MDP Exercise 1.5*. Let $S = \frac{X}{\sqrt{1-Y^2}}$, $T = \frac{Y}{\sqrt{1-X^2}}$. To show that $S \not\perp\!\!\!\perp T$, show that

$$(*) \qquad P[S \le 1/2, T \le 1/2] \ne P[S \le 1/2]P[T \le 1/2].$$

Because $S, T \sim \text{Uniform}(-1, 1)$, the right side $= (3/4)^2 = 9/16$.

Now draw the region on the left side as the intersection of two semi-circular-type regions in the unit disk. The boundaries of the two semicircles intersect at the point $(1/\sqrt{5}, 1/\sqrt{5})$. This intersection includes the entire portion of the disk in the third quadrant, which has probability 1/4. The intersection also includes parts of the disk in the second and fourth quadrants. By the uniform distributions of S and T and by Cavalieri's Principle, each of these has area equal to half of the disc quadrant, hence each has probability 1/8, hence so far we have probability 1/4 + 1/8 +1/8 = 1/2.

Now examine the portion of the intersection that lies in the first quadrant of the disk. By the shape of the semi-circular boundaries, this portion contains the square with lower left vertex (0,0) and upper right vertex $(1/\sqrt{5}, 1/\sqrt{5})$, which square has probability $1/5\pi$. Thus the intersection on the left side of $(*)$ has probability at least $1/2 + 1/5\pi$. But this already exceeds 9/16, the probability of the right side of $(*)$.

512 Practice Exam Question.

Let Z_n be a sequence of random variables such that

$$\sqrt{n}(Z_n - \mu) \xrightarrow{d} N_1(0, \sigma^2)$$

for some constants μ and $\sigma^2 > 0$. Use results in MDP §10 to show that

$$Y_n \xrightarrow{p} \mu.$$

Solution. Apply MDP Theorem 10.6 with $X_n = \sqrt{n}(Z_n - \mu)$, $Y_n = 1/\sqrt{n}$, and $c = 0$.

Exercise 10.6. (i) Assume $X_\lambda \sim \text{Poisson}(\lambda)$. Find a variance-stabilizing transformation for X_λ as $\lambda \to \infty$. That is, find a function h and constant $c > 0$ s.t.

$$[h(X_\lambda) - h(\lambda)] \xrightarrow{d} N_1(0,\, c). \tag{10.52}$$

Use this to obtain an approximate $(1 - \alpha)$-confidence interval for λ.

Hint: First, to find the form of the function h consider $\lambda = n\theta$ where $0 < \theta < \infty$ is fixed and $n \to \infty$. To obtain (10.52) for all values of λ, one approach is to use Slutsky's perturbation result in Theorem 10.4.

(ii) Let $X_\lambda \sim \text{Poisson}(\lambda)$ and $X_\mu \sim \text{Poisson}(\mu)$, where $X_\lambda \perp\!\!\!\perp X_\mu$ and λ and μ are both large. Based on (i), describe an approximate procedure for testing $\lambda = \mu$ vs. $\lambda \neq \mu$.

Solution. (i) From the hint, $X_{n\theta} \sim \text{Poisson}(n\theta)$ so $X_{n\theta} \sim Y_1 + \cdots + Y_n$ where the Y_i are i.i.d. Poisson(θ) rvs. Therefore the CLTh implies that

$$\sqrt{n}(\bar{Y}_n - \theta) \xrightarrow{d} N(0, \theta),$$

so a variance-stabilizing transformation is given by $h(\theta) = \int \frac{d\theta}{\sqrt{\theta}} = \sqrt{\theta}$. Because $h'(\theta) = \frac{1}{2\sqrt{\theta}}$ and $\bar{Y}_n \sim \frac{X_{n\theta}}{n}$,

$$\sqrt{n}(\sqrt{\bar{Y}_n} - \sqrt{\theta}) \xrightarrow{d} N(0, \tfrac{1}{4}),$$

$$\sqrt{n}\Big(\frac{\sqrt{X_{n\theta}}}{\sqrt{n}} - \sqrt{\theta}\Big) \xrightarrow{d} N(0, \tfrac{1}{4}),$$

$$\sqrt{X_{n\theta}} - \sqrt{n\theta} \xrightarrow{d} N(0, \tfrac{1}{4}). \tag{1}$$

Now consider general (non-integral) values $\lambda \to \infty$. Let $[\lambda]$ be the greatest integer $\leq \lambda$. Then from (1) with $\theta = 1$,

$$\sqrt{X_{[\lambda]}} - \sqrt{[\lambda]} \xrightarrow{d} N(0, \tfrac{1}{4}) \qquad \text{as } \lambda \to \infty. \tag{2}$$

By the Intermediate Value Theorem with $h(\lambda) = \sqrt{\lambda}$, for some $\lambda^* \in ([\lambda], \lambda)$,

$$\begin{aligned}\sqrt{\lambda} - \sqrt{[\lambda]} = (\lambda - [\lambda])h'(\lambda^*) =&(\lambda - [\lambda])\tfrac{1}{2\sqrt{\lambda^*}} \\ <&(\lambda - [\lambda])\tfrac{1}{2\sqrt{[\lambda]}} = \tfrac{1}{2}(\tfrac{\lambda}{[\lambda]} - 1) \to 0 \ \text{ as } \lambda \to \infty.\end{aligned} \tag{3}$$

Next, $X_\lambda = X_{[\lambda]} + V_\lambda$ where $V_\lambda \sim \text{Poisson}(\lambda - [\lambda])$ and $X_{[\lambda]} \perp\!\!\!\perp V_\lambda$. As in (3), if $X_{[\lambda]} > 0$ then for some intermediate value $X^* \in (X_{[\lambda]}, X_\lambda)$,

$$\sqrt{X_\lambda} - \sqrt{X_{[\lambda]}} = V_\lambda h'(X^*) = V_\lambda \tfrac{1}{2\sqrt{X^*}} < V_\lambda \tfrac{1}{2\sqrt{X_{[\lambda]}}} \xrightarrow{p} 0 \text{ as } \lambda \to \infty, \tag{4}$$

because $V_\lambda \preceq_{\text{stoch}} \text{Poisson}(1)$ (since $0 \leq \lambda - [\lambda] \leq 1$) and $\sqrt{X_{[\lambda]}} \xrightarrow{p} \infty$. Because $\Pr[X_{[\lambda]} > 0 \to 1$ as $\lambda \to \infty$, it follows from (3) and (4) that

$$(\sqrt{X_\lambda} - \lambda) - (\sqrt{X_{[\lambda]}} - \sqrt{[\lambda]}) \xrightarrow{p} 0 \quad \text{as } \lambda \to \infty, \tag{5}$$

so by (2) and Slutzky's Theorem,

$$\sqrt{X_\lambda} - \sqrt{\lambda} \xrightarrow{d} N(0, \tfrac{1}{4}). \tag{6}$$