STAT 582 FINAL EXAM March 7, 1990

This final is open notes, open books, free use of libraries. It is due on March 12 at noon in my mailbox (across from the Statistics office in Padelford Hall). Work alone: if you find typos or unclarities, please let **me** know. Do as many questions as you can. Please write clean and legible copies of your solutions.

1. Let the sample space be {0,1}, and the parameter space {1,2}. Assume a loss function is given by $L(1,1) = L(2,2) = 0$ and $L(1,2) = a$, $L(2,1) = b$. Suppose that we may observe *either* X or Y where

$$\mathbf{P}_1(X = 1) = 2/3 \qquad \mathbf{P}_2(X = 1) = \tfrac{1}{2}$$
$$\mathbf{P}_1(Y = 1) = 3/4 \qquad \mathbf{P}_2(Y = 1) = \tfrac{1}{2}$$

Let $\boldsymbol{\lambda} = (\lambda, 1 - \lambda)$ be the prior distribution.

(a) Find the Bayes risk when X is observed, and when Y is.

(b) In the case $a = b$, $\boldsymbol{\lambda} = (\frac{1}{2}, \frac{1}{2})$, would it be better to observe X or Y? Find the Bayes rules and corresponding Bayes risks.

(c) For the general case, would it be better to observe X or Y?

2. Suppose that $X_1, \ldots, X_n$ are iid random variables with density

$$f(x; \mu, \kappa) = \frac{1}{2\pi I_0(\kappa)} \exp(\kappa \mu^T x)$$

where μ and x are unit vectors in $\mathbf{R}^2$.

(a) Show that the mle $\hat{\mu} = \sum x_i / r$ where $r = \|\sum x_i\|$ is the resultant.

(b) Show that the distribution of r depends on κ only (so that if κ is known, r is ancillary).

(c) Compute the Barndorff-Nielsen formula for the density of the mle (do not worry about whether or not it applies here) for the case where κ is known.

(d) Show that this is the exact conditional distribution for $\hat{\mu}$, given r.

3. Let $X_1, \ldots, X_n$ given θ be iid $\exp(\theta)$, and let $\theta \sim \Gamma(\alpha, \beta)$.

(a) Find the posterior distribution of θ.

(b) Find the Bayes estimate of θ for the loss function $L(\theta, d) = (\theta - d)^2/\theta^2$ based on the observed values.

(c) Show that the estimate in (b) is Bayes risk consistent, *i.e.,* that the Bayes risk vanishes as $n \to \infty$.

(d) Examine the large sample frequency properties (consistency, asymptotic distribution) of the estimator obtained in (b).

4. Suppose (X, Y) have density $f(x, y) = \exp(-(\theta x + y/\theta))$, $x, y > 0$.

(a) Show that the mle T of θ based on an iid sample of size n from (X, Y) is $(\sum x_i / \sum y_i)^{\frac{1}{2}}$, and that it is not sufficient.

(b) Show that the density of T can be written

$$c_n(\frac{\theta}{t} + t/\theta)^{-2n} t^{-1}, \quad t > 0$$

(you may take as known that $c_n = 2(2n-1)!/((n-1)!)^2$).
Hint: Change variables $(\sum x_i, \sum y_i) \to (t, u)$ where $u = (\sum x_i \sum y_i)^{\frac{1}{2}}$.

(c) Compute the Fisher information from observing T alone, and show that the loss in Fisher information (due to insufficiency) is $\dfrac{2n}{2n+1} \dfrac{1}{\theta^2}$ units.

(d) Show that U is an ancillary statistic.

5. Let $\mathbf{X} \sim \mathrm{Mult}_k(n, \mathbf{p}(\theta))$, where θ is a real-valued parameter and the $p_i(\theta)$, $i = 1, \ldots, k$ are continuously differentiable functions.

(a) Derive an explicit expression for the score test statistic of $H_0\colon \theta = \theta_0$.

(b) What is the large sample distribution of the statistic in (a) when $\theta = \theta_0 + c/n^{\frac{1}{2}}$?

(c) In genetical linkage experiments the null hypothesis is $\theta = \frac{1}{2}$. The following are multinomial cell probabilities from three different linkage experiments:

(i) $(\frac{1}{2}\theta, \frac{1}{2}(1-\theta), \frac{1}{2}\theta, \frac{1}{2}(1-\theta))$

(ii) $((1-\theta)^2/2, 2\theta(1-\theta), \theta^2/2, (\theta^2 + (1-\theta)^2)/2)$

(iii) $(2\theta(1-\theta), \theta^2 + (1-\theta)^2)$.

Which of these experiments is best for testing $\theta = \frac{1}{2}$?