar{X}_n \xrightarrow{a.s.} \mu). \tag{10.19}$$

Under the added assumption of finite second moments, the WLLN is proved easily by means of Chebyshev's inequality (cf. §3.1). The proof of the SLLN is nontrivial.

Next, if we also assume that $\mathrm{Var}(X_i) \equiv \sigma^2$ is finite, then the sample variance s_n^2 converges to σ^2:

$$s_n^2 = \frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}_n^2\right) \xrightarrow{p} (\sigma^2 + \mu^2) - \mu^2 = \sigma^2 \tag{10.20}$$

by (10.19), Thm 10.3b, and the WLLN applied to $X_1^2, X_2^2, \ldots$; also $s_n^2 \xrightarrow{a.s.} \sigma^2$ by the SLLN. (These results also extend to a sequence of rvtrs in $\mathbf{R}^k$.) □

Next let $P, P_1, P_2, \ldots$ be a sequence of probability measures on a common sample space Ω. We say that $A \subseteq \Omega$ is a *P-continuity set* if $P(\partial A) = 0$, where ∂A is the boundary of A:

Definition 10.3. *P_n converges weakly to P* ($P_n \xrightarrow{w} P$) if

$$P_n(A) \to P(A) \qquad \text{as } n \to \infty \ \forall\ P\text{-continuity sets } A. \tag{10.21}$$

Now let X, $X_1, X_2, \ldots$ be a sequence of rvs or rvtrs, *not* necessarily all occurring in the same random experiment, but still with common range $\mathcal{X}$. Let P_X (resp. P_{X_n}) denote the probability distribution of X (resp. X_n).

Definition 10.4. *X_n converges to X in distribution* ($X_n \xrightarrow{d} X$) if $P_{X_n} \xrightarrow{w} P_X$, that is, if

$$P[X_n \in A] \to P[X \in A] \quad \text{as } n \to \infty \ \forall\ P_X\text{-continuity sets } A. \tag{10.22}$$

The need for restriction to P-continuity sets is easily seen: Suppose that X_n is the rv degenerate at $\frac{1}{n}$. Then we want that $X_n \xrightarrow{d} X \equiv 0$, but

$$P[X_n \in (-\infty, 0]] = 0 \not\to 1 = P[X \in (-\infty, 0]].$$

Here the set $A \equiv (-\infty, 0]$ is not a P_X-continuity set, since $\partial A = \{0\}$ so $P(\partial A) = 1 \neq 0$. Thus it does hold that $X_n \xrightarrow{d} X$.

Note that convergence in distribution is a property of the sequence of distributions $P_X, P_{X_1}, P_{X_2}, \ldots$, not of the values of $X, X_1, X_2, \ldots$ themselves. Thus, for example, if $X_n \xrightarrow{d} X$ we cannot conclude anything about the limiting behavior of $X_n - X$ (unless $X \equiv c$).

Theorem 10.1. *(Basic characterization of convergence in distribution.) the following are equivalent:*

(a) $X_n \xrightarrow{d} X$.

(b) $F_{X_n}(x) \to F_X(x)$ *for every continuity point* x *of* F_X.

(c) $\mathrm{E}[g(X_n)] \to \mathrm{E}[g(X)]$ *for every bounded continuous function* g.

Note re (b): F_X *is continuous at* x *iff* $(-\infty,\, x]$ *is a* P_X*-continuity set.*

This theorem, whose proof is omitted, is "basic" because condition (c) is much easier to work with than (a) or (b). Here is an example:

Corollary 10.1. *If* $X_n \xrightarrow{d} X$ *and* h *is continuous, then* $h(X_n) \xrightarrow{d} h(X)$.

Proof. By (c) it suffices to show that $\mathrm{E}[g(h(X_n))] \to \mathrm{E}[g(h(X))]$ for any bounded continuous g. But this holds since $X_n \xrightarrow{d} X$ and $g(h(\cdot))$ is bounded and continuous.

Example 10.2. If $X_n \xrightarrow{d} X \sim N_1(0,1)$, then $X_n^2 \xrightarrow{d} X^2 \sim \chi_1^2$.

Remark 10.1. The conclusion of Corollary 10.1 remains valid if h is not continuous but $P[X \in D_h] = 0$, where D_h is the set of discontinuity points of h [proof omitted]. For example, if $h(x) = \frac{1}{x}$ on $(-\infty, \infty)$, then $D_h = \{0\}$. Thus

$$(10.23) \qquad X_n \xrightarrow{d} X \quad \text{and} \quad P[X = 0] = 0 \quad \Longrightarrow \quad \frac{1}{X_n} \xrightarrow{d} \frac{1}{X}.$$

Remark 10.2. Condition (c) need not hold if $X_n \xrightarrow{d} X$ and g is continuous but unbounded. For example, suppose that $g(x) = x$ on $(-\infty, \infty)$ and take

$$X_n = \begin{cases} n, & \text{with probability } \frac{1}{n}; \\ 0, & \text{with probability } 1 - \frac{1}{n}. \end{cases}$$

Then $X_n \xrightarrow{d} X \equiv 0$ [verify] but $1 = \mathrm{E}(X_n) \not\to \mathrm{E}(X) = 0$. More generally, $X_n \xrightarrow{d} X$ need not imply that the moments of X_n converge to those of X.

Theorem 10.2. *(a)* $X_n \xrightarrow{a.s.} X \;\Rightarrow\; X_n \xrightarrow{p} X \;\Rightarrow\; X_n \xrightarrow{d} X$.

(b) If $X = c$ *then* $X_n \xrightarrow{p} c \iff X_n \xrightarrow{d} c$.

Proof. (a) The first $\Rightarrow$ is immediate by comparing (10.18) to (10.16).

We prove the second $\Rightarrow$ for the case of 1-dimensional rvs, i.e., $\mathcal{X} = \mathbf{R}^1$, by verifying condition (b) of Theorem 10.1. Suppose that $X_n \xrightarrow{p} X$ and let x be a continuity point of F_X. We must show that $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$. For any $x' < x < x''$,

$$\begin{aligned} F_X(x') &= P[X \le x',\, X_n \le x] + P[X \le x',\, X_n > x] \\ &\le P[X_n \le x] + P[X \le x',\, X_n > x]. \end{aligned}$$

But the last probability $\to 0$ since $X_n \xrightarrow{p} X$ and $x' < x$, hence

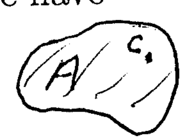
$$F_X(x') \le \liminf_{n\to\infty} P[X_n \le x] \equiv \liminf_{n\to\infty} F_{X_n}(x).$$

Similarly $F_X(x'') \ge \limsup_{n\to\infty} F_{X_n}(x)$, so

$$F_X(x') \le \liminf_{n\to\infty} F_{X_n}(x) \le \limsup_{n\to\infty} F_{X_n}(x) \le F_X(x'').$$

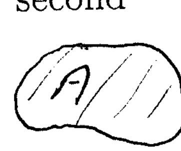
Now let $x' \uparrow x$ and $x'' \downarrow x$. Since F_X is continuous at x, we conclude that $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$, as required.

