STAT 581 FINAL EXAM Dec. 11, 1995

1. Let $X_1, X_2, \ldots, X_n$ be iid positive random variables with $\mathbf{E}\log^2 X < \infty$.

(a) Show that the geometric mean $G_n = (\prod_1^n X_i)^{1/n} \xrightarrow{\mathbf{P}} g$, and find the constant g in terms of the cdf or appropriate moments of functions of X.

(b) Prove that $n^{\frac{1}{2}}(G_n - g) \xrightarrow{d} W$ and $(G_n/g)^{\sqrt{n}} \xrightarrow{d} Z$ for some random variables W and Z, and describe the distributions of W and Z.

2. Let $X_1, X_2, \ldots, X_n$ be iid with probability mass function

$$p(x;\theta) = \frac{2x}{\theta(\theta+1)}, \quad x = 1, 2, \ldots, \theta.$$

(a) Find the mle of θ.
Hint: Look at $L_n(\theta)/L_n(\theta+1)$.

(b) Does classical asymptotic likelihood theory apply to this estimator? If not, why not?

3. Let $X_1, X_2, \ldots, X_n$ be iid random variable with density $f(x;\boldsymbol{\theta})$, where $\boldsymbol{\theta}$ is a p-dimensional parameter. Assume the usual regularity conditions associated with asymptotics of likelihood.

(a) Let $\boldsymbol{\eta} = \boldsymbol{\eta}(\boldsymbol{\theta})$ be a p-dimensional one-to-one differentiable reparametrization. Show that the information $I_n(\boldsymbol{\eta})$ about $\boldsymbol{\eta}$ in the sample can be written

$$I_n(\eta) = \left[\frac{d\theta}{d\eta}\right]^T I_n(\theta)\left[\frac{d\theta}{d\eta}\right]$$

where $\left[\frac{d\theta}{d\eta}\right]$ is the matrix with i,j-element $\partial\theta_i/\partial\eta_j$.

(b) In the case where $\boldsymbol{\theta} = (\tau, \nu)$, where τ is a parameter of interest and ν is a nuisance parameter, it is often desirable to find a partial reparametrization $\boldsymbol{\eta} = \boldsymbol{\nu}(\boldsymbol{\theta}) = (\tau, \mu) = (\tau, \mu(\tau, \nu))$ such that $I_n(\boldsymbol{\eta})$ is a diagonal matrix. This is called an **orthogonal** parametrization. Write down the differential equation that such a reparametrization must satisfy.