Wasserstein gradient flows

Tuesday, January 3, 2023 12:00 PM

Notes

Material taken from textbook by Ambrosio - Gigli - Savaré

Numbers in red

Wasserstein space as an infinite dimensional Riemannian manifold

$\mathcal{P}_2(\mathbb{R}^d)$ - space of all Borel prob distributions with finite second moments.

For $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R}^d)$ $\quad d(\mu_1, \mu_2) = W_2(\mu_1, \mu_2)$.

$(\mathcal{P}_2(\mathbb{R}^d), W_2)$ metric space.

Textbook - AGS Chapters 7 and 8.

How does the topology look like?

Proposition 7.1.5 $(\mathcal{P}_2(\mathbb{R}^d), W_2)$ is a complete separable metric space.

$$\lim_{n\to\infty} W_2(\mu_n, \mu) = 0 \iff \begin{cases} (\mu_n) \text{ converges weakly to } \mu \\ (\mu_n) \text{ has u.i. 2nd moments.} \end{cases}$$

Remark Tightness

$\varepsilon > 0, \quad \exists K_\varepsilon \subseteq \mathbb{R}^d$

$$\sup_n \int_{K_\varepsilon^c} \|x\|^2 \, d\mu_n < \varepsilon.$$

A useful lemma

Approximation by convolutions Lemma 7.1.10

Let $\mu \in \mathcal{P}_2(\mathbb{R}^d)$. Consider a family of mollifiers $(\rho_\varepsilon) \in C_\infty(\mathbb{R}^d)$.

ρ is a prob density $\quad \rho_\varepsilon(x) = \varepsilon^{-d} \rho(x/\varepsilon)$.

Example $\quad \rho(x) = \frac{1}{(\sqrt{2\pi})^d} e^{-\frac{|x|^2}{2}} \quad \rho_\varepsilon(x) \sim N(0, \varepsilon^2 I)$.

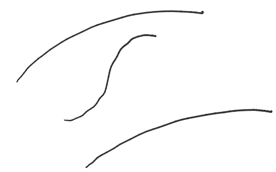
$m = d$

Let $m = \int |x|^2 \rho(x)\,dx$

If $\mu_\varepsilon = \mu * \rho_\varepsilon$, then $W_2(\mu, \mu_\varepsilon) \leq \varepsilon m$.

Thus $\lim_{\varepsilon \downarrow 0} \mu_\varepsilon = \mu$ in $(\mathcal{P}_2, W_2)$.

Absolutely Continuous curves in Wasserstein space. $(\mathcal{P}_2(\mathbb{R}^d), W_2)$



Definition Continuity equation Eqn. 8.1.1

(*) $$\partial_t \mu_t + \nabla\cdot(v_t\,\mu_t) = 0 \quad \text{on } \mathbb{R}^d \times (0,T), \quad T>0.$$

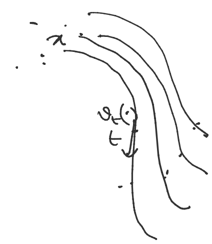
Here (μ_t) is a family of prob measures

and $v:(x,t)\longrightarrow v_t(x)\in\mathbb{R}^d$ is a Borel velocity field.

In the weak sense. $\forall\ \varphi \in C_c^\infty(\mathbb{R}^d)$

$$\frac{d}{dt}\int_{\mathbb{R}^d}\varphi(x)\,d\mu_t(x) = \int_{\mathbb{R}^d}\langle\nabla\varphi(x),\ v_t(x)\rangle\, d\mu_t(x).$$

Interpretation 1 : Flow of push-forwards.



Consider the ODE: $X_0(x) = x \in \mathbb{R}^d$

$$\left[\frac{d}{dt}X_t(x) = v_t(X_t(x)).\right]$$

Given a μ_0, let $\mu_t = (X_t)_\#\mu_0$

Under minimal conditions, Proposition 8.1.8

$\mu_t = (X_t)_\#\mu_0 \iff$ solution of the continuity equation.

Example Take $\mu_0 \in \mathcal{P}_2(\mathbb{R}^d)$. Take $\mu_t = \mu_0 * N(0, tI)$

i.e. $X_0\sim\mu_0$, then

$X_t = X_0 + \sqrt{t}\,Z \sim \mu_t$

(μ_t) is an AC curve

$v_t = -\nabla\log\mu_t$

What are AC curves?

