1. The von Mises distribution on the circle has a probability density for an angle x (in radians) given by

$$f(x;\mu,\kappa) = c(\kappa)\exp(\kappa\cos(x-\mu)) \quad 0 \le x \le 2\pi.$$

(a) Using polar coordinates $(r,\theta) = (f(x),x)$, sketch the density function for some $\kappa \ne 0$. Interpret the parameters μ and κ. Which is the density with $\kappa = 0$?

(b) Express f in exponential family form, and write down the likelihood equations. Here $c(\kappa) = I_0(\kappa)^{-1}$, where I_0 is the modified Bessel function of the first kind and order zero (defined as the appropriate integral, of course). The likelihood equations will depend on $A(x) = (d/dx)\log I_0(x)$.

(c) What is the asymptotic distribution of the mle $(\hat\mu,\hat\kappa)$?

2. Every human being can be classified into one of four blood groups (phenotypes) O, A, B, AB. The inheritance of blood groups is controlled by three genes, O, A, B, of which O is recessive to A and B. If r, p, q are the gene probabilities in the population of O, A, B, respectively, then the probabilities of the six possible combinations (genotypes) in random mating (where two individuals drawn at random from the population contribute one gene each) are

Phenotype	Genotype	Probability
O	OO	r^2
A	AA	p^2
A	AO	$2rp$
B	BB	q^2
B	BO	$2rq$
AB	AB	$2pq$

We observe among N individuals the phenotype frequencies n_O, n_A, n_B, n_{AB}, and wish to estimate the gene probabilities from such data. A simple approach is to regard the observations as incomplete, the complete data set being the genotype frequencies n_{OO}, n_{AA}, n_{AO}, n_{BB}, n_{BO}, n_{AB}.

(a) Derive the EM-algorithm for estimation of (p,q,r).

(b) Estimate (p,q,r) from $n_O = 176$, $n_A = 182$, $n_B = 60$, $n_{AB} = 17$.

(c) Estimate the covariance of $(\hat p,\hat q,\hat r)$.
Hint: Use the asymptotic covariance for $(\hat p,\hat q)$, and note that $\hat r = 1-(\hat p+\hat q)$.

3. Assume the usual regularity conditions for asymptotic normality of the mle. Let $\tilde\theta_n$ be a $\sqrt{n}$-consistent estimator of θ, i.e., $\sqrt{n}(\tilde\theta_n - \theta) = O_P(1)$. Fisher's method of scoring finds the mle by the recursive formula

$$\tilde\theta_n^{(i+1)} = \tilde\theta_n^{(i)} + \frac{l'_n(\tilde\theta_n^{(i)})}{I_n(\tilde\theta_n^{(i)})} \quad i = 1,2,,, \tag{1}$$

where $\tilde\theta_n^{(1)} = \tilde\theta_n$.

(a) Show that the recursion (1) (if it converges) converges to a solution to the likelihood equation.

(b) Show that the one-step estimator obtained by doing just one iteration of (1) is first order efficient, i.e., satisfies

$$\sqrt{n}(\tilde{\theta}_n^{(2)} - \theta - I_n(\theta)^{-1} S_n(\theta)) \xrightarrow{P} 0.$$

(c) Show that $\tilde{\theta}_n^{(2)} \sim \mathrm{AsN}(\theta, I_n^{-1}(\theta))$.

(d) Show that any two first order efficient estimators $\bar{\theta}_1$ and $\bar{\theta}_2$ satisfy $\bar{\theta}_1 - \bar{\theta}_2 = o_P(n^{-\frac{1}{2}})$. In particular, deduce this for any two consistent roots of the likelihood equation.

4. (a) Write the two-parameter normal distribution in exponential family form. Which are the natural parameters (η_1, η_2) in terms of the usual (μ, σ^2)-parametrization? Verify that the mles for the natural parameters correspond to to the usual mles for (μ, σ^2), i.e., that $\hat{\eta}_i = \eta_i(\hat{\mu}, \hat{\sigma}^2)$, $i = 1, 2$.

(b) Now consider the same distribution as a location-scale family. Which parametrization does this correspond to? Verify that the solutions to the likelihood equations for location-scale families (as derived in Ex. 2.1.3 in class) correspond to the usual mles, and compute the information matrix using the formula derived in class.