1. Each research group is to describe their progress and current ideas to me by Friday, October 27 (sooner is better). Just catch me any time (you can look at my schedule on my homepage to see when may be a good time).

2. Let (X_i) be iid (μ, σ^2) with $\mu_4 = \mathbf{E}(X_i - \mu)^4 < \infty$. Let $Y_i = (X_i - \mu)^2$.

(a) Show that $Y_i \sim (\sigma^2, \mu_4 - \sigma^4)$

(b) Show that

$$\sqrt{n}\,\frac{\bar{Y}_n - \sigma^2}{\sqrt{2\sigma^4}} \xrightarrow{d} \mathrm{N}(0, 1 + \gamma_2/2)$$

where $\gamma_2 = (\mu_4 - 3\sigma^4)/\sigma^4$ is the *coefficient of kurtosis*.

(c) Show that

$$\sqrt{n}\,\frac{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2 - \sigma^2}{\sqrt{2\sigma^4}} \xrightarrow{d} \mathrm{N}(0, 1 + \gamma_2/2)$$

Hint: What is $\sum(X_i - \bar{X})^2 - \sum Y_i$?

(d) For testing $H_0\colon \sigma \le \sigma_0$ against $H_A\colon \sigma > \sigma_0$ under normality we reject H_0 if

$$\frac{1}{\sigma_0^2}\sum(X_i - \bar{X})^2 \ge \chi^2_{n-1,\alpha}.$$

What happens if the data are not normal?
Hint: It may help to establish that $\sqrt{n/2}(\chi^2_{n-1,\alpha}/n - 1) \to z_\alpha$, where z_α is the standard normal quantile.

3. Let (X_i, Y_i), $i = 1, \ldots, n$ be iid realizations of $(X, Y) \sim \mathrm{N}_2(\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1 & \rho\\ \rho & 1\end{pmatrix})$, so the density of (X, Y) is

$$f_{X,Y}(x, y; \rho) = \frac{1}{2\pi(1-\rho^2)^{\frac{1}{2}}}\exp(-\frac{(x^2 - 2\rho xy + y^2)}{2(1-\rho^2)}).$$

Define

$$a_{kl} = \frac{1}{n}\sum_{i-1}^{n} X_i^k Y_i^l$$

and

$$m_{kl} = a_{kl} - \bar{X}^k \bar{Y}^l.$$

The sample correlation coefficient is given by

$$r = \frac{m_{11}}{(m_{20}m_{02})^{\frac{1}{2}}}.$$

(a) Show that $n^{\frac{1}{2}}(r-\rho)$ has the same limiting distribution as $n^{\frac{1}{2}}(m_{11}-\rho(m_{02}m_{20})^{\frac{1}{2}})$.

(b) Show that $n^{\frac{1}{2}}(r-\rho)$ has the same limiting distribution as $n^{\frac{1}{2}}(a_{11}-\rho(a_{02}a_{20})^{\frac{1}{2}})$.
Hint: Repeated use of Slutsky.

(c) Show that the covariance matric for $(XY, X^2, Y^2)^T$ is

$$\mathbf{Cov}\begin{pmatrix} XY \\ X^2 \\ Y^2 \end{pmatrix} = \begin{pmatrix} 1+\rho^2 & 2\rho & 2\rho \\ 2\rho & 2 & 2\rho^2 \\ 2\rho & 2\rho^2 & 2 \end{pmatrix}.$$

Hint: You may use without the proof the normal regression result that $(X|Y)$~$\mathrm{N}(\rho Y, 1-\rho^2)$.

(d) Derive the asymptotic distribution of $(a_{11}, a_{20}, a_{02})^T$.

(e) Show that r~$\mathrm{AsN}(\rho, (1-\rho^2)^2/n)$.