Defn. 1.1.1 A curve (μ_t), $t\in(0,T)$, is said to be AC if $\exists$ some $m\in L^1(0,T)$ such that if $\exists$ some $m \in L^1(0,T)$ such that

$$W_2(\mu_s, \mu_t) \le \int_s^t m(u)\,du \qquad 0 < s < t < T$$

- Such a curve is continuous
- If m is a constant, then the curve is $\mathrm{Lip}(m)$
- reparametrize an AC curve to make it Lip.

<u>Metric derivative</u> Given an AC curve the limit

$$|\mu'_t| = \lim_{s \to t} \frac{W_2(\mu_s, \mu_t)}{|s-t|} \quad \text{exists for a.e. } t \text{ in } (0,T).$$

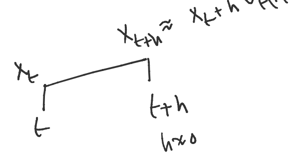
Theorem 1.1.2

Moreover $|\mu'_t| \le m(t)$ a.e.

<u>Theorem</u> 8.3.1 Suppose $(\mu_t,\ t \in (0,T))$ is AC and let $|\mu'_t|$ denote its metric derivative. Then $\exists$ Borel $v_t(x) \in \mathbb{R}^d$ such that $v_t \in L^2(\mu_t)$ and $\|v_t\|_{L^2(\mu_t)} = |\mu'_t|$ t a.e.

and CE holds $\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0.$

Conversely, if (μ_t) satisfies CE for $v_t \in L^2(\mu_t)$, then (μ_t) is AC and $|\mu'_t| \le \|v_t\|_{L^2(\mu_t)}$.



<u>Rmk</u> Given an AC curve, there is an infinite family of v_t that generates the same curve.
But there is a unique one satisfying $\|v_t\|_{L^2(\mu_t)} = |\mu'_t|$.

$$W_2^2(\mu_{h+t}, \mu_t) \cong E(X_{t+h} - X_t)^2 = h\,\|v_t\|_{L^2(\mu_t)}$$

<u>Define</u> (Tangent bundle) Definition 8.4.1

For $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, define $\mathrm{Tan}_\mu \mathcal{P}_2 = \overline{\{\nabla\varphi,\ \varphi \in C_c^\infty(\mathbb{R}^d)\}}^{L^2(\mu)}$

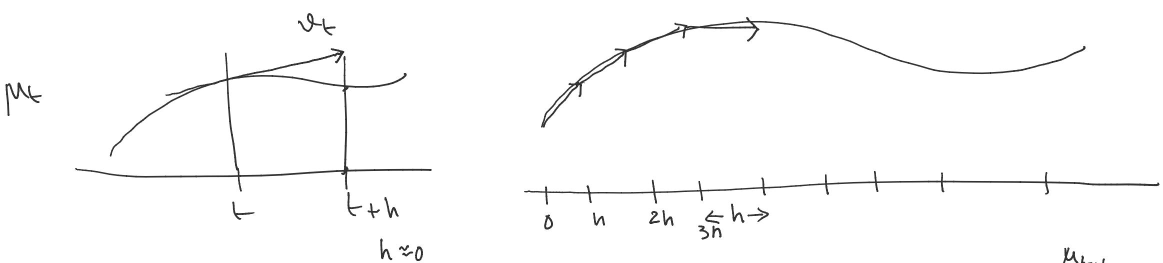
8.4.5 $\exists$ ... unique velocity v_t

For

Proposition 8.4.5 Let (μ_t) be AC. Then the unique velocity v_t satisfying $\|v_t\| = |\mu'_t| \iff v_t \in \mathrm{Tan}_{\mu_t} \mathcal{P}_2$

Proposition 8.4.6 Let (μ_t) be AC and let $v_t \in \mathrm{Tan}_{\mu_t} \mathcal{P}_2(\mathbb{R}^d)$ satisfying $\partial_t \mu_t + \nabla \cdot (v_t \mu_t) = 0.$

Then 1. $$\lim_{h \to 0} \frac{W_2(\mu_{t+h}, (id + h v_t) \# \mu_t)}{|h|} = 0$$



2. For t a.e. in $(0, T)$, let $OT_{\mu_t}^{\mu_{t+h}}$ denote the OT map transporting μ_t to μ_{t+h}, then

$$\underline{\lim_{h \to 0} \frac{1}{h}\left(OT_{\mu_t}^{\mu_{t+h}} - id \right) = v_t} \quad \text{in } L^2(\mu_t).$$

t a.e. in $(0, T)$.

Next time

1. How to define gradients $\nabla_W F(\mu)$
2. Special AC curves "gradient flows".

$$\frac{d}{dt} \mu_t = - \nabla_W F(\mu_t).$$