1. (X_i) is a (weakly) *stationary* sequence if $\mathbf{E}X_i = \mu$, $\mathbf{Var}X_i = \sigma^2$, and $\mathbf{Cov}(X_i, X_{i+k}) = \rho_k$, $k \geq 1$.

(a) Under what conditions on ρ_k does $\bar{X}$ converge in probability to μ?

(b) The sequence is called *exchangeable* if $\rho_k \equiv \rho$. What can you say about the convergence of $\bar{X}$ for exchangeable sequences?
Hint: Look at $P(|\bar{X}_n - (1/n)\sum_{n+k+1}^{2n+k} X_i| > \varepsilon)$.

2. (a) Let Y_n be distributed as N(0,1) with probability π_n, and as N$(0,\tau_n^2)$ with probability $1-\pi_n$. If $\tau_n \to \infty$ and $\pi_n \to \pi$, determine for what values of π the sequence (Y_n) does have a limiting distribution.

(b) Determine to what values Var(Y_n) can tend to under the conditions in (a).

(c) Use (b) to show that the limit of the variance need not coincide with the variance of the limiting distribution.

3. Let $X_1, \ldots, X_n$ be iid, mean μ, k'th central moment $\mu_k = \mathbf{E}(X_1 - \mu)^k$. Define $b_k = \dfrac{1}{n}\sum_1^n (X_i - \mu)^k$, and $m_k = \dfrac{1}{n}\sum_1^n (X_i - \bar{X})^k$.

(a) Show that $b_k \overset{a.s.}{\to} \mu_k$.

(b) For the rest of the problem assume that $\mu_{2k} < \infty$. Show that

$$(b_1, \ldots, b_k)^T \sim \mathrm{AsN}_k((0, \mu_2, \ldots, \mu_k)^T, [(\mu_{i+j} - \mu_i\mu_j)/n])$$

where $[a_{ij}]$ is the matrix with (i,j)-element a_{ij}.

(c) Show that $m_k = \sum_{j=0}^k \binom{k}{j}(-1)^{k-j} b_j b_1^{k-j}$, and deduce that $m_k \overset{a.s.}{\to} \mu_k$.

(d) Show that

$$n^{\frac{1}{2}}(m_i - \mu_i) = n^{\frac{1}{2}}(b_i - \mu_i - i\,\mu_{i-1}b_1) + o_P(1)$$

as $n \to \infty$.

(e) Deduce that

$$n^{\frac{1}{2}}(m_2 - \mu_2, \ldots, m_k - \mu_k)^T \overset{d}{\to} \mathrm{N}_{k-1}(\mathbf{0}, [\sigma_{ij}])$$

where $\sigma_{ij} = \mu_{i+j+2} - \mu_{i+1}\mu_{j+1} - (i+1)\mu_i\mu_{j+2} - (j+1)\mu_{i+2}\mu_j + (i+1)(j+1)\mu_i\mu_j\mu_2$.

(f) Determine the asymptotic distribution of $g_2 = m_4/m_2^2 - 3$.