STAT 581 FINAL EXAM Dec. 8, 1989

1. Let $f(x;\theta) = c(\theta)\exp(-\theta x^2)$, $x \geq 0$, $\theta > 0$.

(a) Show that $c(\theta) = (\theta/\pi)^{\frac{1}{2}}$.
Hint: $\Gamma(\frac{1}{2}) = \pi^{\frac{1}{2}}$.

(b) If X has density f, show that the density of $Y = X^4$ belongs to an exponential family. Exhibit the natural parameter, and the natural sufficient statistic based on n iid observations $Y_1, \dots, Y_n$.

(c) Find the mle for θ. Is it efficient?

2. Let $X_1, \dots, X_n$ be iid, mean μ, variance σ^2, and finite fourth moment μ_4. Let $\bar{X} = (1/n)\sum_{i=1}^n X_i$ and $s_n^2 = (1/n)\sum_{i=1}^n (X_i - \bar{X})$. It is well known that

$$n^{\frac{1}{2}}\begin{pmatrix} \bar{X} - \mu \\ s_n^2 - \sigma^2 \end{pmatrix} \xrightarrow{d} \mathrm{N}_2(\mathbf{0}, \Sigma)$$

where $\Sigma = \begin{pmatrix} \sigma^2 & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4 \end{pmatrix}$.

(a) If $\mu \neq 0$, find the limiting distribution of $n^{\frac{1}{2}}(C_n - c)$ where $C_n = s_n/\bar{X}$ and $c = \sigma/\mu$.

(b) If $\mu = 0$, show that $n^{-\frac{1}{2}}C_n \xrightarrow{d} 1/Z$ where $Z \sim \mathrm{N}(0,1)$.

3. Let $(X_i, Y_i)^T$ be iid $\mathrm{N}_2\left(\begin{pmatrix} \theta \\ \theta^2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$. Find the locally most powerful test of $\theta = 0$ against $\theta > 0$.