(b) (This proof is valid for $\mathcal{X} = \mathbf{R}^k$.) First suppose that $X_n \xrightarrow{p} X \equiv c$. Note that $A \subset \mathbf{R}^k$ is a P_X-continuity set iff $c \notin \partial A$, i.e., iff either $c \in \text{interior}(A)$ or $c \in \text{interior}(\mathbf{R}^k \setminus A)$. In the first case, for all sufficiently small ϵ we have

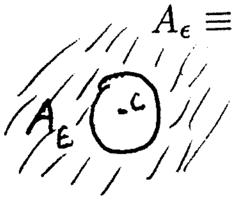


$$1 \ge P[X_n \in A] \ge P[\,\|X_n - X\| \le \epsilon\,] \to 1 = P[X \in A],$$

hence $P[X_n \in A] \to P[X \in A]$. Similarly this limit holds in the second case, hence $X_n \xrightarrow{d} X$.



Now suppose $X_n \xrightarrow{d} X \equiv c$. To show that $X_n \xrightarrow{p} c$, consider the set $A_\epsilon \equiv \{x \mid \|x - c\| > \epsilon\}$. Clearly A_ϵ is a P_X-continuity set $\forall\ \epsilon > 0$, hence



$$P[\,\|X_n - c\| > \epsilon\,] \to P[\,\|X - c\| > \epsilon\,] = 0. \qquad \square$$

Remark 10.3. In Theorem 10.2(a), neither $\Leftarrow$ holds in general. (See CB Example 5.5.8 (as modified by me) for the first counterexample.)

Theorem 10.3. *Let h be a continuous function on $\mathcal{X}$.*

(a) If $X_n \xrightarrow{a.s.} X$ then $h(X_n) \xrightarrow{a.s.} h(X)$.

(b) If $X_n \xrightarrow{p} X$ then $h(X_n) \xrightarrow{p} h(X)$.

Proof. (a) follows easily from the definition (10.17) of a.s. convergence.

(b) Fix $\epsilon > 0$ and select a sufficiently large $\rho(\epsilon)$ such that the ball

$$B_{\rho(\epsilon)} \equiv \{x \in \mathbf{R}^k \mid \|x\| \le \rho(\epsilon)\,\} \subset \mathbf{R}^k$$

satisfies $P[\,X \in B_{\rho(\epsilon)}\,] \ge 1 - \epsilon$. Increase $\rho(\epsilon)$ slightly if necessary to insure that $B_{\rho(\epsilon)}$ is a P_X-continuity set [why possible?]. Since $X_n \xrightarrow{d} X$ by Theorem 10.2a, $\exists\ n(\epsilon)$ s.t.

$$\begin{aligned} n \ge n(\epsilon) \Rightarrow\ & P[\,X_n \in B_{\rho(\epsilon)}] \ge 1 - 2\epsilon \\ \Rightarrow\ & P[\,X_n \in B_{\rho(\epsilon)} \text{ and } X \in B_{\rho(\epsilon)}\,] \\ & \ge P[\,X_n \in B_{\rho(\epsilon)}\,] + P[\,X \in B_{\rho(\epsilon)}\,] - 1 \\ & \ge 1 - 3\epsilon. \end{aligned}$$

Furthermore, h is uniformly continuous on $B_{\rho(\epsilon)}$, that is, $\exists\ \delta(\epsilon) > 0$ s.t.

$$x, y \in B_{\rho(\epsilon)},\ \|x - y\| < \delta(\epsilon) \Rightarrow |h(x) - h(y)| < \epsilon.$$

Also, since $X_n \xrightarrow{p} X$, $\exists\ n'(\epsilon)$ s.t.

$$n \ge n'(\epsilon) \ \Rightarrow\ P[\,\|X_n - X\| \ge \delta(\epsilon)\,] \le \epsilon.$$

Thus $h(X_n) \xrightarrow{p} h(X)$, since for $n \geq \max\{n(\epsilon), n'(\epsilon)\}$,

$$\begin{aligned} P[|h(X_n) - h(X)| \geq \epsilon] =& P[\,|h(X_n) - h(X)| \geq \epsilon,\, X_n \in B_{\rho(\epsilon)},\, X \in B_{\rho(\epsilon)}\,] \\ &+ P[\,|h(X_n) - h(X)| \geq \epsilon,\, \{X_n \in B_{\rho(\epsilon)},\, X \in B_{\rho(\epsilon)}\}^c\,] \\ &\leq P[\,\|X_n - X\| \geq \delta(\epsilon),\, X_n \in B_{\rho(\epsilon)},\, X \in B_{\rho(\epsilon)}\,] \\ &\quad + 1 - P[\,X_n \in B_{\rho(\epsilon)} \text{ and } X \in B_{\rho(\epsilon)}\,] \\ &\leq 4\epsilon. \end{aligned}$$

Exercise 10.1. It can be shown that if $X_n \xrightarrow{p} X$, there is a subsequence $\{n'\} \subseteq \{n\}$ s.t. $X_{n'} \xrightarrow{a.s.} X$. Use this to give another proof of (b).

Theorem 10.4 (Slutsky). *If $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} 0$, then $X_n + Y_n \xrightarrow{d} X$.*

Exercise 10.2. Prove Theorem 10.4 for rvs in $\mathbf{R}^1$.

Hint: Similar to the proof of the second $\Rightarrow$ in Theorem 10.2a.

Theorem 10.5. *Suppose that $X_n \xrightarrow{d} X$ in $\mathbf{R}^k$ and $Y_n \xrightarrow{p} c$ in $\mathbf{R}^l$. Then $(X_n, Y_n) \xrightarrow{d} (X, c)$ in $\mathbf{R}^{k+l}$.*

Proof. Write $(X_n, Y_n) = (X_n, c) + (0, Y_n - c)$. By Theorem 10.1c, $(X_n, c) \xrightarrow{d} (X, c)$, and clearly $(0, Y_n - c) \xrightarrow{d} (0, 0)$, so the result follows from Theorem 10.4.

Theorem 10.6 (Slutsky). *Suppose that $X_n \xrightarrow{d} X$ in $\mathbf{R}^k$ and $Y_n \xrightarrow{p} c$ in $\mathbf{R}^l$. If $h(x, y)$ is continuous then $h(X_n, Y_n) \xrightarrow{d} h(X, c)$.*

Proof. Apply Theorem 10.5 and Corollary 10.1.

Remark 10.4. If h is not continuous but $P[\,(X, c) \in D_h\,] = 0$, then $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{p} c$ still imply $h(X_n, Y_n) \xrightarrow{d} h(X, c)$ (use Remark 10.1).

Example 10.3. If $X_n \xrightarrow{d} N_1(0, 1)$ and $Y_n \to c \neq 0$ then by Remark 10.4,

$$h(X_n, Y_n) \equiv \frac{X_n}{Y_n} \xrightarrow{d} \frac{N_1(0, 1)}{c} = N_1\left(0, \frac{1}{c^2}\right).$$

In particular, if $X_1, \ldots, X_n$ are i.i.d. rvs with $\mathrm{E}(X_i) = \mu$ and $\mathrm{Var}(X_i) = \sigma^2$,

$$\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \xrightarrow{d} N_1(0,\, 1) \quad \text{and} \quad \frac{s_n}{\sigma} \xrightarrow{p} 1$$

by the CLT and (10.20), so

$$(10.24) \qquad t_n \equiv \frac{\sqrt{n}(\bar{X}_n - \mu)}{s_n} = \frac{\sqrt{n}(\bar{X}_n - \mu)/\sigma}{s_n/\sigma} \xrightarrow{d} N_1(0,\, 1).$$

[This is a "robustness" property of the Student t-statistic, since it shows that the large-sample distribution of t_n does not depend on the actual distribution being sampled. (As long as it has finite variance.)]

