1. In astronomy at the turn of the century two estimates of variance of a normal distribution were in common use: Peter's mean absolute deviation $s_1 = (\pi/2)^{\frac{1}{2}}(1/n)\sum|X_i - \bar{X}|$, and Bessel's root mean squared deviation $s_2 = ((1/n)\sum(X_i - \bar{X})^2)^{\frac{1}{2}}$. A. S. Eddington claimed (in his book *Stellar Movemets*) that s_1 was the better estimate on grounds of having the smaller variance and being a safer criterion, "especially if any doubtful observations have been rejected."

(a) Show that $\mathbf{E}s_1 = \sigma(1 - 1/n)^{\frac{1}{2}}$.

(b) Show that $\mathbf{E}s_2 = \sigma(2/n)^{\frac{1}{2}}\Gamma(n/2)/\Gamma((n-1)/2)$.

(c) Compare the bias to order $1/n$ of s_1 and s_2.

(d) In a paper by Fisher (1920) the variance of s_1 was shown to be $(\pi/2 - 1)\sigma^2/n + O(n^{-2})$. Show that the variance of s_2 is $\sigma^2/2n + O(n^{-2})$.

(e) On the basis of these calculations, which estimator would you prefer in large samples?

2. Let $X_1, \ldots, X_n$ be iid $\mathrm{N}(\mu, \sigma^2)$, where μ is an integer and σ^2 is known.

(a) Show that the mle of μ is $[\bar{X} + .5]$ where $[\bullet]$ is the integer part function, and that it is unbiased.
Hint: Use symmetry.

(b) Show that if $n \geq 16\sigma^2$, then $\mathbf{P}(\hat{\mu} = \mu) \geq .95$.

(c) Show that $\hat{\mu}$ is consistent.

(d) Show that $\mathbf{Var}(\hat{\mu}) = 2\sum_1^\infty (2j-1)(1 - \Phi((j - \frac{1}{2})n^{\frac{1}{2}}/\sigma))$.

(e) Verify that $\mathbf{Var}(\hat{\mu})/((8\sigma^2/\pi n)^{\frac{1}{2}}\exp(-n/8\sigma^2)) \to 1$ as $n \to \infty$.
Hint: Use the fact that for large x $1 - \Phi(x) = \phi(x)(1/x + O(1/x^3))$. Then show that $\sum \exp(-j(j-1)n/2\sigma^2) = 1 + O(\exp(-n/\sigma^2))$ as $n \to \infty$.

3. (a) Let $\Sigma = [\sigma_{ij}]$ be a symmetric matrix. Show that

$$\frac{\partial|\Sigma|}{\partial\sigma_{ii}} = \Sigma_{ii} \qquad \frac{\partial|\Sigma|}{\partial\sigma_{ij}} = 2\Sigma_{ij}$$

where $|\Sigma|$ is the determinant of Σ, and Σ_{ij} is the (i, j)-cofactor of Σ.

(b) Show that

$$\frac{\partial\mathbf{x}^T\Sigma\mathbf{x}}{\partial\mathbf{x}} = 2\Sigma\mathbf{x}$$

where $\partial/\partial\mathbf{x}$ denotes the partial derivatives with respect to each component of the vector $\mathbf{x}$ arranged in a column vector.

(c) Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be vectors in $\mathbf{R}^p$, and $\bar{\mathbf{x}}$ the componentwise average. Show that for any vector $\mathbf{b}$

$$\sum_1^N (\mathbf{x}_i - \mathbf{b})(\mathbf{x}_i - \mathbf{b})^T = \sum_1^N (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T + \sum_1^N (\bar{\mathbf{x}} - \mathbf{b})(\bar{\mathbf{x}} - \mathbf{b})^T$$

(d) Let $\mathbf{x}_1, \ldots, \mathbf{x}_N$ be observations of $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$, where Σ is non-singular. Find the mle of $\boldsymbol{\mu}$.
Hint: Show that

$$\sum_1^N (\mathbf{x}_i - \boldsymbol{\mu})^T \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu}) = \mathrm{tr} \sum_1^N \Sigma^{-1} (\mathbf{x}_i - \boldsymbol{\mu})(\mathbf{x}_i - \boldsymbol{\mu})^T .$$

(e) Let $f(C) = \frac{1}{2} N \log |C| - \frac{1}{2} \sum_{i,j=1}^{p} c_{ij} d_{ij}$ where $C = [c_{ij}]$ is positive semi-definite and $D = [d_{ij}]$ is positive definite. Show that the maximum of $f(C)$ is obtained at $C = ND^{-1}$.

(f) Show that the mle of Σ for the data in (d) is $(1/N) \sum (\mathbf{x}_i - \bar{\mathbf{x}})(\mathbf{x}_i - \bar{\mathbf{x}})^T$.
Hint: First find the mle of Σ^{-1}.