STAT 581 Homework 6 Due Nov. 13

This homework must be handed in for grading.

1. Show that the saddlepoint approximation for the arithmetic mean of n iid N(μ, σ^2) variates is exact without renormalization.

2. Let the density of X have exponential family form

$$f_X(x;\eta) = \exp(\eta T(x) - A(\eta))\, h(x).$$

(a) Show that the exponential tilting of $f_{T(X)}$ is the same exponential family with parameter $\theta + \eta$.

(b) Determine the saddlepoint approximation to the density of $T = \sum T(X_i)$, where the X_i are iid with density f_X.

(c) Show that the expression in (b) can be written

$$f_T(t;\eta) = (2\pi)^{-1/2} J_n(\hat\eta)^{-\frac{1}{2}} \frac{L_n(\eta)}{L_n(\hat\eta)}$$

where $J(\eta) = -(\partial^2/\partial\eta^2)\log L(\eta)$.

3. Let $\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathrm{N}_2(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$. Find the likelihood equation for estimating ρ from n independent observations of $(X,Y)^T$. Show that it always has a solution in [-1,1], and that the solution is unique for large enough values of n.

4. (a) Suppose that $\hat\theta$ solves the likelihood equation. Let $\eta = q(\theta)$ be a differentiable one-to-one reparametrization of the problem. Show that $\hat\eta = q(\hat\theta)$ is the mle of η.

(b) Let, again, $\eta = q(\theta)$ be a reparametrization, but only assume that q is a function (not necessarily one-to-one or differentiable). Assume that the likelihood function $L(\theta)$ is bounded for $\theta \in \Theta$. Define the likelihood function **induced by** q as

$$L^q(\eta) = \sup_{\theta: q(\theta)=\eta} L(\theta).$$

Show that $\hat\eta = q(\hat\theta)$ maximizes $L^q(\eta)$ over $\eta \in H = \{q(\theta) : \theta \in \Theta\}$, where $\hat\theta$ maximizes $L(\theta)$.

(c) Suppose that $X_1, \ldots, X_n$ are iid Bern(p). Determine the mle of $\mathbf{Var} X_1$.