Example 10.4. (Asymptotic distribution of the sample variance.) Let $X_1, \ldots, X_n$ be an i.i.d. sample from a univariate distribution with *finite fourth moment.* Set $\mu = \mathrm{E}(X_i)$, $\sigma^2 = \mathrm{Var}(X_i)$, and

$$(10.25) \qquad \lambda_4 = \mathrm{Var}\left[\left(\frac{X_i - \mu}{\sigma}\right)^2\right] = \mathrm{E}\left[\left(\frac{X_i - \mu}{\sigma}\right)^4\right] - \frac{(\sigma^2)^2}{\sigma^4} \equiv \kappa_4 - 1.$$

Then the sample variance

$$s_n^2 \equiv \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2 = \frac{1}{n-1}\left[\sum_{i=1}^n X_i^2 - n\bar{X}_n^2\right]$$

is an unbiased, consistent estimator for σ^2 (cf. Examples 8.1 and 10.1). We wish to find the limiting distribution of s_n^2 (suitably normalized) as $n \to \infty$:

Let $V_i = X_i - \mu$. We have the following normal approximation for s_n^2:

$$\begin{aligned}\sqrt{n}(s_n^2 - \sigma^2) &= \sqrt{n}\left[\frac{n}{n-1}\left(\frac{1}{n}\sum_{i=1}^n V_i^2 - \bar{V}_n^2\right) - \sigma^2\right] \\ &= \frac{n}{n-1}\left[\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n V_i^2 - \sigma^2\right)\right] + \frac{\sqrt{n}\sigma^2}{n-1} - \frac{n}{\sqrt{n}\,(n-1)}\left(\sqrt{n}\bar{V}_n\right)^2 \\ (10.26) \qquad &\xrightarrow{d} N_1(0,\, \lambda_4\,\sigma^4)\end{aligned}$$

by Slutsky's Theorem, the CLT, and the fact that $\sqrt{n}\bar{V}_n \xrightarrow{d} N_1(0,\, \sigma^2)$. [Since λ_4 depends on the distribution being sampled, this shows that s_n^2, unlike t_n, is not robust to departures from normality.] In the special case that $X_i \sim N_1(\mu, \sigma^2)$, $\lambda_4 = \mathrm{Var}(\chi_1^2) = 2$ [verify], so

$$(10.27) \qquad \sqrt{n}(s_n^2 - \sigma^2) \xrightarrow{d} N_1(0,\, 2\sigma^4). \qquad \square$$

10.3. Propagation of error $\equiv$ δ-method $\equiv$ Taylor approximation.

Theorem 10.7. *(a) Let $\{Y_n\}$ be a sequence of rvs in $\mathbf{R}^1$ such that*

$$(10.28) \qquad \sqrt{n}(Y_n - \mu) \xrightarrow{d} N_1(0,\, \sigma^2), \quad \sigma^2 \geq 0.$$

If $g(y)$ is differentiable at μ and $g'(\mu) \neq 0$ then

$$(10.29) \qquad \sqrt{n}\,[g(Y_n) - g(\mu)] \xrightarrow{d} N_1\left(0,\, [g'(\mu)]^2\sigma^2\right).$$

(b) Let $\{Y_n\}$ be a sequence of rvtrs in $\mathbf{R}^k$ such that

$$(10.30) \qquad \sqrt{n}(Y_n - \mu) \xrightarrow{d} N_k(0,\, \Sigma), \quad \Sigma \text{ pd.}$$

If $g(y_1, \ldots, y_k)$ is differentiable at $\mu \equiv (\mu_1, \ldots, \mu_k)$, then

$$(10.31) \qquad \sqrt{n}\,[g(Y_n) - g(\mu)] \xrightarrow{d} N_1\big(0,\, (\nabla g)\,\Sigma\,(\nabla g)^t\big),$$

where $\nabla g = \left(\frac{\partial g}{\partial y_1}, \ldots, \frac{\partial g}{\partial y_k}\right)$ and the partial derivatives are evaluated at $y = \mu$, provided that at least one $\frac{\partial g}{\partial y_i} \neq 0$.

Proof. (a) Since g is differentiable at μ, its first-order Taylor expansion is

$$(10.32) \qquad g(y) = g(\mu) + (y - \mu)g'(\mu) + O(|y - \mu|^2).$$

Thus by (10.28) and Slutsky's theorems,

$$\begin{aligned}\sqrt{n}\,[g(Y_n) - g(\mu)] &= \sqrt{n}(Y_n - \mu)g'(\mu) + O(\sqrt{n}\,|Y_n - \mu|^2) \\ &\xrightarrow{d} N_1\left(0,\, [g'(\mu)]^2\sigma^2\right),\end{aligned}$$

since $\sqrt{n}(Y_n - \mu) \xrightarrow{d} N_1(0, \sigma^2) \;\Rightarrow\; \sqrt{n}\,|Y_n - \mu|^2 = O_p\left(\frac{1}{\sqrt{n}}\right) \xrightarrow{p} 0.$

(b) The multivariate first-order Taylor approximation of g at $y = \mu$ is

$$
\begin{aligned}
g(y) &= g(\mu) + \sum_{i=1}^{k} (y_i - \mu_i) \frac{\partial g}{\partial y_i}\Big|_{\mu} + O(\|y - \mu\|^2) \\
&= g(\mu) + \left(\frac{\partial g}{\partial y_1}, \ldots, \frac{\partial g}{\partial y_k} \right) (y - \mu) + O(\|y - \mu\|^2) \\
&\equiv g(\mu) + (\nabla g)(y - \mu) + O(\|y - \mu\|^2).
\end{aligned}
$$

Thus by (10.30) and Slutsky's theorems,

$$
\begin{aligned}
\sqrt{n}\,[g(Y_n) - g(\mu)] &= (\nabla g)\sqrt{n}(Y_n - \mu) + O(\sqrt{n}\,\|Y_n - \mu\|^2) \\
&\xrightarrow{d} N_1\left(0,\, (\nabla g)\,\Sigma\,(\nabla g)^t\right),
\end{aligned}
$$

since $n\,\|Y_n - \mu\|^2 \xrightarrow{d} \|N_k(0,\, \Sigma)\|^2 < \infty \;\Rightarrow\; O(\sqrt{n}\,\|Y_n - \mu\|^2) \xrightarrow{p} 0.$ □

Remark 10.5. Often $Y_n = \bar{X}_n$, a sample mean of i.i.d. rvs or rvtrs, but Y_n also may be a sample median (see §10.6), a maximum likelihood estimator (see §14.3), etc.

Example 10.5. (a) Let $g(y) = \frac{1}{y}$. Then g is differentiable at $y = \mu$ with $g'(\mu) = -\frac{1}{\mu^2}$ (provided that $\mu \neq 0$), so (10.29) becomes

(10.33)
$$\sqrt{n}\left(\frac{1}{Y_n} - \frac{1}{\mu}\right) \xrightarrow{d} N_1\left(0,\, \frac{\sigma^2}{\mu^4}\right).$$

(b) Similarly, if $g(y) = \log y$ and $\mu > 0$, then $g'(\mu) = \frac{1}{\mu}$, hence

(10.34)
$$\sqrt{n}\,(\log Y_n - \log \mu) \xrightarrow{d} N_1\left(0,\, \frac{\sigma^2}{\mu^2}\right).$$

Exercise 10.3. Assume that $\sqrt{n}(Y_n - \mu) \xrightarrow{d} N_1(0, \sigma^2)$.

(i) Find the asymptotic distribution of $\sqrt{n}\,(Y_n^2 - \mu^2)$ when $\mu \neq 0$.

