STAT 581 Homework 2 Due Oct 16

This homework must be handed in for grading.

1. Suppose that $X_n \sim \mathrm{AsN}(\mu_n, \sigma_n^2)$.

(a) Show that $X_n \sim \mathrm{AsN}(m_n, s_n^2)$ iff $s_n/\sigma_n \to 1$ and $(\mu_n - m_n)/\sigma_n \to 0$.

(b) Show that $a_n X_n + b_n \sim \mathrm{AsN}(\mu_n, \sigma_n^2)$ iff $a_n \to 1$ and $(\mu_n(a_n - 1) + b_n)/\sigma_n \to 0$.

(c) Let $X_n \sim \chi^2(n)$, so that $X_n \sim \mathrm{AsN}(n, 2n)$. Is the asymptotic distribution of $(1 - 1/n)X_n$ the same as that of X_n? How about that of $(1 - 1/\sqrt{n})X_n$?

2. A sequence U_n is **bounded in probability** if for any $\varepsilon > 0$ there are integers M_ε, N_ε such that

$$F_{U_n}(M_\varepsilon) - F_{U_n}(-M_\varepsilon) > 1 - \varepsilon$$

for all $n > N_\varepsilon$. We say that $X_n = O_P(Y_n)$ if X_n/Y_n is bounded in probability. In particular, for the sequence U_n above we have that $U_n = O_P(1)$. Also, $X_n = o_P(Y_n)$ if $X_n/Y_n \xrightarrow{P} 0$.

(a) Show that if $X_n \xrightarrow{d} X$, then $X_n = O_P(1)$.

(b) Show that if $X_n = o_P(Y_n)$, then $X_n = O_P(Y_n)$.

(c) Suppose that $X_n \sim \mathrm{AsN}(\mu_n, \sigma_n^2)$ and that $Y_n/\sigma_n \xrightarrow{P} 1$. Show that $X_n - \mu_n = O_p(Y_n)$.

3. Suppose that $X_n \sim \mathrm{AsN}(\theta, \sigma_n^2(\theta))$, where θ is a realvalued parameter. A differentiable function g is called a **variance stabilizing transformation** for X_n if $g(X_n) \sim \mathrm{AsN}(g(\theta), \gamma_n)$, where γ_n does not depend on θ.

(a) Assume that $\sigma_n(\theta) = h(\theta)v_n$, where $v_n \to 0$. Show that a class of variance stabilizing transformations satisfies the differential equation $g'(x) = c/h(x)$, where c is a constant.

(b) Find a variance stabilizing transformation for $X_n \sim \mathrm{Bin}(n, p)$.

4. Let $X_n \sim \mathrm{AsN}(\mu, \sigma_n^2)$, with $\sigma_n \to 0$. Let g be a realvalued function, differentiable m times at $x = \mu$, with $g^{(m)}(\mu) \neq 0$, but $g^{(j)}(\mu) = 0$ for $j < m$. Show that

$$\frac{g(X_n) - g(\mu)}{\frac{1}{m!}\, g^{(m)}(\mu)\sigma_n^m} \xrightarrow{d} (\mathrm{N}(0,1))^m.$$