(ii) When $\mu = 0$, show that this asymptotic distribution is degenerate (constant). Find the (non-degenerate!) asymptotic distribution of $n\,(Y_n^2 - 0)$. *Note:* if $\mu = 0$ the first-order (linear) term in the Taylor expansion of $g(y) \equiv y^2$ at $y = 0$ vanishes, so the second-order (quadratic) term determines the limiting distribution – see CB Theorem 5.5.26. Of course, here no expansion is needed [why?].

Example 10.6. For a bivariate example, take $g(x, y) = xy$ and suppose

$$\sqrt{n}\left[\begin{pmatrix} X_n \\ Y_n \end{pmatrix} - \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}\right] \xrightarrow{d} N_2\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}\right]. \tag{10.35}$$

Then $\frac{\partial g}{\partial x} = \mu_y$ and $\frac{\partial g}{\partial y} = \mu_x$ at (μ_x, μ_y), so if $(\mu_x, \mu_y) \neq (0, 0)$, (10.31) yields

$$\begin{aligned} \sqrt{n}(X_n Y_n - \mu_x \mu_y) &\xrightarrow{d} N_1\left[0, \, (\mu_y, \, \mu_x) \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix} \begin{pmatrix} \mu_y \\ \mu_x \end{pmatrix}\right] \\ &= N_1(0, \, \mu_y^2 \sigma_x^2 + \mu_x^2 \sigma_y^2 + 2\mu_y \mu_x \sigma_{xy}) \end{aligned} \tag{10.36}$$

In particular, if X_n and Y_n are asymptotically independent, i.e., if $\sigma_{xy} = 0$,

$$\sqrt{n}(X_n Y_n - \mu_x \mu_y) \xrightarrow{d} N_1(0, \, \mu_y^2 \sigma_x^2 + \mu_x^2 \sigma_y^2). \tag{10.37}$$

[Note the interchange of the subscripts x and y – can you explain this?]

Exercise 10.4. In Example 10.6, suppose that $(\mu_x, \mu_y) = (0, 0)$. Show that $\sqrt{n}(X_n Y_n - 0) \xrightarrow{d} 0$ but that $n X_n Y_n$ has a non-degenerate limiting distribution. Express this limiting distribution in terms of normal variates and find its mean and variance.

Exercise 10.5. (i) Repeat Example 10.6 with $g(x, \, y) = \frac{x}{y}$. (Take $\mu_y \neq 0$.)

(ii) Let $F_{m,n}$ denote a rv having the F-distribution with m and n degrees of freedom. (See CB Definition 5.36 and especially page 624.) First suppose that $m = n$. Show that as $n \to \infty$,

$$\sqrt{n}\,(F_{n,n} - 1) \xrightarrow{d} N_1(0, \, 4). \tag{10.38}$$

(iii) Now let $m \to \infty$ and $n \to \infty$ s.t. $\frac{m}{n} \to \gamma$ $(0 < \gamma < \infty)$. Show that

$$\sqrt{m}\,(F_{m,n} - 1) \xrightarrow{d} N_1(0,\, 2 + 2\gamma), \tag{10.39}$$

$$\sqrt{n}\,(F_{m,n} - 1) \xrightarrow{d} N_1\left(0,\, 2 + \frac{2}{\gamma}\right). \tag{10.40}$$

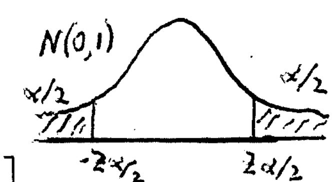
10.4. Variance-stabilizing transformations.

Let $\{Y_n\}$ be a consistent, asymptotically normal (CAN) sequence of estimators of a real-valued statistical parameter θ, e.g., $\theta = p$ in the Binomial(n, p) model, $\theta = \lambda$ in the Poisson(λ) model, $\theta = \lambda$ or $\frac{1}{\lambda}$ in the Exponential(λ) model, $\theta = \mu$ or σ or $\frac{\mu}{\sigma}$ in the normal model $N(\mu,\, \sigma^2)$. Assume that the asymptotic variance of Y_n depends only on θ:

$$\sqrt{n}\,(Y_n - \theta) \xrightarrow{d} N_1\left(0,\, \sigma^2(\theta)\right). \tag{10.41}$$

Define $z_\beta = \Phi^{-1}(1-\beta)$, the $(1-\beta)$-quantile of $N_1(0,\, 1)$. Then (10.41) gives

$$\begin{aligned} 1 - \alpha &\approx P\left[-z_{\frac{\alpha}{2}} \le \frac{\sqrt{n}(Y_n - \theta)}{\sigma(\theta)} \le z_{\frac{\alpha}{2}}\right] \\ &= P\left[Y_n - \frac{\sigma(\theta)}{\sqrt{n}} z_{\frac{\alpha}{2}} \le \theta \le Y_n + \frac{\sigma(\theta)}{\sqrt{n}} z_{\frac{\alpha}{2}}\right]. \end{aligned} \tag{10.42}$$



hence $Y_n \pm \frac{\sigma(\theta)}{\sqrt{n}} z_{\frac{\alpha}{2}}$ is an approximate $(1-\alpha)$-confidence interval for θ. However, this confidence interval has two drawbacks:

(i) If $\sigma(\theta)$ is not constant but varies with θ, it must be estimated,[14] which may be difficult and introduces additional variability in the confidence limits, so the actual confidence probability will be *less* the nominal $1 - \alpha$.

(ii) The accuracy of the normal approximation (10.41) may vary with θ. That is, how large n must be to insure the accuracy of (10.41) may depend on the unknown parameter θ, hence is not subject to control.

[14] Sometimes the inequalities in (10.42) may be solved directly for θ. This occurs. e.g., in the Binomial(n, p) model with $\theta = p$, $\sigma^2(p) = p(1-p)$ - see Example 10.8.

These two drawbacks can be resolved by a *variance-stabilizing transformation* $g(Y_n)$, found as follows: For any g that is differentiable at $y = \theta$, it follows from (10.41) and the propagation of error formula that

$$\sqrt{n}[g(Y_n) - g(\theta)] \xrightarrow{d} N_1\left(0,\, [g'(\theta)]^2\sigma^2(\theta)\right). \tag{10.43}$$

Therefore, if we can find a differentiable function g such that

$$[g'(\theta)]^2\sigma^2(\theta) = 1 \quad \text{(not depending on } \theta\text{)}, \tag{10.44}$$

then (10.40) becomes

$$\sqrt{n}[g(Y_n) - g(\theta)] \xrightarrow{d} N_1(0,\, 1). \tag{10.45}$$

This averts the difficulty (i), since it implies that $g(Y_n) \pm \sqrt{\frac{1}{n}} z_{\frac{\alpha}{2}}$ is an approximate $(1-\alpha)$-confidence interval for $g(\theta)$ that does not involve the unknown θ. If in addition $g(\theta)$ is monotone in θ, this interval can be converted to a confidence interval for θ. Furthermore, it may also alleviate difficulty (ii) since the normal approximation (10.45) is usually accurate over a wider range of θ-values uniformly in n. (See Example 10.8.)

To find g that satisfies (10.44), simply note that (10.44) yields

$$g'(\theta) = \frac{1}{\sigma(\theta)},$$

$$g(\theta) = \int \frac{d\theta}{\sigma(\theta)} \qquad \text{(an indefinite integral)}. \tag{10.46}$$

If we can solve this for g, then $g(Y_n)$ will satisfy (10.45).

Example 10.7. Suppose that $X_1, \ldots, X_n$ is an i.i.d. sample from the Exponential$(\lambda = \frac{1}{\theta})$ distribution. Then $\mathrm{E}(X_i) = \theta$, $\mathrm{Var}(X_i) = \theta^2$, hence $\bar{X}_n$ is a CAN estimator of θ:

$$\sqrt{n}\,(\bar{X}_n - \theta) \xrightarrow{d} N_1(0,\, \theta^2). \tag{10.47}$$

Here $\sigma^2(\theta) = \theta^2$, so the variance-stabilizing function $g(\theta)$ in (10.46) becomes

$$g(\theta) = \int \frac{d\theta}{\theta} = \log\theta.$$

We conclude that

$$\sqrt{n}\,\left[\log \bar{X}_n - \log\theta\right] \xrightarrow{d} N_1(0,\,1), \tag{10.48}$$

which yields confidence intervals for $\log\theta$ and thence for θ. □

Note: If $X \sim \text{Expo}(\frac{1}{\theta})$ then θ is a scale parameter, i.e., $X \sim \theta Y$ where $Y \sim \text{Expo}(1)$. Thus

$$\log X \sim \log Y + \log\theta,$$

which easily shows why $\log \bar{X}_n$ stabilizes the variance for $0 < \theta < \infty$.

Example 10.8. Suppose we want a confidence interval for $p \in (0,1)$ based on $X_n \sim \text{Binomial}(n,\,p)$. Then $\hat{p}_n \equiv \frac{X_n}{n}$ is a CAN estimator of p:

$$\sqrt{n}\,(\hat{p}_n - p) \xrightarrow{d} N_1\left(0,\, p(1-p)\right). \tag{10.49}$$

Here $\sigma(p) = \sqrt{p(1-p)}$, so the function $g(p)$ in (10.46) is given by

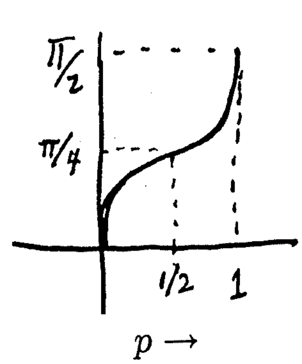
$$g(p) = \int \frac{dp}{\sqrt{p(1-p)}} = 2\arcsin(\sqrt{p}) \qquad \text{[verify!]}$$

Thus $\arcsin(\sqrt{\hat{p}_n})$ stabilizes the variance of $\hat{p}_n$ (see the following figures):

$$\sqrt{n}\,\left[\arcsin(\sqrt{\hat{p}_n}) - \arcsin(\sqrt{p})\right] \xrightarrow{d} N_1\left(0,\,\frac{1}{4}\right), \tag{10.50}$$

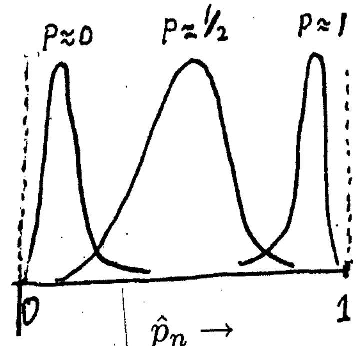
which yields confidence intervals for $\arcsin(\sqrt{p})$ and thence for p.

arcsin($\sqrt{p}$) :



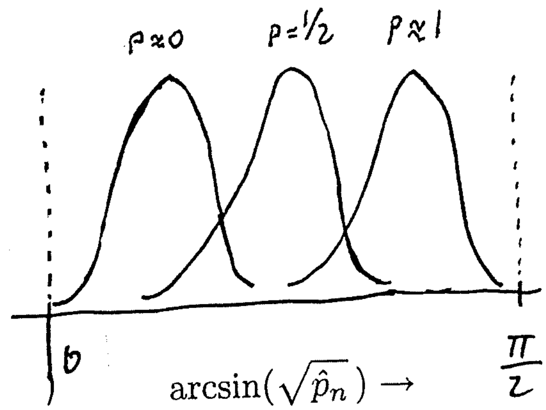
This non-linear transformation "stretches" the interval $(0,1)$ more near $p = 0$ and $p = 1$ than near $p = 1/2$.

Asymptotic distn. of $\hat{p}_n$:



$\text{Var}(\hat{p}_n) = \frac{p(1-p)}{n}$ depends on p. It is very small for $p \approx 0, 1$, where the distribution of $\hat{p}_n$ is relatively *skewed* due to its truncation at the endpoint 0 or 1 and the normal approximation is not very good unless n is very large.

Asymptotic distn. of arcsin($\sqrt{\hat{p}_n}$):



$\text{Var}(\arcsin(\sqrt{\hat{p}_n})) \approx \frac{1}{4n}$ does not depend on p. The distribution of $\arcsin(\sqrt{\hat{p}_n})$ is not very skewed for $p \approx 0, 1$ and the normal approximation is fairly good uniformly in p for moderately large n.

Remark 10.6. A variance-stabilizing transformation can be used to make comparisons between two or more parameters based on independent samples. For example, if $m\hat{p}_1 \sim \text{Binomial}(m, p_1)$ and $n\hat{p}_2 \sim \text{Binomial}(n, p_2)$ with $m \to \infty$ and $n \to \infty$ s.t. $\frac{m}{n} \to \gamma$ $(0 < \gamma < \infty)$, then [verify]

$$\sqrt{n}\left[\left(\arcsin(\sqrt{\hat{p}_1}\,) - \arcsin(\sqrt{\hat{p}_2}\,)\right) - \left(\arcsin(\sqrt{p_1}\,) - \arcsin(\sqrt{p_2}\,)\right)\right]$$

$$\xrightarrow{d} N_1\left(0,\, \frac{1}{4} + \frac{1}{4\gamma}\right). \tag{10.51}$$

Thus $\left(\arcsin(\sqrt{\hat{p}_1}\,) - \arcsin(\sqrt{\hat{p}_2}\,)\right) \pm \sqrt{\frac{\gamma+1}{4n\gamma}}\, z_{\frac{\alpha}{2}}$ is an approximate $(1-\alpha)$-confidence interval for $(\arcsin(\sqrt{p_1}\,) - \arcsin(\sqrt{p_2}\,))$, which can in turn be used to test the hypothesis $p_1 = p_2$ vs. $p_1 \neq p_2$.

Exercise 10.6. (i) Assume $X_\lambda \sim \text{Poisson}(\lambda)$. Find a variance-stabilizing transformation for X_λ as $\lambda \to \infty$. That is, find a function h and constant $c > 0$ s.t.

$$[h(X_\lambda) - h(\lambda)] \xrightarrow{d} N_1(0,\, c). \tag{10.52}$$

Use this to obtain an approximate $(1-\alpha)$-confidence interval for λ.

Hint: Write $\lambda = n\theta$ with $n \to \infty$ and θ the fixed unknown parameter.

(ii) Let $X_\lambda \sim \text{Poisson}(\lambda)$ and $X_\mu \sim \text{Poisson}(\mu)$, where $X_\lambda \perp\!\!\!\perp X_\mu$ and λ and μ are both large. Based on (i), describe an approximate procedure for testing $\lambda = \mu$ vs. $\lambda \neq \mu$.

10.5. Asymptotic distribution of a sample covariance matrix.

Let $W_1 \equiv \begin{pmatrix} X_1 \\ Y_1 \end{pmatrix}, \ldots, W_n \equiv \begin{pmatrix} X_n \\ Y_n \end{pmatrix}$ be an i.i.d. sample from a *bivariate* distribution with *finite fourth moments.* Let

$$\mathrm{E}\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} \mu \\ \nu \end{pmatrix} \equiv \xi,$$

(10.53)

$$\mathrm{Cov}\begin{pmatrix} X_i \\ Y_i \end{pmatrix} = \begin{pmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{pmatrix} \equiv \begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix}\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\begin{pmatrix} \sigma & 0 \\ 0 & \tau \end{pmatrix} \equiv \Sigma.$$

The *sample covariance matrix* is defined to be

$$(10.54)\quad S_n = \frac{1}{n-1}\sum_{i=1}^n (W_i - \bar{W}_n)(W_i - \bar{W}_n)'$$

$$(10.55)\quad = \frac{1}{n-1}\begin{pmatrix} \sum(X_i - \bar{X}_n)^2 & \sum(X_i - \bar{X}_n)(Y_i - \bar{Y}_n) \\ \sum(X_i - \bar{X}_n)(Y_i - \bar{Y}_n) & \sum(Y_i - \bar{Y}_n)^2 \end{pmatrix}.$$

First, S_n is an unbiased, consistent estimator of Σ. To see this, set

$$V_i = \begin{pmatrix} X_i - \mu \\ Y_i - \nu \end{pmatrix} \equiv W_i - \xi,$$

so $\mathrm{E}(V_i) = 0,\ \mathrm{Cov}(V_i) = \Sigma,\ \mathrm{Cov}(\bar{V}_n) = \frac{1}{n}\Sigma$. Then from (10.54),

(10.56)

$$\begin{aligned}
\mathrm{E}(S_n) &= \frac{1}{n-1}\mathrm{E}\left[\sum(V_i - \bar{V}_n)(V_i - \bar{V}_n)'\right] \\
&= \frac{1}{n-1}\mathrm{E}\left[\sum V_i V_i' - n\bar{V}_n\bar{V}_n'\right] \\
&= \frac{1}{n-1}\left[n\,\mathrm{Cov}(V_i) - n\,\mathrm{Cov}(\bar{V}_n)\right] \\
&= \frac{1}{n-1}\left[n\,\Sigma - n\cdot\frac{1}{n}\Sigma\right] \\
&= \Sigma,
\end{aligned}$$

so S_n is unbiased. Also, S_n is consistent: by the WLLN applied twice,

$$S_n = \frac{n}{n-1}\left(\frac{1}{n}\sum V_i V_i' - \bar{V}_n \bar{V}_n'\right) \xrightarrow{p} \Sigma. \tag{10.57}$$

We wish to determine the limiting distribution of $\sqrt{n}(S_n - \Sigma)$ as $n \to \infty$ (recall Example 10.4 for the limiting distribution of $\sqrt{n}(s_n^2 - \sigma^2)$). From this we can determine, for example, the limiting distribution of the sample correlation coefficient. Note that S_n and Σ, being symmetric 2×2 matrices, should be viewed as 3-dimensional vectors. As in (10.26) we can write

$$\begin{aligned}
&\sqrt{n}(S_n - \Sigma) \\
&= \sqrt{n}\left[\frac{1}{n-1}\left(\sum V_i V_i' - n\bar{V}_n \bar{V}_n'\right) - \Sigma\right] \\
&= \frac{n}{n-1}\left[\sqrt{n}\left(\frac{1}{n}\sum V_i V_i' - \Sigma\right)\right] + \frac{\sqrt{n}}{n-1}\Sigma - \frac{n}{n-1}\frac{(\sqrt{n}\bar{V}_n)(\sqrt{n}\bar{V}_n)'}{\sqrt{n}}
\end{aligned}$$

Because $\frac{n}{n-1} \to 1$ and $\sqrt{n}\bar{V}_n \xrightarrow{d} N_2(0, \Sigma)$, the second and third terms $\to 0$ [verify], hence by Slutzky's Theorem, $\sqrt{n}(S_n - \Sigma)$ has the same limiting distribution as

$$\sqrt{n}\left(\frac{1}{n}\sum V_i V_i' - \Sigma\right).$$

However, $V_1 V_1', \ldots, V_n V_n'$ can be viewed as i.i.d. 3-dimensional random vectors, so the Central Limit Theorem can be applied as follows:

Clearly $\mathrm{E}(V_i V_i') = \Sigma$. Next,

$$\mathrm{Cov}(V_i V_i') \equiv \mathrm{Cov}\left[\begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma\tau & 0 \\ 0 & 0 & \tau^2 \end{pmatrix} \begin{pmatrix} (\frac{X_i-\mu}{\sigma})^2 \\ (\frac{X_i-\mu}{\sigma})(\frac{Y_i-\nu}{\tau}) \\ (\frac{Y_i-\nu}{\tau})^2 \end{pmatrix}\right] \tag{10.58}$$

$$= D(\sigma,\tau)\,\mathrm{Cov}\begin{pmatrix} (\frac{X_i-\mu}{\sigma})^2 \\ (\frac{X_i-\mu}{\sigma})(\frac{Y_i-\nu}{\tau}) \\ (\frac{Y_i-\nu}{\tau})^2 \end{pmatrix} D(\sigma,\tau), \tag{10.59}$$

where

$$D(\sigma,\tau) = \begin{pmatrix} \sigma^2 & 0 & 0 \\ 0 & \sigma\tau & 0 \\ 0 & 0 & \tau^2 \end{pmatrix}. \tag{10.60}$$

But

$$(10.61) \qquad \operatorname{Cov}\begin{pmatrix} (\frac{X_i-\mu}{\sigma})^2 \\ (\frac{X_i-\mu}{\sigma})(\frac{Y_i-\nu}{\tau}) \\ (\frac{Y_i-\nu}{\tau})^2 \end{pmatrix} = K - R(\rho)R(\rho)',$$

where

$$(10.62) \quad K = \begin{pmatrix} \kappa_{40} & \kappa_{31} & \kappa_{22} \\ \kappa_{31} & \kappa_{22} & \kappa_{13} \\ \kappa_{22} & \kappa_{13} & \kappa_{04} \end{pmatrix}, \quad \kappa_{jk} = \mathrm{E}\left[\left(\frac{X_i-\mu}{\sigma}\right)^j \left(\frac{Y_i-\nu}{\tau}\right)^k\right],$$

$$(10.63) \qquad R(\rho) = \mathrm{E}\begin{pmatrix} \left(\frac{X_i-\mu}{\sigma}\right)^2 \\ \left(\frac{X_i-\mu}{\sigma}\right)\left(\frac{Y_i-\nu}{\tau}\right) \\ \left(\frac{Y_i-\nu}{\tau}\right)^2 \end{pmatrix} = \begin{pmatrix} 1 \\ \rho \\ 1 \end{pmatrix}.$$

Therefore from (10.59) and (10.61),

$$(10.64) \qquad \begin{aligned} \operatorname{Cov}(V_iV_i') &= D(\sigma,\tau)\ [K - R(\rho)R(\rho)']\ D(\sigma,\tau) \\ &\equiv \Lambda, \end{aligned}$$

(recall (10.25)). The factorization (10.64) shows how the scale parameters σ and τ, the standardized fourth moments $\kappa_{j,k}$, and the correlation ρ contribute to the covariance matrix of V_iV_i', hence to that of S_n. Thus the CLT yields

$$(10.65) \qquad \sqrt{n}(S_n - \Sigma) \xrightarrow{d} N_3(0,\, \Lambda).$$

Example 10.8 (Consistency and asymptotic normality of the sample correlation coefficient and Fisher's z-transform.) The sample correlation r_n is a consistent estimator of the population correlation ρ: because

$$(10.66) \qquad r_n \equiv \frac{\frac{1}{n-1}\sum(X_i - \bar{X}_n)(Y_i - \bar{Y}_n)}{\sqrt{\frac{1}{n-1}\sum(X_i - \bar{X}_n)^2}\sqrt{\frac{1}{n-1}\sum(Y_i - \bar{Y}_n)^2}} = g(S_n),$$

where

$$(10.67) \qquad g(x, y, z) \equiv \frac{y}{\sqrt{x}\sqrt{z}}$$

is continuous for $x, y > 0$, and because $S_n \xrightarrow{p} \Sigma$ by (10.57), it follows that

$$(10.68) \qquad r_n \xrightarrow{p} \rho \qquad \text{as } n \to \infty.$$

Exercise 10.7. (i) Apply (10.65) to find the asymptotic distribution of $\sqrt{n}\,(r_n - \rho)$. Express the asymptotic variance in terms of ρ and the moments κ_{jk} in (10.62). (Since r_n is invariant under location and scale changes, its distribution does not depend on μ, ν, σ, or τ.)

(ii) Specialize the result in (i) to the *bivariate normal* case, i.e., assume that

$$\begin{pmatrix} X_i \\ Y_i \end{pmatrix} \sim N_2\left[\begin{pmatrix} \mu \\ \nu \end{pmatrix}, \begin{pmatrix} \sigma^2 & \rho\sigma\tau \\ \rho\sigma\tau & \tau^2 \end{pmatrix} \equiv \Sigma\right]$$

[Evaluate the κ_{jk} in (10.62).]

(iii) In (ii), find a variance-stabilizing tranformation for r_n. That is, find a function $g(r_n)$ such that $\sqrt{n}\,[g(r_n) - g(\rho)] \xrightarrow{d} N_1(0,\, c)$ where c does not depend on ρ (specify c).

10.6. Asymptotic distribution of sample quantiles.

Let $X_1, \ldots, X_n$ be an i.i.d. sample from a *continuous* distribution on $\mathbf{R}^1$ with unknown cdf F. We shall show that for $0 < p < 1$, the p-th sample quantile $X_{([np]+1)} \equiv F_n^{-1}(p)$ is a CAN estimator of the p-th population quantile $\equiv F^{-1}(p)$. That is, we shall show:

$$X_{([np]+1)} \xrightarrow{p} F^{-1}(p) \tag{10.69}$$

provided that F^{-1} exists and is continuous at p;

$$\sqrt{n}\,\left(X_{([np]+1)} - F^{-1}(p)\right) \xrightarrow{d} N_1\left(0,\, \frac{p(1-p)}{[f(F^{-1}(p))]^2}\right) \tag{10.70}$$

provided that F^{-1} exists and is differentiable at p. (This requires that $f(F^{-1}(p)) > 0$.)

To derive (10.69) and (10.70), first consider the case where $U_1, \ldots, U_n$ are i.i.d. Uniform(0,1) rvs with cdf $G(u) = u$, $0 \le u \le 1$ (recall §10.1). From (10.4) and (10.5) we know that the empirical cdf $G_n(u) \sim \frac{1}{n}\text{Binomial}(n,\, u)$ is a CAN estimator of $G(u)$:

$$G_n(u) \xrightarrow{p} G(u) = u, \tag{10.71}$$

$$\sqrt{n}\,[G_n(u) - G(u)] \xrightarrow{d} N_1\left(0,\, u(1-u)\right). \tag{10.72}$$

Let $0 < U_{(1)} < \cdots < U_{(n)} < 1$ be the order statistics based on $U_1, \ldots, U_n$.

Proposition 10.1. $U_{([np]+1)} \xrightarrow{p} p \quad (\equiv G^{-1}(p))$.

Proof. Fix $\epsilon > 0$. Then

$$\begin{aligned} \left\{U_{([np]+1)} \le p - \epsilon\right\} &= \{[np] + 1 \text{ or more } U_i's \le p - \epsilon\} \\ &= \left\{G_n(p-\epsilon) \ge \frac{[np]+1}{n}\right\}. \end{aligned}$$

But $G_n(p-\epsilon) \xrightarrow{p} p - \epsilon$ and $\frac{[np]+1}{n} \to p$ (since $|np - ([np]+1)| \le 1$), so

$$P\left[U_{([np]+1)} \le p - \epsilon\right] \to 0 \qquad \text{as } n \to \infty.$$

Similarly,

$$P\left[U_{([np]+1)} \ge p + \epsilon\right] \to 0 \qquad \text{as } n \to \infty.$$

Proof of (10.69): Since $(X_1, \ldots, X_n) \overset{\text{distn}}{=} (F^{-1}(U_1), \ldots, F^{-1}(U_n))$,

(10.73) $$(X_{(1)}, \ldots, X_{(n)}) \overset{\text{distn}}{=} (F^{-1}(U_{(1)}), \ldots, F^{-1}(U_{(n)}))$$

because $F$is increasing, so

(10.74) $$X_{([np]+1)} \overset{\text{distn}}{=} F^{-1}(U_{([np]+1)}).$$

Since F^{-1} is assumed continuous at p, $X_{([np]+1)} \xrightarrow{p} F^{-1}(p)$ by Prop. 10.1.

Proposition 10.2.

(10.75) $$\sqrt{n}\,\left(U_{([np]+1)} - p\right) \xrightarrow{d} N_1\,(0,\, p(1-p))\,.$$

Proof: First, for any $t \in \mathbf{R}^1$,

$$\begin{aligned}\{\sqrt{n}\,\left(U_{([np]+1)} - p\right) \le t\} &= \left\{U_{([np]+1)} \le p + \frac{t}{\sqrt{n}}\right\} \\ &= \left\{[np] + 1 \text{ or more } U_i's \ \le\ p + \frac{t}{\sqrt{n}}\right\} \\ &= \left\{G_n\left(p + \frac{t}{\sqrt{n}}\right) \ge \frac{[np]+1}{n}\right\} \\ &= \{A_n + B_n + C_n \ge -t\}\,,\end{aligned}$$

where

(10.76) $$A_n \equiv \sqrt{n}\left[G_n\left(p + \frac{t}{\sqrt{n}}\right) - G_n(p) - \frac{t}{\sqrt{n}}\right] \xrightarrow{p} 0, \quad \text{[see below]}$$

(10.77) $$B_n \equiv \sqrt{n}\,[G_n(p) - p] \xrightarrow{d} N_1\,(0,\, p(1-p))\,, \qquad \text{[by (10.72)]}$$

(10.78) $$C_n \equiv \sqrt{n}\left[p - \left(\frac{[np]+1}{n}\right)\right] = \frac{|np - ([np]+1)|}{\sqrt{n}} \le \frac{1}{\sqrt{n}} \to 0.$$

Thus the asserted result follows by Slutsky and symmetry:

$$\begin{aligned} P\left[\sqrt{n}\,\left(U_{([np]+1)} - p\right) \leq t\right] &\to P[N_1(0,\, p(1-p)\,) \geq -t] \\ &= P[N_1(0,\, p(1-p)\,) \leq t]. \end{aligned}$$

To verify (10.76): if $t > 0$ then for $\sqrt{n} > t$,

$$A_n \overset{\text{distn}}{=} \sqrt{n}\left[\frac{\text{Binomial}(n, \frac{t}{\sqrt{n}})}{n} - \frac{t}{\sqrt{n}}\right],$$

hence

$$\mathrm{E}(A_n) = 0, \qquad \mathrm{Var}(A_n) = \frac{t}{\sqrt{n}}\left(1 - \frac{t}{\sqrt{n}}\right) \to 0 \quad \text{as } n \to \infty,$$

so (10.76) follows from Chebyshev's inequality. The proof is similar if $t < 0$.

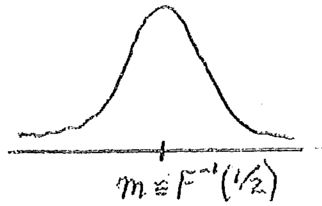
Proof of (10.70): By (10.74), (10.76), and propagation of error,

$$\begin{aligned} \sqrt{n}\,\left(X_{([np]+1)} - F^{-1}(p)\,\right) &\overset{\text{distn}}{=} \sqrt{n}\,\left[F^{-1}(U_{([np]+1)}) - F^{-1}(p)\,\right] \\ &\xrightarrow{d} N_1\left(0, \left[(F^{-1})'(p)\right]^2 p(1-p)\right) \\ &= N_1\left(0, \frac{p(1-p)}{[f(F^{-1}(p))]^2}\right) \qquad [\text{since } F' = f]. \end{aligned}$$

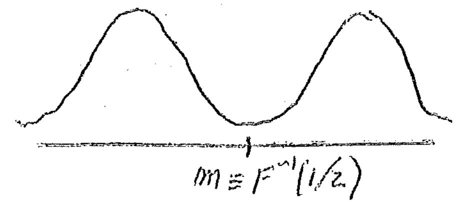
Example 10.9. Asymptotic distribution of the sample median. When $p = \frac{1}{2}$, $F^{-1}(1/2) \equiv m$ is the population median, $X_{([n/2]+1)} \equiv \tilde{X}_n$ is the sample median, and (10.70) becomes

$$\sqrt{n}\,\left(\tilde{X}_n - m\right) \xrightarrow{d} N_1\left(0, \frac{1}{4\,[f(m)]^2}\right). \tag{10.79}$$

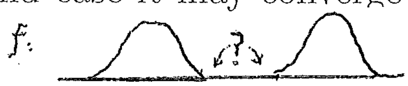
This shows that the precision $\equiv$ accuracy of the sample median as an estimator of the population median is directly proportional to $f(m)$. This makes sense because the larger $f(m)$ is, the more the observations X_i will accrue in the vicinity of m (see following figure):



sample median
has high precision



sample median
has low precision

Remark 10.7. (a) If $f(F^{-1}(p)) = \infty$ or $= 0$ then (10.70) is inapplicable. In the first case $X_{([np]+1)}$ may converge to $F^{-1}(p)$ at a rate faster than $\frac{1}{\sqrt{n}}$, while in the second case it may converge at a slower rate or may not converge at all: 

(b) The asymptotic normality (10.70) of the p-th sample quantile $X_{([np]+1)}$ is valid when p is fixed with $0 < p < 1$. It is not valid when $p \equiv p(n) \to 0$ or 1, e.g., the extreme order statistics $X_{(1)} \equiv X_{\min}$ and $X_{(n)} \equiv X_{\max}$ are not asymptotically normal. For example, in the Uniform$(0,1)$ case of Example 9.1, the variance of these extreme order statistics are $O\left(\frac{1}{n^2}\right)$ rather than $O\left(\frac{1}{n}\right)$, thus to obtain a nontrivial limiting distribution they must be multiplied by an "inflation factor" n rather than $\sqrt{n}$ – see Exercise 9.4.

10.7. Asymptotic efficiency of sample mean vs. sample median as estimators of the center of a symmetric distribution.

Suppose that $X_1, \ldots, X_n$ is an i.i.d. sample from a distribution with pdf $f_\theta(x) \equiv f(x-\theta)$ on $\mathbf{R}^1$, where θ is an unknown location parameter. Suppose that f is *symmetric* about 0, i.e., $f(x) = f(-x)\ \forall\ x$, so f_θ is symmetric about θ. Thus θ serves as both the population mean (provided it exists) and the population median. Thus it is natural to compare the sample mean $\bar{X}_n$ and the sample median $\tilde{X}_n$ as estimators[15] of θ.

Suppose that $\tau^2 \equiv \mathrm{Var}(X_i) \equiv \int x^2 f(x)dx < \infty$ and $f(0) > 0$. Then from the CLT and (10.79),

$$\sqrt{n}\left(\bar{X}_n - \theta\right) \xrightarrow{d} N_1(0, \tau^2), \tag{10.80}$$

$$\sqrt{n}\left(\tilde{X}_n - \theta\right) \xrightarrow{d} N_1\left(0, \frac{1}{4[f(0)]^2}\right). \tag{10.81}$$

[15] Neither $\bar{X}_n$ nor $\tilde{X}_n$ need be the (asymptotically) optimal estimator of θ. The maximum likelihood estimator usually is asympotically optimal (Theorem 14.9), and it need not be equivalent to $\bar{X}_n$ or $\tilde{X}_n$; e.g., if f is a Cauchy density (Example 14.5).

Thus *the asymptotic efficiency of* $\tilde{X}_n$ *relative to* $\bar{X}_n$ *(as measured by the ratio of their asymptotic variances) is* $4[f(0)]^2\tau^2$.

Example 10.10. If $X_i \sim N_1(\theta, 1)$ (so $\tau^2 = 1$), then $f(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, so

$$4[f(0)]^2\tau^2 = \frac{2}{\pi} \approx 0.637, \tag{10.82}$$

hence $\tilde{X}_n$ is (asymptotically) less efficient than $\bar{X}_n$: in the normal case the precision of the sample median based on n observations is about the same as that of the sample mean based on only $.637n$ observations.

Actually, this should *not* be interpreted as a strong argument in favor of $\bar{X}_n$. The efficiency of the sample median $\tilde{X}_n$ relative to $\bar{X}_n$ is about 64%, which, while a significant loss, is not catastrophic. If, however, our assumption of a normal model is wrong, then the performance of the sample mean $\bar{X}_n$ may itself be catastrophic. For example, if $f(x) \equiv \frac{1}{\pi(1+x^2)}$ is the standard Cauchy density then the asymptotic variance of $\sqrt{n}(\tilde{X}_n - \theta)$ is

$$\frac{1}{4[f(0)]^2} = \frac{\pi^2}{4} \approx 2.47, \tag{10.83}$$

but $\tau^2 = \infty$ so the asymptotic variance of $\bar{X}_n$ is infinite. In fact, $\bar{X}_n$ is not even a consistent estimator of θ. Because of this, we say that the sample median is a *robust* estimator of the location parameter θ (for heavy-tailed departures from normality), whereas the sample mean is not robust.

Note: in the Cauchy case the sample median, while robust, is not optimal: the MLE is better (see Exercise 14.36(i)). □

Exercise 10.8. Let $f(x) \equiv \frac{1}{2}e^{-|x|}$ be the standard double exponential density on $\mathbf{R}^1$, so $f_\theta(x) = \frac{1}{2}e^{-|x-\theta|}$.

(i) Find the asymptotic efficiency of the sample median $\tilde{X}_n$ relative to the sample mean $\bar{X}_n$ as estimators of θ.

(ii) Find the MLE $\hat{\theta}$ for θ, i.e., the value of θ that maximizes the joint pdf

$$f_\theta(x_1, \ldots, x_n) \equiv \frac{1}{2^n}\prod_{i=1}^{n} e^{-|x_i-\theta|}. \tag{10.84}$$